Classical Algebraic Geometry: a modern view

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Preface

The main purpose of the present treatise is to give an account of some of the topics in algebraic geometry which while having occupied the minds of many mathematicians in previous generations have fallen out of fashion in modern times. Often in the history of mathematics new ideas and techniques make the work of previous generations of researchers obsolete, especially this applies to the foundations of the subject and the fundamental general theoretical facts used heavily in research. Even the greatest achievements of the past generations which can be found for example in the work of F. Severi on algebraic cycles or in the work of O. Zariski's in the theory of algebraic surfaces have been greatly generalized and clarified so that they now remain only of historical interest. In contrast, the fact that a nonsingular cubic surface has 27 lines or that a plane quartic has 28 bitangents is something that cannot be improved upon and continues to fascinate modern geometers. One of the goals of this present work is then to save from oblivion the work of many mathematicians who discovered these classic tenets and many other beautiful results.

In writing this book the greatest challenge the author has faced was distilling the material down to what should be covered. The number of concrete facts, examples of special varieties and beautiful geometric constructions that have accumulated during the classical period of development of algebraic geometry is enormous and what the reader is going to find in the book is really only the tip of the iceberg; a work that is like a taste sampler of classical algebraic geometry. It avoids most of the material found in other modern books on the subject, such as, for example, [10] where one can find many of the classical results on algebraic curves. Instead, it tries to assemble or, in other words, to create a compendium of material that either cannot be found, is too dispersed to be found easily, or is simply not treated adequately by contemporary research papers. On the other hand, while most of the material treated in the book exists in classical treatises in algebraic geometry, their somewhat archaic terminology

and what is by now completely forgotten background knowledge makes these books useful to but a handful of experts in the classical literature. Lastly, one must admit that the personal taste of the author also has much sway in the choice of material.

The reader should be warned that the book is by no means an introduction to algebraic geometry. Although some of the exposition can be followed with only a minimum background in algebraic geometry, for example, based on Shafarevich's book [530], it often relies on current cohomological techniques, such as those found in Hartshorne's book [282]. The idea was to reconstruct a result by using modern techniques but not necessarily its original proof. For one, the ingenious geometric constructions in those proofs were often beyond the authors abilities to follow them completely. Understandably, the price of this was often to replace a beautiful geometric argument with a dull cohomological one. For those looking for a less demanding sample of some of the topics covered in the book, the recent beautiful book [39] may be of great use.

No attempt has been made to give a complete bibliography. To give an idea of such an enormous task one could mention that the report on the status of topics in algebraic geometry submitted to the National Research Council in Washington in 1928 [535] contains more than 500 items of bibliography by 130 different authors only in the subject of planar Cremona transformations (covered in one of the chapters of the present book.) Another example is the bibliography on cubic surfaces compiled by J. E. Hill [295] in 1896 which alone contains 205 titles. Meyer's article [385] cites around 130 papers published 1896-1928. The title search in MathSciNet reveals more than 200 papers refereed since 1940, many of them published only in the past 20 years. How sad it is when one considers the impossibility of saving from oblivion so many names of researchers of the past who have contributed so much to our subject.

A word about exercises: some of them are easy and follow from the definitions, some of them are hard and are meant to provide additional facts not covered in the main text. In this case we indicate the sources for the statements and solutions.

I am very grateful to many people for their comments and corrections to many previous versions of the manuscript. I am especially thankful to Sergey Tikhomirov whose help in the mathematical editing of the book was essential for getting rid of many mistakes in the previous versions. For all the errors still found in the book the author bears sole responsibility.

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1

Polarity

1.1 Polar hypersurfaces

1.1.1 The polar pairing

We will take $\mathbb{C}$ as the base field, although many constructions in this book work over an arbitrary algebraically closed field.

We will usually denote by E a Its dual vector space will be denoted by $E^\vee$.

Let $S(E)\binom{d+n}{n}$. The image of a tensor $v_1 \otimes \cdots \otimes v_d$ in $S^d(E)$ is denoted by $v_1 \cdots v_d$.

The permutation group $\mathfrak{S}_d$

$$S^d(E) \to S_d(E).$$

Replacing E by its dual space $E^\vee$, we obtain a natural isomorphism

$$\mathfrak{p}_d : S^d(E^\vee) \to S_d(E^\vee). \tag{1.1}$$

Under the identification of $(E^\vee)^{\otimes d}$ with the space $(E^{\otimes d})^\vee$, we will be able to identify $S_d(E^\vee)$ with the space $\mathrm{Hom}(E^d, \mathbb{C})^{\mathfrak{S}_d}$ of symmetric d-multilinear functions $E^d \to \mathbb{C}$. The isomorphism $\mathfrak{p}_d$ is classically known as the *total polarization map*.

Next we use that the quotient map $E^{\otimes d} \to S^d(E)$ is a universal symmetric d-multilinear map, i.e. any linear map $E^{\otimes d} \to F$ with values in some vector space F factors through a linear map $S^d(E) \to F$. If $F = \mathbb{C}$, this gives a natural isomorphism

$$(E^{\otimes d})^\vee = S_d(E^\vee) \to S^d(E)^\vee.$$

Composing it with $\mathfrak{p}_d$, we get a natural isomorphism

$$S^d(E^\vee) \to S^d(E)^\vee. \tag{1.2}$$

It can be viewed as a perfect bilinear pairing, the *polar pairing*

$$\langle , \rangle : S^d(E^\vee) \otimes S^d(E) \to \mathbb{C}. \tag{1.3}$$

This pairing extends the natural pairing between E and $E^\vee$ to the symmetric powers. Explicitly,

$$\langle l_1 \cdots l_d, w_1 \cdots w_d \rangle = \sum_{\sigma \in \mathfrak{S}_d} l_{\sigma^{-1}(1)}(w_1) \cdots l_{\sigma^{-1}(d)}(w_d).$$

One can extend the total polarization isomorphism to a *partial polarization map*

$$\langle , \rangle : S^d(E^\vee) \otimes S^k(E) \to S^{d-k}(E^\vee), \quad k \le d, \tag{1.4}$$

$$\langle l_1 \cdots l_d, w_1 \cdots w_k \rangle = \sum_{1 \le i_1 \le \ldots \le i_k \le n} \langle l_{i_1} \cdots l_{i_k}, w_1 \cdots w_k \rangle \prod_{j \ne i_1, \ldots, i_k} l_j.$$

In coordinates, if we choose a basis $(\xi_0, \ldots, \xi_n)$ in E and its dual basis $t_0, \ldots, t_n$ in $E^\vee$, then we can identify $S(E^\vee)$ with the polynomial algebra $\mathbb{C}[t_0, \ldots, t_n]$ and $S^d(E^\vee)$ with the space $\mathbb{C}[t_0, \ldots, t_n]_d$ of homogeneous polynomials of degree d. Similarly, we identify $S^d(E)$ with $\mathbb{C}[\xi_0, \ldots, \xi_n]$. The polarization isomorphism extends by linearity of the pairing on monomials

$$\langle t_0^{i_0} \cdots t_n^{i_n}, \xi_0^{j_0} \cdots \xi_n^{j_n} \rangle = \begin{cases} i_0! \cdots i_n! & \text{if } (i_0, \ldots, i_n) = (j_0, \ldots, j_n), \\ 0 & \text{otherwise.} \end{cases}$$

One can give an explicit formula for pairing (1.4) in terms of differential operators. Since $\langle t_i, \xi_j \rangle = \delta_{ij}$, it is convenient to view a basis vector ξ_j as the partial derivative operator $\partial_j = \frac{\partial}{\partial t_j}$.

$$D_\psi = \psi(\partial_0, \ldots, \partial_n).$$

The pairing (1.4) becomes

$$\langle \psi(\xi_0, \ldots, \xi_n), f(t_0, \ldots, t_n) \rangle = D_\psi(f).$$

For any monomial $\boldsymbol{\partial}^{\mathbf{i}} = \partial_0^{i_0} \cdots \partial_n^{i_n}$ and any monomial $\mathbf{t}^{\mathbf{j}} = t_0^{j_0} \cdots t_n^{j_n}$, we have

$$\boldsymbol{\partial}^{\mathbf{i}}(\mathbf{t}^{\mathbf{j}}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} \mathbf{t}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j} - \mathbf{i} \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Here and later we use the vector notation:

$$\mathbf{i}! = i_0! \cdots i_n!, \quad \binom{k}{\mathbf{i}} = \frac{k!}{\mathbf{i}!}, \quad |\mathbf{i}| = i_0 + \cdots + i_n.$$

The total polarization $\tilde{f}$ of a polynomial f is given explicitly by the following formula:

$$\tilde{f}(v_1, \ldots, v_d) = D_{v_1 \cdots v_d}(f) = (D_{v_1} \circ \ldots \circ D_{v_d})(f).$$

Taking $v_1 = \ldots = v_d = v$, we get

$$\tilde{f}(v, \ldots, v) = d!f(v) = D_{v^d}(f) = \sum_{|\mathbf{i}|=d} \tbinom{d}{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f. \tag{1.6}$$

Remark 1.1.1 The polarization isomorphism was known in the classical literature as the *symbolic method*. Suppose $f = l^d$ is a d-th power of a linear form. Then $D_v(f) = dl(v)^{d-1}$ and

$$D_{v_1} \circ \ldots \circ D_{v_k}(f) = d(d-1)\cdots(d-k+1)l(v_1)\cdots l(v_k)l^{d-k}.$$

In classical notation, a linear form $\sum a_i x_i$ on $\mathbb{C}^{n+1}$ is denoted by a_x and the dot-product of two vectors a, b is denoted by (ab). Symbolically, one denotes any homogeneous form by a_x^d and the right-hand side of the previous formula reads as $d(d-1)\cdots(d-k+1)(ab)^k a_x^{d-k}$.

Let us take $E = S^m(U^\vee)$ for some vector space U and consider the linear space $S^d(S^m(U^\vee)^\vee)$. Using the polarization isomorphism, we can identify $S^m(U^\vee)^\vee$ with $S^m(U)$. Let $(\xi_0, \ldots, \xi_r)$ be a basis in U and $(t_0, \ldots, t_{r+1})$ be the dual basis in $U^\vee$. Then we can take for a basis of $S^m(U)$ the monomials $\boldsymbol{\xi}^{\mathbf{j}}$. The dual basis in $S^m(U^\vee)$ is formed by the monomials $\frac{1}{\mathbf{i}!}\mathbf{x}^{\mathbf{i}}$. Thus, for any $f \in S^m(U^\vee)$, we can write

$$m!f = \sum_{|\mathbf{i}|=m} \tbinom{m}{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}. \tag{1.7}$$

In symbolic form, $m!f = (a_x)^m$. Consider the matrix

$$\Xi = \begin{pmatrix} \xi_0^{(1)} & \cdots & \xi_0^{(d)} \\ \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \cdots & \xi_r^{(d)} \end{pmatrix},$$

where $(\xi_0^{(k)}, \ldots, \xi_r^{(k)})$ is a copy of a basis in U. Then the space $S^d(S^m(U))$ is equal to the subspace of the polynomial algebra $\mathbb{C}[(\xi_j^{(i)})]$ in $d(r+1)$ variables $\xi_j^{(i)}$ of polynomials which are homogeneous of degree m in each column of the matrix and symmetric with respect to permutations of the columns. Let $J \subset \{1, \ldots, d\}$ with $\#J = r+1$ and (J) be the corresponding maximal minor of the matrix Ξ. Assume $r+1$ divides dm. Consider a product of $k = \frac{dm}{r+1}$ such minors in which each column participates exactly m times. Then a sum of such products which is invariant with respect to permutations of columns

represents an element from $S^d(S^m(U))$ which has an additional property that it is invariant with respect to the group $\mathrm{SL}(U) \cong \mathrm{SL}(r+1, \mathbb{C})$ We can interpret elements of $S^d(S^m(U^\vee)^\vee)$ as polynomials in coefficients of $a_{\mathfrak{i}}$ of a homogeneous form of degree d in $r+1$ variables written in the form (1.7). We write symbolically an invariant in the form $(J_1)\cdots(J_k)$ meaning that it is obtained as sum of such products with some coefficients. If the number d is small, we can use letters, say $a, b, c, \dots$, instead of numbers $1, \dots, d$. For example, $(12)^2(13)^2(23)^2 = (ab)^2(bc)^2(ac)^2$ represents an element in $S^3(S^4(\mathbb{C}^2))$.

In a similar way, one considers the matrix

$$\begin{pmatrix} \xi_0^{(1)} & \dots & \xi_0^{(d)} & t_0^{(1)} & \dots & t_0^{(s)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \dots & \xi_r^{(d)} & t_r^{(1)} & \dots & t_r^{(s)} \end{pmatrix}.$$

The product of k maximal minors such that each of the first d columns occurs exactly k times and each of the last s columns occurs exactly p times represents a *covariant* of degree p and order k. For example, $(ab)^2 a_x b_x$ represents the *Hessian determinant*

$$\mathrm{He}(f) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

of a ternary cubic form f.

The *projective space* of lines in E will be denoted by $|E|$.

A basis $\xi_0, \dots, \xi_n$ in E defines an isomorphism $E \cong \mathbb{C}^{n+1}$ and identifies $|E|$ with the projective space $\mathbb{P}^n := |\mathbb{C}^{n+1}|$.

The projective space comes with the tautological invertible sheaf $\mathcal{O}_{|E|}(1)$ whose space of global sections is identified with the dual space $E^\vee$. Its d-th tensor power is denoted by $\mathcal{O}_{|E|}(d)$. Its space of global sections is identified with the symmetric d-th power $S^d(E^\vee)$.

For any $f \in S^d(E^\vee), d > 0$, we denote by $V(f)$ the corresponding effective divisor from $|\mathcal{O}_{|E|}(d)|$, considered as a closed subscheme of $|E|$, not necessarily reduced. We call $V(f)$ a *hypersurface* of degree d in $|E|$ defined by equation $f = 0$[1] A hypersurface of degree 1 is a *hyperplane*. By definition, $V(0) = |E|$ and $V(1) = \emptyset$. The projective space $|S^d(E^\vee)|$ can be viewed as the projective space of hypersurfaces in $|E|$. It is equal to the complete linear system $|\mathcal{O}_{|E|}(d)|$. Using isomorphism (1.2), we may identify the projective

[1] This notation should not be confused with the notation of the closed subset in Zariski topology defined by the ideal (f). It is equal to $V(f)_{\mathrm{red}}$.

space $|S^d(E)|$ of hypersurfaces of degree d in $|E^\vee|$ with the dual of the projective space $|S^dE^\vee|$. A hypersurface of degree d in $|E^\vee|$ is classically known as an *envelope* of *class* d.

The natural isomorphisms

$$(E^\vee)^{\otimes d} \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d}),\ S_d(E^\vee) \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d}$$

allow one to give the following geometric interpretation of the polarization isomorphism. Consider the diagonal embedding $\delta_d : |E| \hookrightarrow |E|^d$. Then the total polarization map is the inverse of the isomorphism

$$\delta_d^* : H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d} \to H^0(|E|, \mathcal{O}_{|E|}(d)).$$

We view $a_0\partial_0 + \cdots + a_n\partial_n \neq 0$ as a point $a \in |E|$ with projective coordinates $[a_0, \ldots, a_n]$.

Definition 1.1.2 *Let $X = V(f)$ be a hypersurface of degree d in $|E|$ and $x = [v]$ be a point in $|E|$. The hypersurface*

$$P_{a^k}(X) := V(D_{v^k}(f))$$

of degree $d - k$ is called the k-th polar hypersurface *of the point a with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).*

Example 1.1.3 Let $d = 2$, i.e.

$$f = \sum_{i=0}^{n} \alpha_{ii} t_i^2 + 2 \sum_{0 \leq i < j \leq n} \alpha_{ij} t_i t_j$$

is a quadratic form on $\mathbb{C}^{n+1}$. For any $x = [a_0, \ldots, a_n] \in \mathbb{P}^n$, $P_x(V(f)) = V(g)$, where

$$g = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i} = 2 \sum_{0 \leq i < j \leq n} a_i \alpha_{ij} t_j, \quad \alpha_{ji} = \alpha_{ij}.$$

The linear map $v \mapsto D_v(f)$ is a map from $\mathbb{C}^{n+1}$ to $(\mathbb{C}^{n+1})^\vee$ which can be identified with the *polar bilinear form* associated to f with matrix $2(\alpha_{ij})$.

Let us give another definition of the polar hypersurfaces $P_{x^k}(X)$. Choose two different points $a = [a_0, \ldots, a_n]$ and $b = [b_0, \ldots, b_n]$ in $\mathbb{P}^n$ and consider the line $\ell = \overline{ab}$ spanned by the two points as the image of the map

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n, \quad [u_0, u_1] \mapsto u_0 a + u_1 b := [a_0 u_0 + b_0 u_1, \ldots, a_n u_0 + b_n u_1]$$

(a parametric equation of ℓ). The intersection $\ell \cap X$ is isomorphic to the positive divisor on $\mathbb{P}^1$ defined by the degree d homogeneous form

$$\varphi^*(f) = f(u_0 a + u_1 b) = f(a_0 u_0 + b_0 u_1, \ldots, a_n u_0 + b_n u_1).$$

Using the Taylor formula at $(0,0)$, we can write

$$\varphi^*(f) = \sum_{k+m=d} \frac{1}{k!m!} u_0^k u_1^m A_{km}(a,b), \tag{1.8}$$

where

$$A_{km}(a,b) = \frac{\partial^d \varphi^*(f)}{\partial u_0^k \partial u_1^m}(0,0).$$

Using the Chain Rule, we get

$$A_{km}(a,b) = \sum_{|\mathbf{i}|=k,|\mathbf{j}|=m} \tbinom{k}{\mathbf{i}} \tbinom{m}{\mathbf{j}} \mathbf{a}^{\mathbf{i}} \mathbf{b}^{\mathbf{j}} \boldsymbol{\partial}^{\mathbf{i}+\mathbf{j}} f = D_{a^k b^m}(f). \tag{1.9}$$

Observe the symmetry

$$A_{km}(a,b) = A_{mk}(b,a). \tag{1.10}$$

When we fix a and let b vary in $\mathbb{P}^n$ we obtain a hypersurface $V(A(a,x))$ of degree $d-k$ which is the k-th polar hypersurface of $X = V(f)$ with respect to the point a. When we fix b and vary a in $\mathbb{P}^n$, we obtain the m-th polar hypersurface $V(A(x,b))$ of X with respect to the point b.

Note that

$$D_{a^k b^m}(f) = D_{a^k}(D_{b^m}(f)) = D_{b^m}(a) = D_{b^m}(D_{a^k}(f)) = D_{a^k}(f)(b). \tag{1.11}$$

This gives the symmetry property of polars

$$b \in P_{a^k}(X) \Leftrightarrow a \in P_{b^{d-k}}(X). \tag{1.12}$$

Since we are in characteristic 0, if $m \leq d$, $D_{a^m}(f)$ cannot be zero for all a. To see this we use the *Euler formula*:

$$df = \sum_{i=0}^{n} t_i \frac{\partial f}{\partial t_i}.$$

Applying this formula to the partial derivatives, we obtain

$$d(d-1)\cdots(d-k+1)f = \sum_{|\mathbf{i}|=k} \tbinom{k}{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f \tag{1.13}$$

(also called the Euler formula). It follows from this formula that, for all $k \leq d$,

$$a \in P_{a^k}(X) \Leftrightarrow a \in X. \tag{1.14}$$

This is known as the *reciprocity theorem*.

Example 1.1.4 Let M_d be the vector space of complex square matrices of size d with coordinates t_{ij}. We view the determinant function $\det : M_d \to \mathbb{C}$ as an element of $S^d(M_d^\vee)$, i.e. a polynomial of degree d in the variables t_{ij}. Let $C_{ij} = \frac{\partial \det}{\partial t_{ij}}$. For any point $A = (a_{ij})$ in M_d the value of C_{ij} at A is equal to the ij-th cofactor of A. Applying (1.6), for any $B = (b_{ij}) \in M_d$, we obtain

$$D_{A^{d-1}B}(\det) = D_A^{d-1}(D_B(\det)) = D_A^{d-1}(\sum b_{ij}C_{ij}) = (d-1)!\sum b_{ij}C_{ij}(A).$$

Thus $D_A^{d-1}(\det)$ is a linear function $\sum t_{ij}C_{ij}$ on M_d. The linear map

$$S^{d-1}(M_n) \to M_d^\vee, \quad A \mapsto \frac{1}{(d-1)!}D_A^{d-1}(\det),$$

can be identified with the function $A \mapsto \mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the cofactor matrix (classically called the *adjugate matrix* of A, but not the adjoint matrix as it is often called in modern text-books).

1.1.2 First polars

Let us consider some special cases. Let $X = V(f)$ be a hypersurface of degree d. Obviously, any 0-th polar of X is equal to X and, by (1.12), the d-th polar $P_{a^d}(X)$ is empty if $a \notin X$. and equals $\mathbb{P}^n$ if $a \in X$. Now take $k = 1, d-1$. By using (1.6), we obtain

$$D_a(f) = \sum_{i=0}^n a_i \frac{\partial f}{\partial t_i},$$

$$\frac{1}{(d-1)!}D_{a^{d-1}}(f) = \sum_{i=0}^n \frac{\partial f}{\partial t_i}(a)t_i.$$

Together with (1.12) this implies the following.

Theorem 1.1.5 *For any smooth point $x \in X$, we have*

$$P_{x^{d-1}}(X) = \mathbb{T}_x(X).$$

If x is a singular point of X, $P_{x^{d-1}}(X) = \mathbb{P}^n$. Moreover, for any $a \in \mathbb{P}^n$,

$$X \cap P_a(X) = \{x \in X : a \in \mathbb{T}_x(X)\}.$$

Here and later on we denote by $\mathbb{T}_x(X)$ the *embedded tangent space* of a projective subvariety $X \subset \mathbb{P}^n$ at its point x. It is a linear subspace of $\mathbb{P}^n$ equal to the projective closure of the affine Zariski tangent space $T_x(X)$ of X at x (see [278], p. 181).

In classical terminology, the intersection $X \cap P_a(X)$ is called the *apparent*

boundary of X from the point a. If one projects X to $\mathbb{P}^{n-1}$ from the point a, then the apparent boundary is the ramification divisor of the projection map.

The following picture makes an attempt to show what happens in the case when X is a conic.

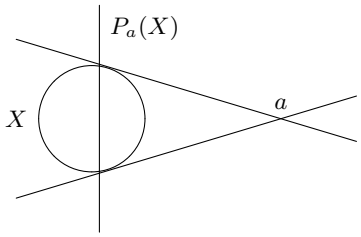


Figure 1.1 Polar line of a conic

The set of first polars $P_a(X)$ defines a linear system contained in the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$. The dimension of this linear system $\leq n$. We will be freely using the language of linear systems and divisors on algebraic varieties (see [282]).

Proposition 1.1.6 *The dimension of the linear system of first polars $\leq r$ if and only if, after a linear change of variables, the polynomial f becomes a polynomial in $r+1$ variables.*

Proof Let $X = V(f)$. It is obvious that the dimension of the linear system of first polars $\leq r$ if and only if the linear map $E \to S^{d-1}(E^\vee), v \mapsto D_v(f)$ has kernel of dimension $\geq n-r$. Choosing an appropriate basis, we may assume that the kernel is generated by vectors $(1,0,\ldots,0)$, *etc.* Now, it is obvious that f does not depend on the variables $t_0,\ldots,t_{n-r-1}$. □

It follows from Theorem 1.1.5 that the first polar $P_a(X)$ of a point a with respect to a hypersurface X passes through all singular points of X. One can say more.

Proposition 1.1.7 *Let a be a singular point of X of multiplicity m. For each $r \leq \deg X - m$, $P_{a^r}(X)$ has a singular point at a of multiplicity m and the tangent cone of $P_{a^r}(X)$ at a coincides with the tangent cone $\mathrm{TC}_a(X)$ of X at a. For any point $b \neq a$, the r-th polar $P_{b^r}(X)$ has multiplicity $\geq m-r$ at a and its tangent cone at a is equal to the r-th polar of $\mathrm{TC}_a(X)$ with respect to b.*

Proof Let us prove the first assertion. Without loss of generality, we may

assume that $a = [1, 0, \dots, 0]$. Then $X = V(f)$, where

$$f = t_0^{d-m} f_m(t_1, \dots, t_n) + t_0^{d-m-1} f_{m+1}(t_1, \dots, t_n) + \cdots + f_d(t_1, \dots, t_n). \tag{1.15}$$

The equation $f_m(t_1, \dots, t_n) = 0$ defines the tangent cone of X at b. The equation of $P_{a^r}(X)$ is

$$\frac{\partial^r f}{\partial t_0^r} = r! \sum_{i=0}^{d-m-r} \binom{d-m-i}{r} t_0^{d-m-r-i} f_{m+i}(t_1, \dots, t_n) = 0.$$

It is clear that $[1, 0, \dots, 0]$ is a singular point of $P_{a^r}(X)$ of multiplicity m with the tangent cone $V(f_m(t_1, \dots, t_n))$.

Now we prove the second assertion. Without loss of generality, we may assume that $a = [1, 0, \dots, 0]$ and $b = [0, 1, 0, \dots, 0]$. Then the equation of $P_{b^r}(X)$ is

$$\frac{\partial^r f}{\partial t_1^r} = t_0^{d-m} \frac{\partial^r f_m}{\partial t_1^r} + \cdots + \frac{\partial^r f_d}{\partial t_1^r} = 0.$$

The point a is a singular point of multiplicity $\geq m - r$. The tangent cone of $P_{b^r}(X)$ at the point a is equal to $V(\frac{\partial^r f_m}{\partial t_1^r})$ and this coincides with the r-th polar of $\mathrm{TC}_a(X) = V(f_m)$ with respect to b. □

We leave it to the reader to see what happens if $r > d - m$.

Keeping the notation from the previous proposition, consider a line ℓ through the point a such that it intersects X at some point $x \neq a$ with multiplicity larger than one. The closure $\mathrm{EC}_a(X)$ of the union of such lines is called the *enveloping cone* of X at the point a. If X is not a cone with vertex at a, the branch divisor of the projection $p : X \setminus \{a\} \to \mathbb{P}^{n-1}$ from a is equal to the projection of the enveloping cone. Let us find the equation of the enveloping cone.

As above, we assume that $a = [1, 0, \dots, 0]$. Let H be the hyperplane $t_0 = 0$. Write ℓ in a parametric form $ua + vx$ for some $x \in H$. Plugging in Equation (1.15), we get

$$P(t) = t^{d-m} f_m(x_1, \dots, x_n) + t^{d-m-1} f_{m+1}(x_1, \dots, x_m) + \cdots + f_d(x_1, \dots, x_n) = 0,$$

where $t = u/v$.

We assume that $X \neq \mathrm{TC}_a(X)$, i.e. X is not a cone with vertex at a (otherwise, by definition, $\mathrm{EC}_a(X) = \mathrm{TC}_a(X)$). The image of the tangent cone under the projection $p : X \setminus \{a\} \to H \cong \mathbb{P}^{n-1}$ is a proper closed subset of H. If $f_m(x_1, \dots, x_n) \neq 0$, then a multiple root of $P(t)$ defines a line in the enveloping cone. Let $\mathsf{D}_k(A_0, \dots, A_k)$ be the discriminant of a general poly-

nomial $P = A_0T^k + \cdots + A_k$ of degree k. Recall that

$$A_0\mathsf{D}_k(A_0,\ldots,A_k) = (-1)^{k(k-1)/2}\mathrm{Res}(P,P')(A_0,\ldots,A_k),$$

where $\mathrm{Res}(P,P')$ is the resultant of P and its derivative P'. It follows from the known determinant expression of the resultant that

$$\mathsf{D}_k(0,A_1,\ldots,A_k) = (-1)^{\frac{k^2-k+2}{2}}A_0^2\mathsf{D}_{k-1}(A_1,\ldots,A_k).$$

The equation $P(t) = 0$ has a multiple zero with $t \neq 0$ if and only if

$$\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x)) = 0.$$

So, we see that

$$\begin{aligned} \mathrm{EC}_a(X) &\subset V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x))), \\ \mathrm{EC}_a(X)\cap\mathrm{TC}_a(X) &\subset V(\mathsf{D}_{d-m-1}(f_{m+1}(x),\ldots,f_d(x))). \end{aligned} \tag{1.16}$$

It follows from the computation of $\frac{\partial^r f}{\partial t_0^r}$ in the proof of the previous Proposition that the hypersurface $V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x)))$ is equal to the projection of $P_a(X)\cap X$ to H.

Suppose $V(\mathsf{D}_{d-m-1}(f_{m+1}(x),\ldots,f_d(x)))$ and $\mathrm{TC}_a(X)$ do not share an irreducible component. Then

$$V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x)))\setminus\mathrm{TC}_a(X)\cap V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x)))$$

$$= V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x)))\setminus V(\mathsf{D}_{d-m-1}(f_{m+1}(x),\ldots,f_d(x)))\subset \mathrm{EC}_a(X),$$

gives the opposite inclusion of (1.16), and we get

$$\mathrm{EC}_a(X) = V(\mathsf{D}_{d-m}(f_m(x),\ldots,f_d(x))). \tag{1.17}$$

Note that the discriminant $\mathsf{D}_{d-m}(A_0,\ldots,A_k)$ is an invariant of the group $\mathrm{SL}(2)$ in its natural representation on degree k binary forms. Taking the diagonal subtorus, we immediately infer that any monomial $A_0^{i_0}\cdots A_k^{i_k}$ entering in the discriminant polynomial satisfies

$$k\sum_{s=0}^{k} i_s = 2\sum_{s=0}^{k} si_s.$$

It is also known that the discriminant is a homogeneous polynomial of degree $2k-2$. Thus, we get

$$k(k-1) = \sum_{s=0}^{k} si_s.$$

In our case $k = d - m$, we obtain that

$$\deg V(\mathsf{D}_{d-m}(f_m(x), \dots, f_d(x))) = \sum_{s=0}^{d-m} (m+s) i_s$$

$$= m(2d - 2m - 2) + (d-m)(d-m-1) = (d+m)(d-m-1).$$

This is the expected degree of the enveloping cone.

Example 1.1.8 Assume $m = d - 2$, then

$$\mathsf{D}_2(f_{d-2}(x), f_{d-1}(x), f_d(x)) = f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x),$$
$$\mathsf{D}_2(0, f_{d-1}(x), f_d(x)) = f_{d-2}(x) = 0.$$

Suppose $f_{d-2}(x)$ and f_{d-1} are coprime. Then our assumption is satisfied, and we obtain

$$\mathrm{EC}_a(X) = V(f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x)).$$

Observe that the hypersurfaces $V(f_{d-2}(x))$ and $V(f_d(x))$ are everywhere tangent to the enveloping cone. In particular, the quadric tangent cone $\mathrm{TC}_a(X)$ is everywhere tangent to the enveloping cone along the intersection of $V(f_{d-2}(x))$ with $V(f_{d-1}(x))$.

For any nonsingular quadric Q, the map $x \mapsto P_x(Q)$ defines a projective isomorphism from the projective space to the dual projective space. This is a special case of a correlation.

According to classical terminology, a projective automorphism of $\mathbb{P}^n$ is called a *collineation*. An isomorphism from $|E|$ to its dual space $\mathbb{P}(E)$ is called a *correlation*. A correlation $\mathfrak{c} : |E| \to \mathbb{P}(E)$ is given by an invertible linear map $\phi : E \to E^\vee$ defined uniquely up to proportionality. A correlation transforms points in $|E|$ to hyperplanes in $|E|$. A point $x \in |E|$ is called *conjugate* to a point $y \in |E|$ with respect to the correlation $\mathfrak{c}$ if $y \in \mathfrak{c}(x)$. The transpose of the inverse map ${}^t\phi^{-1} : E^\vee \to E$ transforms hyperplanes in $|E|$ to points in $|E|$. It can be considered as a correlation between the dual spaces $\mathbb{P}(E)$ and $|E|$. It is denoted by $\mathfrak{c}^\vee$ and is called the *dual correlation*. It is clear that $(\mathfrak{c}^\vee)^\vee = \mathfrak{c}$. If H is a hyperplane in $|E|$ and x is a point in H, then point $y \in |E|$ conjugate to x under $\mathfrak{c}$ belongs to any hyperplane H' in $|E|$ conjugate to H under $\mathfrak{c}^\vee$.

A correlation can be considered as a line in $(E \otimes E)^\vee$ spanned by a nondegenerate bilinear form, or, in other words as a nonsingular correspondence of type $(1,1)$ in $|E| \times |E|$. The dual correlation is the image of the divisor under the switch of the factors. A pair $(x, y) \in |E| \times |E|$ of conjugate points is just a point on this divisor.

We can define the *composition of correlations* $\mathfrak{c}' \circ \mathfrak{c}^\vee$. Collineations and

correlations form a group $\Sigma\mathrm{PGL}(E)$ isomorphic to the group of outer automorphisms of $\mathrm{PGL}(E)$. The subgroup of collineations is of index 2.

A correlation $\mathfrak{c}$ of order 2 in the group $\Sigma\mathrm{PGL}(E)$ is called a *polarity*. In linear representative, this means that ${}^t\phi = \lambda\phi$ for some nonzero scalar λ. After transposing, we obtain $\lambda = \pm 1$. The case $\lambda = 1$ corresponds to the (quadric) polarity with respect to a nonsingular quadric in $|E|$ which we discussed in this section. The case $\lambda = -1$ corresponds to a *null-system* (or *null polarity*) which we will discuss in Chapters 2 and 10. In terms of bilinear forms, a correlation is a quadric polarity (resp. null polarity) if it can be represented by a symmetric (skew-symmetric) bilinear form.

Theorem 1.1.9 *Any projective automorphism is equal to the product of two quadric polarities.*

Proof Choose a basis in E to represent the automorphism by a Jordan matrix J. Let $J_k(\lambda)$ be its block of size k with λ at the diagonal. Let

$$B_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then

$$C_k(\lambda) = B_k J_k(\lambda) = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda \\ 0 & 0 & \dots & \lambda & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda & \dots & 0 & 0 \\ \lambda & 1 & \dots & 0 & 0 \end{pmatrix}.$$

Observe that the matrices B_k^{-1} and $C_k(\lambda)$ are symmetric. Thus each Jordan block of J can be written as the product of symmetric matrices, hence J is the product of two symmetric matrices. It follows from the definition of composition in the group $\Sigma\mathrm{PGL}(E)$ that the product of the matrices representing the bilinear forms associated to correlations coincides with the matrix representing the projective transformation equal to the composition of the correlations. □

1.1.3 Polar quadrics

A $(d-2)$-polar of $X = V(f)$ is a quadric, called the *polar quadric* of X with respect to $a = [a_0, \dots, a_n]$. It is defined by the quadratic form

$$q = D_{a^{d-2}}(f) = \sum_{|\mathbf{i}|=d-2} \binom{d-2}{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f.$$

Using Equation (1.9), we obtain

$$q = \sum_{|\mathbf{i}|=2} \binom{2}{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f(a).$$

By (1.14), each $a \in X$ belongs to the polar quadric $P_{a^{d-2}}(X)$. Also, by Theorem 1.1.5,

$$\mathbb{T}_a(P_{a^{d-2}}(X)) = P_a(P_{a^{d-2}}(X)) = P_{a^{d-1}}(X) = \mathbb{T}_a(X). \tag{1.18}$$

This shows that the polar quadric is tangent to the hypersurface at the point a.

Consider the line $\ell = \overline{ab}$ through two points a, b. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ be its parametric equation, i.e. a closed embedding with the image equal to ℓ. It follows from (1.8) and (1.9) that

$$i(X, \overline{ab})_a \geq s+1 \Longleftrightarrow b \in P_{a^{d-k}}(X), \quad k \leq s. \tag{1.19}$$

For $s = 0$, the condition means that $a \in X$. For $s = 1$, by Theorem 1.1.5, this condition implies that b, and hence ℓ, belongs to the tangent plane $\mathbb{T}_a(X)$. For $s = 2$, this condition implies that $b \in P_{a^{d-2}}(X)$. Since ℓ is tangent to X at a, and $P_{a^{d-2}}(X)$ is tangent to X at a, this is equivalent to that ℓ belongs to $P_{a^{d-2}}(X)$.

It follows from (1.19) that a is a singular point of X of multiplicity $\geq s+1$ if and only if $P_{a^{d-k}}(X) = \mathbb{P}^n$ for $k \leq s$. In particular, the quadric polar $P_{a^{d-2}}(X) = \mathbb{P}^n$ if and only if a is a singular point of X of multiplicity ≥ 3.

Definition 1.1.10 *A line is called an* inflection tangent *to X at a point a if*

$$i(X, \ell)_a > 2.$$

Proposition 1.1.11 *Let ℓ be a line through a point a. Then ℓ is an inflection tangent to X at a if and only if it is contained in the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$.*

Note that the intersection of an irreducible quadric hypersurface $Q = V(q)$ with its tangent hyperplane H at a point $a \in Q$ is a cone in H over the quadric $\bar{Q}$ in the image $\bar{H}$ of H in $|E/[a]|$.

Corollary 1.1.12 *Assume $n \geq 3$. For any $a \in X$, there exists an inflection tangent line. The union of the inflection tangents containing the point a is the cone $\mathbb{T}_a(X) \cap P_{a^{d-2}}(X)$ in $\mathbb{T}_a(X)$.*

Example 1.1.13 Assume a is a singular point of X. By Theorem 1.1.5, this is equivalent to that $P_{a^{d-1}}(X) = \mathbb{P}^n$. By (1.18), the polar quadric Q is also singular at a and therefore it must be a cone over its image under the projection from a. The union of inflection tangents is equal to Q.

Example 1.1.14 Assume a is a nonsingular point of an irreducible surface X in $\mathbb{P}^3$. A tangent hyperplane $\mathbb{T}_a(X)$ cuts out in X a curve C with a singular point a. If a is an ordinary double point of C, there are two inflection tangents corresponding to the two branches of C at a. The polar quadric Q is nonsingular at a. The tangent cone of C at the point a is a cone over a quadric $\bar{Q}$ in $\mathbb{P}^1$. If $\bar{Q}$ consists of two points, there are two inflection tangents corresponding to the two branches of C at a. If $\bar{Q}$ consists of one point (corresponding to non-reduced hypersurface in $\mathbb{P}^1$), then we have one branch. The latter case happens only if Q is singular at some point $b \neq a$.

1.1.4 The Hessian hypersurface

Let $Q(a)$ be the polar quadric of $X = V(f)$ with respect to some point $a \in \mathbb{P}^n$. The symmetric matrix defining the corresponding quadratic form is equal to the *Hessian matrix* of second partial derivatives of f

$$\mathrm{He}(f) = \Big(\frac{\partial^2 f}{\partial t_i \partial t_j}\Big)_{i,j=0,\dots,n},$$

evaluated at the point a.The quadric $Q(a)$ is singular if and only if the determinant of the matrix is equal to zero (the locus of singular points is equal to the projectivization of the null-space of the matrix). The hypersurface

$$\mathrm{He}(X) = V(\det \mathrm{He}(f))$$

describes the set of points $a \in \mathbb{P}^n$ such that the polar quadric $P_{a^{d-2}}(X)$ is singular. It is called the *Hessian hypersurface* of X. Its degree is equal to $(d-2)(n+1)$ unless it coincides with $\mathbb{P}^n$.

Proposition 1.1.15 *The following is equivalent:*

(i) $\mathrm{He}(X) = \mathbb{P}^n$*;*
(ii) there exists a nonzero polynomial $g(z_0, \dots, z_n)$ such that

$$g(\partial_0 f, \dots, \partial_n f) \equiv 0.$$

Proof This is a special case of a more general result about the *Jacobian determinant* (also known as the *functional determinant*) of $n+1$ polynomial functions $f_0, \dots, f_n$ defined by

$$J(f_0, \dots, f_n) = \det\big((\frac{\partial f_i}{\partial t_j})\big).$$

Suppose $J(f_0, \dots, f_n) \equiv 0$. Then the map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by the functions $f_0, \dots, f_n$ is degenerate at each point (i.e. df_x is of rank $< n+1$ at each point x). Thus the closure of the image is a proper closed subset of $\mathbb{C}^{n+1}$. Hence there is an irreducible polynomial that vanishes identically on the image.

Conversely, assume that $g(f_0, \dots, f_n) \equiv 0$ for some polynomial g which we may assume to be irreducible. Then

$$\frac{\partial g}{\partial t_i} = \sum_{j=0}^{n} \frac{\partial g}{\partial z_j}(f_0, \dots, f_n)\frac{\partial f_j}{\partial t_i} = 0,\ i = 0, \dots, n.$$

Since g is irreducible, its set of zeros is nonsingular on a Zariski open set U. Thus the vector

$$\big(\frac{\partial g}{\partial z_0}(f_0(x), \dots, f_n(x)), \dots, \frac{\partial g}{\partial z_n}(f_0(x), \dots, f_n(x)\big)$$

is a nontrivial solution of the system of linear equations with matrix $(\frac{\partial f_i}{\partial t_j}(x))$, where $x \in U$. Therefore, the determinant of this matrix must be equal to zero. This implies that $J(f_0, \dots, f_n) = 0$ on U, hence it is identically zero. □

Remark 1.1.16 It was claimed by O. Hesse that the vanishing of the Hessian implies that the partial derivatives are linearly dependent. Unfortunately, his attempted proof was wrong. The first counterexample was given by P. Gordan and M. Noether in [253]. Consider the polynomial

$$f = t_2t_0^2 + t_3t_1^2 + t_4t_0t_1.$$

Note that the partial derivatives

$$\frac{\partial f}{\partial t_2} = t_0^2, \quad \frac{\partial f}{\partial t_3} = t_1^2, \quad \frac{\partial f}{\partial t_4} = t_0t_1$$

are algebraically dependent. This implies that the Hessian is identically equal to zero. We have

$$\frac{\partial f}{\partial t_0} = 2t_0t_2 + t_4t_1, \quad \frac{\partial f}{\partial t_1} = 2t_1t_3 + t_4t_0.$$

Suppose that a linear combination of the partials is equal to zero. Then

$$c_0t_0^2 + c_1t_1^2 + c_2t_0t_1 + c_3(2t_0t_2 + t_4t_1) + c_4(2t_1t_3 + t_4t_0) = 0.$$

Collecting the terms in which t_2, t_3, t_4 enter, we get

$$2c_3t_0 = 0, \quad 2c_4t_1 = 0, \quad c_3t_1 + c_4t_0 = 0.$$

This gives $c_3 = c_4 = 0$. Since the polynomials t_0^2, t_1^2, t_0t_1 are linearly independent, we also get $c_0 = c_1 = c_2 = 0$.

The known cases when the assertion of Hesse is true are $d = 2$ (any n) and $n \leq 3$ (any d) (see [253], [370], [102]).

Recall that the set of singular quadrics in $\mathbb{P}^n$ is the *discriminant hypersurface* $\mathsf{D}_2(n)$ in $\mathbb{P}^{n(n+3)/2}$ defined by the equation

$$\det \begin{pmatrix} t_{00} & t_{01} & \dots & t_{0n} \\ t_{01} & t_{11} & \dots & t_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{0n} & t_{1n} & \dots & t_{nn} \end{pmatrix} = 0.$$

By differentiating, we easily find that its singular points are defined by the determinants of $n \times n$ minors of the matrix. This shows that the singular locus of $\mathsf{D}_2(n)$ parameterizes quadrics defined by quadratic forms of rank $\leq n-1$ (or corank ≥ 2). Abusing the terminology, we say that a quadric $\mathcal{Q}$ is of rank k if the corresponding quadratic form is of this rank. Note that

$$\dim \mathrm{Sing}(Q) = \mathrm{corank}\ Q - 1.$$

Assume that $\mathrm{He}(f) \neq 0$. Consider the rational map $p : |E| \to |S^2(E^\vee)|$ defined by $a \mapsto P_{a^{d-2}}(X)$. Note that $P_{a^{d-2}}(f) = 0$ implies $P_{a^{d-1}}(f) = 0$ and hence $\sum_{i=0}^n b_i \partial_i f(a) = 0$ for all b. This shows that a is a singular point of X. Thus p is defined everywhere except maybe at singular points of X. So the map p is regular if X is nonsingular, and the preimage of the discriminant hypersurface is equal to the Hessian of X. The preimage of the singular locus $\mathrm{Sing}(\mathsf{D}_2(n))$ is the subset of points $a \in \mathrm{He}(f)$ such that $\mathrm{Sing}(P_{a^{d-2}}(X))$ is of positive dimension.

Here is another description of the Hessian hypersurface.

Proposition 1.1.17 *The Hessian hypersurface* $\mathrm{He}(X)$ *is the locus of singular points of the first polars of* X.

Proof Let $a \in \mathrm{He}(X)$ and let $b \in \mathrm{Sing}(P_{a^{d-2}}(X))$. Then

$$D_b(D_{a^{d-2}}(f)) = D_{a^{d-2}}(D_b(f)) = 0.$$

Since $D_b(f)$ is of degree $d-1$, this means that $\mathbb{T}_a(P_b(X)) = \mathbb{P}^n$, i.e., a is a singular point of $P_b(X)$.

Conversely, if $a \in \mathrm{Sing}(P_b(X))$, then $D_{a^{d-2}}(D_b(f)) = D_b(D_{a^{d-2}}(f)) = 0$. This means that b is a singular point of the polar quadric with respect to a. Hence $a \in \mathrm{He}(X)$. □

Let us find the affine equation of the Hessian hypersurface. Applying the Euler formula (1.13), we can write

$$t_0 f_{0i} = (d-1)\partial_i f - t_1 f_{1i} - \cdots - t_n f_{ni},$$

$$t_0 \partial_0 f = df - t_1 \partial_1 f - \cdots - t_n \partial_n f,$$

where f_{ij} denote the second partial derivative. Multiplying the first row of the Hessian determinant by t_0 and adding to it the linear combination of the remaining rows taken with the coefficients t_i, we get the following equality:

$$\det(\mathrm{He}(f)) = \frac{d-1}{t_0} \det \begin{pmatrix} \partial_0 f & \partial_1 f & \dots & \partial_n f \\ f_{10} & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n0} & f_{n1} & \dots & f_{nn} \end{pmatrix}.$$

Repeating the same procedure but this time with the columns, we finally get

$$\det(\mathrm{He}(f)) = \frac{(d-1)^2}{t_0^2} \det \begin{pmatrix} \frac{d}{d-1} f & \partial_1 f & \dots & \partial_n f \\ \partial_1 f & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_n f & f_{n1} & \dots & f_{nn} \end{pmatrix}. \tag{1.20}$$

Let $\phi(z_1, \dots, z_n)$ be the dehomogenization of f with respect to t_0, i.e.,

$$f(t_0, \dots, t_d) = t_0^d \phi(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}).$$

We have

$$\frac{\partial f}{\partial t_i} = t_0^{d-1} \phi_i(z_1, \dots, z_n), \ \frac{\partial^2 f}{\partial t_i \partial t_j} = t_0^{d-2} \phi_{ij}(z_1, \dots, z_n), \quad i, j = 1, \dots, n,$$

where

$$\phi_i = \frac{\partial \phi}{\partial z_i}, \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial z_i \partial z_j}.$$

Plugging these expressions in (1.20), we obtain, that up to a nonzero constant

factor,

$$t_0^{-(n+1)(d-2)} \det(\mathrm{He}(\phi)) = \det \begin{pmatrix} \frac{d}{d-1}\phi(z) & \phi_1(z) & \dots & \phi_n(z) \\ \phi_1(z) & \phi_{11}(z) & \dots & \phi_{1n}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_n(z) & \phi_{n1}(z) & \dots & \phi_{nn}(z) \end{pmatrix}, \tag{1.21}$$

where $z = (z_1, \dots, z_n), z_i = t_i/t_0,\ i = 1, \dots, n.$

Remark 1.1.18 If $f(x, y)$ is a real polynomial in three variables, the value of (1.21) at a point $v \in \mathbb{R}^n$ with $[v] \in V(f)$ multiplied by $\frac{-1}{f_1(a)^2+f_2(a)^2+f_3(a)^2}$ is equal to the *Gauss curvature* of $X(\mathbb{R})$ at the point a (see [221]).

1.1.5 Parabolic points

Let us see where $\mathrm{He}(X)$ intersects X. We assume that $\mathrm{He}(X)$ is a hypersurface of degree $(n+1)(d-2) > 0$. A glance at the expression (1.21) reveals the following fact.

Proposition 1.1.19 *Each singular point of X belongs to* $\mathrm{He}(X)$.

Let us see now when a nonsingular point $a \in X$ lies in its Hessian hypersurface $\mathrm{He}(X)$.

By Corollary 1.1.12, the inflection tangents in $\mathbb{T}_a(X)$ sweep the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$. If $a \in \mathrm{He}(X)$, then the polar quadric is singular at some point b.

If $n = 2$, a singular quadric is the union of two lines, so this means that one of the lines is an inflection tangent. A point a of a plane curve X such that there exists an inflection tangent at a is called an *inflection point* of X.

If $n > 2$, the inflection tangent lines at a point $a \in X \cap \mathrm{He}(X)$ sweep a cone over a singular quadric in $\mathbb{P}^{n-2}$ (or the whole $\mathbb{P}^{n-2}$ if the point is singular). Such a point is called a *parabolic point* of X. The closure of the set of parabolic points is the *parabolic hypersurface* in X (it could be the whole X).

Theorem 1.1.20 *Let X be a hypersurface of degree $d > 2$ in $\mathbb{P}^n$. If $n = 2$, then $\mathrm{He}(X) \cap X$ consists of inflection points of X. In particular, each nonsingular curve of degree ≥ 3 has an inflection point, and the number of inflections points is either infinite or less than or equal to $3d(d-2)$. If $n > 2$, then the set $X \cap \mathrm{He}(X)$ consists of parabolic points. The parabolic hypersurface in X is either the whole X or a subvariety of degree $(n+1)d(d-2)$ in $\mathbb{P}^n$.*

Example 1.1.21 Let X be a surface of degree d in $\mathbb{P}^3$. If a is a parabolic point of X, then $\mathbb{T}_a(X) \cap X$ is a singular curve whose singularity at a is of

multiplicity higher than 3 or it has only one branch. In fact, otherwise X has at least two distinct inflection tangent lines which cannot sweep a cone over a singular quadric in $\mathbb{P}^1$. The converse is also true. For example, a nonsingular quadric has no parabolic points, and all nonsingular points of a singular quadric are parabolic.

A generalization of a quadratic cone is a *developable surface*. It is a special kind of a *ruled surface* which characterized by the condition that the tangent plane does not change along a ruling. We will discuss these surfaces later in Chapter 10. The Hessian surface of a developable surface contains this surface. The residual surface of degree $2d - 8$ is called the *pro-Hessian surface*. An example of a developable surface is the quartic surface

$$(t_0t_3-t_1t_2)^2-4(t_1^2-t_0t_2)(t_2^2-t_1t_3) = -6t_0t_1t_2t_3+4t_1^3t_3+4t_0t_2^3+t_0^2t_3^2-3t_1^2t_2^2 = 0.$$

It is the surface swept out by the tangent lines of a rational normal curve of degree 3. It is also the *discriminant surface* of a binary cubic, i.e. the surface parameterizing binary cubics $a_0u^3 + 3a_1u^2v + 3a_2uv^2 + a_3v^3$ with a multiple root. The pro-Hessian of any quartic developable surface is the surface itself [84].

Assume now that X is a curve. Let us see when it has infinitely many inflection points. Certainly, this happens when X contains a line component; each of its points is an inflection point. It must be also an irreducible component of $\mathrm{He}(X)$. The set of inflection points is a closed subset of X. So, if X has infinitely many inflection points, it must have an irreducible component consisting of inflection points. Each such component is contained in $\mathrm{He}(X)$. Conversely, each common irreducible component of X and $\mathrm{He}(X)$ consists of inflection points.

We will prove the converse in a little more general form taking care of not necessarily reduced curves.

Proposition 1.1.22 *A polynomial $f(t_0, t_1, t_2)$ divides its Hessian polynomial $\mathrm{He}(f)$ if and only if each of its multiple factors is a linear polynomial.*

Proof Since each point on a non-reduced component of $X_{\mathrm{red}} \subset V(f)$ is a singular point (i.e. all the first partials vanish), and each point on a line component is an inflection point, we see that the condition is sufficient for $X \subset \mathrm{He}(f)$. Suppose this happens and let R be a reduced irreducible component of the curve X which is contained in the Hessian. Take a nonsingular point of R and consider an affine equation of R with coordinates (x, y). We may assume that $\mathcal{O}_{R,x}$ is included in $\hat{\mathcal{O}}_{R,x} \cong \mathbb{C}[[t]]$ such that $x = t, y = t^r\epsilon$, where $\epsilon(0) = 1$.

Thus the equation of R looks like

$$f(x,y) = y - x^r + g(x,y), \tag{1.22}$$

where $g(x,y)$ does not contain terms $cy, c \in \mathbb{C}$. It is easy to see that $(0,0)$ is an inflection point if and only if $r > 2$ with the inflection tangent $y = 0$.

We use the affine equation of the Hessian (1.21), and obtain that the image of

$$h(x,y) = \det \begin{pmatrix} \frac{d}{d-1} f & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}$$

in $\mathbb{C}[[t]]$ is equal to

$$\det \begin{pmatrix} 0 & -rt^{r-1} + g_1 & 1 + g_2 \\ -rt^{r-1} + g_1 & -r(r-1)t^{r-2} + g_{11} & g_{12} \\ 1 + g_2 & g_{12} & g_{22} \end{pmatrix}.$$

Since every monomial entering in g is divisible by y^2, xy or $x^i, i > r$, we see that $\frac{\partial g}{\partial y}$ is divisible by t and $\frac{\partial g}{\partial x}$ is divisible by t^{r-1}. Also g_{11} is divisible by t^{r-1}. This shows that

$$h(x,y) = \det \begin{pmatrix} 0 & at^{r-1} + \cdots & 1 + \cdots \\ at^{r-1} + \cdots & -r(r-1)t^{r-2} + \cdots & g_{12} \\ 1 + \cdots & g_{12} & g_{22} \end{pmatrix},$$

where $\cdots$ denotes terms of higher degree in t. We compute the determinant and see that it is equal to $r(r-1)t^{r-2} + \cdots$. This means that its image in $\mathbb{C}[[t]]$ is not equal to zero, unless the equation of the curve is equal to $y = 0$, i.e. the curve is a line. □

In fact, we have proved more. We say that a nonsingular point of X is an inflection point of *order* $r-2$ and denote the order by $\mathrm{ordfl}_x X$ if one can choose an equation of the curve as in (1.22) with $r \geq 3$. It follows from the previous proof that $r-2$ is equal to the multiplicity $i(X, \mathrm{He})_x$ of the intersection of the curve and its Hessian at the point x. It is clear that $\mathrm{ordfl}_x X = i(\ell, X)_x - 2$, where ℓ is the inflection tangent line of X at x. If X is nonsingular, we have

$$\sum_{x \in X} i(X, \mathrm{He})_x = \sum_{x \in X} \mathrm{ordfl}_x X = 3d(d-2). \tag{1.23}$$

1.1.6 The Steinerian hypersurface

Recall that the Hessian hypersurface of a hypersurface $X = V(f)$ is the locus of points a such that the polar quadric $P_{a^{d-2}}(X)$ is singular. The *Steinerian*

hypersurface $\mathrm{St}(X)$ of X is the locus of singular points of the polar quadrics. Thus

$$\mathrm{St}(X) = \bigcup_{a \in \mathrm{He}(X)} \mathrm{Sing}(P_{a^{d-2}}(X)). \tag{1.24}$$

The proof of Proposition 1.1.17 shows that it can be equivalently defined as

$$\mathrm{St}(X) = \{a \in \mathbb{P}^n : P_a(X) \text{ is singular}\}. \tag{1.25}$$

We also have

$$\mathrm{He}(X) = \bigcup_{a \in \mathrm{St}(X)} \mathrm{Sing}(P_a(X)). \tag{1.26}$$

A point $b = [b_0, \dots, b_n] \in \mathrm{St}(X)$ satisfies the equation

$$\mathrm{He}(f)(a) \cdot \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = 0, \tag{1.27}$$

where $a \in \mathrm{He}(X)$. This equation defines a subvariety

$$\mathrm{HS}(X) \subset \mathbb{P}^n \times \mathbb{P}^n \tag{1.28}$$

given by $n+1$ equations of bidegree $(d-2, 1)$.

The following argument confirms our expectation. It is known (see, for example, [239]) that the locus of singular hypersurfaces of degree d in $|E|$ is a hypersurface

$$\mathsf{D}_d(n) \subset |S^d(E^\vee)|$$

of degree $(n+1)(d-1)^n$ defined by the *discriminant* of a general degree d homogeneous polynomial in $n+1$ variables (the *discriminant hypersurface*). Let L be the projective subspace of $|S^{d-1}(E^\vee)|$ that consists of first polars of X. Assume that no polar $P_a(X)$ is equal to $\mathbb{P}^n$. Then

$$\mathrm{St}(X) \cong L \cap \mathsf{D}_n(d-1).$$

So, unless L is contained in $\mathsf{D}_n(d-1)$, we get a hypersurface. Moreover, we obtain

$$\deg(\mathrm{St}(X)) = (n+1)(d-2)^n. \tag{1.29}$$

Assume that the quadric $P_{a^{d-2}}(X)$ is of corank 1. Then it has a unique singular point b with the coordinates $[b_0, \dots, b_n]$ proportional to any column or a row of the adjugate matrix $\mathrm{adj}(\mathrm{He}(f))$ evaluated at the point a. Thus,

$\mathrm{St}(X)$ coincides with the image of the Hessian hypersurface under the rational map

$$\mathrm{st} : \mathrm{He}(X) \dashrightarrow \mathrm{St}(X), \quad a \mapsto \mathrm{Sing}(P_{a^{d-2}}(X)),$$

given by polynomials of degree $n(d-2)$. We call it the *Steinerian map*. Of course, it is not defined when all polar quadrics are of corank > 1. Also, if the first polar hypersurface $P_a(X)$ has an isolated singular point for a general point a, we get a rational map

$$\mathrm{st}^{-1} : \mathrm{St}(X) \dashrightarrow \mathrm{He}(X), \quad a \mapsto \mathrm{Sing}(P_a(X)).$$

These maps are obviously inverse to each other. It is a difficult question to determine the sets of indeterminacy points for both maps.

Proposition 1.1.23 *Let X be a reduced hypersurface. The Steinerian hypersurface of X coincides with $\mathbb{P}^n$ if X has a singular point of multiplicity ≥ 3. The converse is true if we additionally assume that X has only isolated singular points.*

Proof Assume that X has a point of multiplicity ≥ 3. We may harmlessly assume that the point is $p = [1, 0, \dots, 0]$. Write the equation of X in the form

$$f = t_0^k g_{d-k}(t_1, \dots, t_n) + t_0^{k-1} g_{d-k+1}(t_1, \dots, t_n) + \cdots + g_d(t_1, \dots, t_n) = 0, \tag{1.30}$$

where the subscript indicates the degree of the polynomial. Since the multiplicity of p is greater than or equal to 3, we must have $d - k \geq 3$. Then a first polar $P_a(X)$ has the equation

$$a_0 \sum_{i=0}^{k} (k-i) t_0^{k-1-i} g_{d-k+i} + \sum_{s=1}^{n} a_s \sum_{i=0}^{k} t_0^{k-i} \frac{\partial g_{d-k+i}}{\partial t_s} = 0. \tag{1.31}$$

It is clear that the point p is a singular point of $P_a(X)$ of multiplicity $\geq d - k - 1 \geq 2$.

Conversely, assume that all polars are singular. By Bertini's Theorem (see [278], Theorem 17.16), the singular locus of a general polar is contained in the base locus of the linear system of polars. The latter is equal to the singular locus of X. By assumption, it consists of isolated points, hence we can find a singular point of X at which a general polar has a singular point. We may assume that the singular point is $p = [1, 0, \dots, 0]$ and (1.30) is the equation of X. Then the first polar $P_a(X)$ is given by Equation (1.31). The largest power of t_0 in this expression is at most k. The degree of the equation is $d - 1$. Thus the point p is a singular point of $P_a(X)$ if and only if $k \leq d - 3$, or, equivalently, if p is at least triple point of X. □

Example 1.1.24 The assumption on the singular locus is essential. First, it is easy to check that $X = V(f^2)$, where $V(f)$ is a nonsingular hypersurface has no points of multiplicity ≥ 3 and its Steinerian coincides with $\mathbb{P}^n$. An example of a reduced hypersurface X with the same property is a surface of degree 6 in $\mathbb{P}^3$ given by the equation

$$(\sum_{i=0}^{3} t_i^3)^2 + (\sum_{i=0}^{3} t_i^2)^3 = 0.$$

Its singular locus is the curve $V(\sum_{i=0}^{3} t_i^3) \cap V(\sum_{i=0}^{3} t_i^2)$. Each of its points is a double point on X. Easy calculation shows that

$$P_a(X) = V\big((\sum_{i=0}^{3} t_i^3)\sum_{i=0}^{3} a_i t_i^2 + (\sum_{i=0}^{3} t_i^2)^2 \sum_{i=0}^{3} a_i t_i\big).$$

and

$$V(\sum_{i=0}^{3} t_i^3) \cap V(\sum_{i=0}^{3} t_i^2) \cap V(\sum_{i=0}^{3} a_i t_i^2) \subset \mathrm{Sing}(P_a(X)).$$

By Proposition 1.1.7, $\mathrm{Sing}(X)$ is contained in $\mathrm{St}(X)$. Since the same is true for $\mathrm{He}(X)$, we obtain the following.

Proposition 1.1.25 *The intersection* $\mathrm{He}(X) \cap \mathrm{St}(X)$ *contains the singular locus of* X.

One can assign one more variety to a hypersurface $X = V(f)$. This is the *Cayleyan variety*. It is defined as the image $\mathrm{Cay}(X)$ of the rational map

$$\mathrm{HS}(X) \dashrightarrow G_1(\mathbb{P}^n), \quad (a, b) \mapsto \overline{ab},$$

where $G_r(\mathbb{P}^n)$ denotes the Grassmannian of r-dimensional subspaces in $\mathbb{P}^n$.

Note that in the case $n = 2$, the Cayleyan variety is a plane curve in the dual plane, the *Cayleyan curve* of X.

Proposition 1.1.26 *Let* X *be a general hypersurface of degree* $d \geq 3$. *Then*

$$\deg \mathrm{Cay}(X) = \begin{cases} \sum_{i=1}^{n} (d-2)^i \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d > 3, \\ \frac{1}{2}\sum_{i=1}^{n} \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d = 3, \end{cases}$$

where the degree is considered with respect to the Plücker embedding of the Grassmannian $G_1(\mathbb{P}^n)$.

Proof Since $\mathrm{St}(X) \neq \mathbb{P}^n$, the correspondence $\mathrm{HS}(X)$ is a complete intersection of $n+1$ hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(d-2, 1)$. Since

$a \in \mathrm{Sing}(P_a(X))$ implies that $a \in \mathrm{Sing}(X)$, the intersection of $\mathrm{HS}(X)$ with the diagonal is empty. Consider the regular map

$$r : \mathrm{HS}(X) \to G_1(\mathbb{P}^n), \quad (a, b) \mapsto \overline{ab}. \tag{1.32}$$

It is given by the linear system of divisors of type $(1, 1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ restricted to $\mathrm{HS}(X)$. The genericity assumption implies that this map is of degree 1 onto the image if $d > 3$ and of degree 2 if $d = 3$ (in this case the map factors through the involution of $\mathbb{P}^n \times \mathbb{P}^n$ that switches the factors).

It is known that the set of lines intersecting a codimension 2 linear subspace Λ is a hyperplane section of the Grassmannian $G_1(\mathbb{P}^n)$ in its Plücker embedding. Write $\mathbb{P}^n = |E|$ and $\Lambda = |L|$. Let $\omega = v_1 \wedge \ldots \wedge v_{n-1}$ for some basis $(v_1, \ldots, v_{n-1})$ of L. The locus of pairs of points $(a, b) = ([w_1], [w_2])$ in $\mathbb{P}^n \times \mathbb{P}^n$ such that the line $\overline{ab}$ intersects Λ is given by the equation $w_1 \wedge w_2 \wedge \omega = 0$. This is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^n \times \mathbb{P}^n$. This shows that the map (1.32) is given by a linear system of divisors of type $(1, 1)$. Its degree (or twice of the degree) is equal to the intersection $((d-2)h_1 + h_2)^{n+1}(h_1 + h_2)^{n-1}$, where h_1, h_2 are the natural generators of $H^2(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$. We have

$$((d-2)h_1 + h_2)^{n+1}(h_1 + h_2)^{n-1} =$$

$$\Big(\sum_{i=0}^{n+1} \tbinom{n+1}{i}(d-2)^i h_1^i h_2^{n+1-i}\Big)\Big(\sum_{j=0}^{n-1} \tbinom{n-1}{j} h_1^{n-1-j} h_2^j\Big)$$

$$= \sum_{i=1}^{n} (d-2)^i \tbinom{n+1}{i} \tbinom{n-1}{i-1}.$$

□

For example, if $n = 2, d > 3$, we obtain a classical result

$$\deg \mathrm{Cay}(X) = 3(d-2) + 3(d-2)^2 = 3(d-2)(d-1),$$

and $\deg \mathrm{Cay}(X) = 3$ if $d = 3$.

Remark 1.1.27 The homogeneous forms defining the Hessian and Steinerian hypersurfaces of $V(f)$ are examples of *covariants* of f. We already discussed them in the case $n = 1$. The form defining the Cayleyan of a plane curve is an example of a *contravariant* of f.

1.1.7 The Jacobian hypersurface

In the previous sections we discussed some natural varieties attached to the linear system of first polars of a hypersurface. We can extend these constructions

to arbitrary n-dimensional linear systems of hypersurfaces in $\mathbb{P}^n = |E|$. We assume that the linear system has no fixed components, i.e. its general member is an irreducible hypersurface of some degree d. Let $L \subset S^d(E^\vee)$ be a linear subspace of dimension $n+1$ and $|L|$ be the corresponding linear system of hypersurfaces of degree d. Note that, in the case of linear system of polars of a hypersurface X of degree $d+1$, the linear subspace L can be canonically identified with E and the inclusion $|E| \subset |S^d(E^\vee)|$ corresponds to the polarization map $a \mapsto P_a(X)$.

Let $\mathsf{D}_d(n) \subset |S^d(E^\vee)|$ be the discriminant hypersurface.

$$\mathsf{D}(|L|) = |L| \cap \mathsf{D}_d(n)$$

is called the *discriminant hypersurface* of $|L|$.

$$\widetilde{\mathsf{D}}(|L|) = \{(x, D) \in \mathbb{P}^n \times |L| : x \in \mathrm{Sing}(D)\}$$

with two projections $p : \widetilde{\mathsf{D}} \to \mathsf{D}(|L|)$ and $q : \widetilde{\mathsf{D}} \to |L|$. We define the *Jacobian hypersurface* of $|L|$ as

$$\mathrm{Jac}(|L|) = q(\widetilde{\mathsf{D}}(|L|)).$$

It parameterizes singular points of singular members of $|L|$. Again, it may coincide with the whole $\mathbb{P}^n$. In the case of polar linear systems, the discriminant hypersurface is equal to the Steinerian hypersurface, and the Jacobian hypersurface is equal to the Hessian hypersurface.

The *Steinerian hypersurface* $\mathrm{St}(|L|)$ $x \in \cap_{D\in|L|} P_{a^{n-1}}(D)$. Since $\dim L = n+1$, the intersection is empty, unless there exists D such that $P_{a^{n-1}}(D) = 0$. Thus $P_{a^n}(D) = 0$ and $a \in \mathrm{Sing}(D)$, hence $a \in \mathrm{Jac}(|L|)$ and $D \in \mathsf{D}(|L|)$. Conversely, if $a \in \mathrm{Jac}(|L|)$, then $\cap_{D\in|L|} P_{a^{n-1}}(D) \neq \emptyset$ and it is contained in $\mathrm{St}(|L|)$. By duality (1.12),

$$x \in \bigcap_{D\in|L|} P_{a^{n-1}}(D) \Leftrightarrow a \in \bigcap_{D\in|L|} P_x(D).$$

Thus the Jacobian hypersurface is equal to the locus of points which belong to the intersection of the first polars of divisors in $|L|$ with respect to some point $x \in \mathrm{St}(X)$. Let

$$\mathrm{HS}(|L|) = \{(a, b) \in \mathrm{He}(|L|) \times \mathrm{St}(|L|) : a \in \bigcap_{D\in|L|} P_b(D)\}$$

$$= \{(a, b) \in \mathrm{He}(|L|) \times \mathrm{St}(|L|) : b \in \bigcap_{D\in|L|} P_{a^{d-1}}(D)\}.$$

It is clear that $\mathrm{HS}(|L|) \subset \mathbb{P}^n \times \mathbb{P}^n$ is a complete intersection of $n+1$ divisors of type $(d-1,1)$. In particular,

$$\omega_{\mathrm{HS}(|L|)} \cong \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^n}((d-2)(n+1))). \tag{1.33}$$

One expects that, for a general point $x \in \mathrm{St}(|L|)$, there exists a unique $a \in \mathrm{Jac}(|L|)$ and a unique $D \in \mathsf{D}(|L|)$ as above. In this case, the correspondence $\mathrm{HS}(|L|)$ defines a birational isomorphism between the Jacobian and Steinerian hypersurface. Also, it is clear that $\mathrm{He}(|L|) = \mathrm{St}(|L|)$ if $d = 2$.

Assume that $|L|$ has no base points. Then $\mathrm{HS}(|L|)$ does not intersect the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. This defines a map

$$\mathrm{HS}(|L|) \to G_1(\mathbb{P}^n), \quad (a,b) \mapsto \overline{ab}.$$

Its image $\mathrm{Cay}(|L|)$ is called the *Cayleyan variety* of $|L|$.

A line $\ell \in \mathrm{Cay}(|L|)$ is called a *Reye line* of $|L|$. It follows from the definitions that a Reye line is characterized by the property that it contains a point such that there is a hyperplane in $|L|$ of hypersurfaces tangent to ℓ at this point. For example, if $d = 2$ this is equivalent to the property that ℓ is contained is a linear subsystem of $|L|$ of codimension 2 (instead of expected codimension 3).

The proof of Proposition 1.1.26 applies to our more general situation to give the degree of $\mathrm{Cay}(|L|)$ for a general n-dimensional linear system $|L|$ of hypersurfaces of degree d.

$$\deg \mathrm{Cay}(|L|) = \begin{cases} \sum_{i=1}^n (d-1)^i \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d > 2, \\ \frac{1}{2}\sum_{i=1}^n \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d = 2. \end{cases} \tag{1.34}$$

Let $\underline{f} = (f_0, \dots, f_n)$ be a basis of L. Choose coordinates in $\mathbb{P}^n$ to identify $S^d(E^\vee)$ with the polynomial ring $\mathbb{C}[t_0, \dots, t_n]$. A well-known fact from the complex analysis asserts that $\mathrm{Jac}(|L|)$ is given by the determinant of the Jacobian matrix

$$J(\underline{f}) = \begin{pmatrix} \partial_0 f_0 & \partial_1 f_0 & \dots & \partial_n f_0 \\ \partial_0 f_1 & \partial_1 f_1 & \dots & \partial_n f_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_0 f_n & \partial_1 f_n & \dots & \partial_n f_n \end{pmatrix}.$$

In particular, we expect that

$$\deg \mathrm{Jac}(|L|) = (n+1)(d-1).$$

If $a \in \mathrm{Jac}(|L|)$, then a nonzero vector in the null-space of $J(\underline{f})$ defines a point x such that $P_x(f_0)(a) = \dots = P_x(f_n)(a) = 0$. Equivalently,

$$P_{a^{n-1}}(f_0)(x) = \dots = P_{a^{n-1}}(f_n)(x) = 0.$$

This shows that $\mathrm{St}(|L|)$ is equal to the projectivization of the union of the null-spaces $\mathrm{Null}(\mathrm{Jac}(\underline{f}(a)))$, $a \in \mathbb{C}^{n+1}$. Also, a nonzero vector in the null space of the transpose matrix ${}^tJ(\underline{f})$ defines a hypersurface in $\mathsf{D}(|L|)$ with singularity at the point a.

Let $\mathrm{Jac}(|L|)^0$ be the open subset of points where the corank of the jacobian matrix is equal to 1. We assume that it is a dense subset of $\mathrm{Jac}(|L|)$. Then, taking the right and the left kernels of the Jacobian matrix, defines two maps

$$\mathrm{Jac}(|L|)^0 \to \mathsf{D}(|L|), \quad \mathrm{Jac}(|L|)^0 \to \mathrm{St}(|L|).$$

Explicitly, the maps are defined by the nonzero rows (resp. columns) of the adjugate matrix $\mathrm{adj}(\mathrm{He}(\underline{f}))$.

Let $\phi_{|L|} : \mathbb{P}^n \dashrightarrow |L^\vee|$ be the rational map defined by the linear system $|L|$. Under some assumptions of generality which we do not want to spell out, one can identify $\mathrm{Jac}(|L|)$ with the ramification divisor of the map and $\mathsf{D}(|L|)$ with the dual hypersurface of the branch divisor.

Let us now define a new variety attached to a n-dimensional linear system in $\mathbb{P}^n$. Consider the inclusion map $L \hookrightarrow S^d(E^\vee)$ and let

$$L \hookrightarrow S^d(E)^\vee, \quad f \mapsto \tilde{f},$$

be the restriction of the total polarization map (1.2) to L. Now we can consider $|L|$ as a n-dimensional linear system $\widetilde{|L|}$ on $|E|^d$ of divisors of type $(1,\dots,1)$. Let

$$\mathrm{PB}(|L|) = \bigcap_{D \in \widetilde{|L|}} D \subset |E|^d$$

be the base scheme of $\widetilde{|L|}$. We call it the *polar base locus* of $|L|$. It is equal to the complete intersection of $n+1$ effective divisors of type $(1,\dots,1)$. By the adjunction formula,

$$\omega_{\mathrm{PB}(|L|)} \cong \mathcal{O}_{\mathrm{PB}(|L|)}.$$

If smooth, $\mathrm{PB}(|L|)$ is a *Calabi-Yau variety* of dimension $(d-1)n-1$.

For any $f \in L$, let $N(f)$ be the set of points $x = ([v^{(1)}],\dots,[v^{(d)}]) \in |E|^d$ such that

$$\tilde{f}(v^{(1)},\dots,v^{(j-1)},v,v^{(j+1)},\dots,v^{(d)}) = 0$$

for every $j = 1,\dots,d$ and $v \in E$. Since

$$\tilde{f}(v^{(1)},\dots,v^{(j-1)},v,v^{(j+1)},\dots,v^{(d)}) = D_{v^{(1)}\dots v^{(j-1)}v^{(j+1)}\dots v^{(d)}}(D_v(f)),$$

This can be also expressed in the form

$$\widetilde{\partial_j f}(v^{(1)},\dots,v^{(j-1)},v^{(j+1)},\dots,v^{(d)}) = 0,\ j = 0,\dots,n. \tag{1.35}$$

Choose coordinates $u_0, \ldots, u_n$ in L and coordinates $t_0, \ldots, t_n$ in E. Let $\underline{\tilde{f}}$ be the image of a basis $\underline{f}$ of L in $(E^\vee)^d$. Then $\mathrm{PB}(|L|)$ is a subvariety of $(\mathbb{P}^n)^d$ given by a system of d multilinear equations

$$\tilde{f}_0(t^{(1)}, \ldots, t^{(d)}) = \ldots = \tilde{f}_n(t^{(1)}, \ldots, t^{(d)}) = 0,$$

where $t^{(j)} = (t_0^{(j)}, \ldots, t_n^{(j)}), j = 1, \ldots, d$. For any $\lambda = (\lambda_0, \ldots, \lambda_n)$, set $\tilde{f}_\lambda = \sum_{i=0}^n \lambda_i \tilde{f}_i$.

Proposition 1.1.28 *The following is equivalent:*

(i) $x \in \mathrm{PB}(|L|)$ *is a singular point,*
(ii) $x \in N(\tilde{f}_\lambda)$ *for some* $\lambda \neq 0$.

Proof The variety $\mathrm{PB}(|L|)$ is smooth at a point x if and only if the rank of the $d(n+1) \times (n+1)$-size matrix

$$(a_{ij}^k) = \big(\frac{\partial \tilde{f}_k}{\partial t_i^{(j)}}(x)\big)_{i,k=0,\ldots,n,j=1,\ldots,d}$$

is equal to $n+1$. Let $\tilde{f}_u = u_0\tilde{f}_0 + \cdots + u_n\tilde{f}_n$, where $u_0, \ldots, u_n$ are unknowns. Then the nullspace of the matrix is equal to the space of solutions $u = (\lambda_0, \ldots, \lambda_n)$ of the system of linear equations

$$\frac{\partial \tilde{f}_u}{\partial u_0}(x) = \ldots = \frac{\partial \tilde{f}_u}{\partial u_n}(x) = \frac{\partial \tilde{f}_u}{\partial t_i^{(j)}}(x) = 0. \tag{1.36}$$

For a fixed λ, in terminology of [239], p. 445, the system has a solution x in $|E|^d$ if $\tilde{f}_\lambda = \sum \lambda_i \tilde{f}_i$ is a *degenerate multilinear form*. By Proposition 1.1 from Chapter 14 of loc.cit., $\tilde{f}_\lambda$ is degenerate if and only if $N(\tilde{f}_\lambda)$ is non-empty. This proves the assertion. □

For any non-empty subset I of $\{1, \ldots, d\}$, let Δ_I be the subset of points $x \in |E|^d$ with equal projections to i-th factors with $i \in I$. Let Δ_k be the union of Δ_I with $\#I = k$. The set Δ_d is denoted by Δ (the small diagonal).

Observe that $\mathrm{PB}(|L|) = \mathrm{HS}(|L|)$ if $d = 2$ and $\mathrm{PB}(|L|) \cap \Delta_{d-1}$ consists of d copies isomorphic to $\mathrm{HS}(|L|)$ if $d > 2$.

Definition 1.1.29 *A* n*-dimensional linear system* $|L| \subset |S^d(E^\vee)|$ *is called* regular *if* $\mathrm{PB}(|L|)$ *is smooth at each point of* Δ_{d-1}.

Proposition 1.1.30 *Assume* $|L|$ *is regular. Then*

(i) $|L|$ *has no base points,*
(ii) $\widetilde{\mathsf{D}}(|L|)$ *is smooth.*

Proof (i) Assume that $x = ([v_0], \dots, [v_0]) \in \mathrm{PB}(|L|) \cap \Delta$. Consider the linear map $L \to E$ defined by evaluating $\tilde{f}$ at a point $(v_0, \dots, v_0, v, v_0, \dots, v_0)$, where $v \in E$. This map factors through a linear map $L \to E/[v_0]$, and hence has a nonzero f in its kernel. This implies that $x \in N(f)$, and hence x is a singular point of $\mathrm{PB}(|L|)$.

(ii) In coordinates, the variety $\widetilde{\mathsf{D}}(|L|)$ is a subvariety of type $(1, d-1)$ of $\mathbb{P}^n \times \mathbb{P}^n$ given by the equations

$$\sum_{k=0}^{n} u_k \frac{\partial f_k}{\partial t_0} = \dots = \sum_{k=0}^{n} u_k \frac{\partial f_k}{\partial t_n} = 0.$$

The tangent space at a point $([\lambda], [a])$ is given by the system of $n+1$ linear equations in $2n+2$ variables $(X_0, \dots, X_n, Y_0, \dots, Y_n)$

$$\sum_{k=0}^{n} \frac{\partial f_k}{\partial t_i}(a) X_k + \sum_{j=0}^{n} \frac{\partial^2 f_\lambda}{\partial t_i \partial t_j}(a) Y_j = 0, \quad i = 0, \dots, n, \tag{1.37}$$

where $f_\lambda = \sum_{k=0}^{n} \lambda_k f_k$. Suppose $([\lambda], [a])$ is a singular point. Then the equations are linearly dependent. Thus there exists a nonzero vector $v = (\alpha_0, \dots, \alpha_n)$ such that

$$\sum_{i=0}^{n} \alpha_i \frac{\partial f_k}{\partial t_i}(a) = D_v(f_k)(a) = \tilde{f}_k(a, \dots, a, v) = 0, \ k = 0, \dots, n$$

and

$$\sum_{i}^{n} \alpha_i \frac{\partial^2 f_\lambda}{\partial t_i \partial t_j}(a) = D_v(\frac{\partial f_\lambda}{\partial t_j})(a) = D_{a^{d-2}v}(\frac{\partial f_\lambda}{\partial t_j}) = 0, \ j = 0, \dots, n,$$

where $f_\lambda = \sum \lambda_k f_k$. The first equation implies that $x = ([a], \dots, [a], [v])$ belongs to $\mathrm{PB}(|L|)$. Since $a \in \mathrm{Sing}(f_\lambda)$, we have $D_{a^{d-1}}(\frac{\partial f_\lambda}{\partial t_j}) = 0$, $j = 0, \dots, n$. By (1.35), this and the second equation now imply that $x \in N(f_\lambda)$. By Proposition 1.1.28, $\mathrm{PB}(|L|)$ is singular at x, contradicting the assumption. □

Corollary 1.1.31 *Suppose $|L|$ is regular. Then the projection*

$$q : \widetilde{\mathsf{D}}(|L|) \to \mathsf{D}(|L|)$$

is a resolution of singularities.

Consider the projection $p : \widetilde{\mathsf{D}}(|L|) \to \mathrm{Jac}(|L|), (D, x) \mapsto x$. Its fibres are linear spaces of divisors in $|L|$ singular at the point $[a]$. Conversely, suppose $\mathsf{D}(|L|)$ contains a linear subspace, in particular, a line. Then, by Bertini's Theorem all singular divisors parameterized by the line have a common singular

point. This implies that the morphism p has positive dimensional fibres. This simple observation gives the following.

Proposition 1.1.32 *Suppose* $\mathsf{D}(|L|)$ *does not contain lines. Then* $\widetilde{\mathsf{D}}(|L|)$ *is smooth if and only if* $\mathrm{Jac}(|L|)$ *is smooth. Moreover,* $\mathrm{HS}(|L|) \cong \mathrm{St}(|L|) \cong \mathrm{Jac}(|L|)$.

Remark 1.1.33 We will prove later in Example 1.2.3 that the tangent space of the discriminant hypersurface $\mathsf{D}_d(n)$ at a point corresponding to a hypersurface $X = V(f)$ with only one ordinary double point x is naturally isomorphic to the linear space of homogeneous forms of degree d vanishing at the point x modulo $\mathbb{C}f$. This implies that $\mathsf{D}(|L|)$ is nonsingular at a point corresponding to a hypersurface with one ordinary double point unless this double point is a base point of $|L|$. If $|L|$ has no base points, the singular points of $\mathsf{D}(|L|)$ are of two sorts: either they correspond to divisors with worse singularities than one ordinary double point, or the linear space $|L|$ is tangent to $\mathsf{D}_d(n)$ at its nonsingular point.

Consider the natural action of the symmetric group $\mathfrak{S}_d$ on $(\mathbb{P}^n)^d$. It leaves $\mathrm{PB}(|L|)$ invarian. The quotient variety

$$\mathrm{Rey}(|L|) = \mathrm{PB}(|L|)/\mathfrak{S}_d$$

is called the *Reye variety* of $|L|$. If $d > 2$ and $n > 1$, the Reye variety is singular.

Example 1.1.34 Assume $d = 2$. Then $\mathrm{PB}(|L|) = \mathrm{HS}(|L|)$ and $\mathrm{Jac}(|L|) = \mathrm{St}(|L|)$. Moreover, $\mathrm{Rey}(|L|) \cong \mathrm{Cay}(|L|)$. We have

$$\deg \mathrm{Jac}(|L|) = \deg \mathsf{D}(|L|) = n+1, \quad \deg \mathrm{Cay}(|L|) = \sum_{i=1}^{n} \tbinom{n+1}{i}\tbinom{n-1}{i-1}.$$

The linear system is regular if and only if $\mathrm{PB}(|L|)$ is smooth. This coincides with the notion of regularity of a web of quadrics in $\mathbb{P}^3$ discussed in [132].

A Reye line ℓ is contained in a codimension 2 subspace $\Lambda(\ell)$ of $|L|$, and is characterized by this condition. The linear subsystem $\Lambda(\ell)$ of dimension $n-2$ contains ℓ in its base locus. The residual component is a curve of degree $2^{n-1}-1$ which intersects ℓ at two points. The points are the two ramification points of the pencil $Q \cap \ell, Q \in |L|$. The two singular points of the base locus of $\Lambda(\ell)$ define two singular points of the intersection $\Lambda(\ell) \cap \mathsf{D}(|L|)$. Thus $\Lambda(\ell)$ is a codimension 2 subspace of $|L|$ which is tangent to the determinantal hypersurface at two points.

If $|L|$ is regular and $n = 3$, $\mathrm{PB}(|L|)$ is a K3 surface, and its quotient $\mathrm{Rey}(|L|)$ is an Enriques surface. The Cayley variety is a congruence (i.e. a surface) of

lines in $G_1(\mathbb{P}^3)$ of order 7 and class 3 (this means that there are 7 Reye lines through a general point in $\mathbb{P}^3$ and there 3 Reye lines in a general plane). The Reye lines are bitangents of the quartic surface $\mathsf{D}(|L|)$. The quartic surface has 10 nodes and is called *Cayley quartic symmetroid*. We refer for the details to [132]. The *Reye congruence* of lines is also discussed in [267].

1.2 The dual hypersurface

1.2.1 The polar map

Let $X = V(f)$ for some $f \in S^d(E^\vee)$. We assume that it is not a cone. The polarization map

$$E \to S^{d-1}(E^\vee), \quad v \mapsto D_v(f),$$

allows us to identify $|E|$ with an n-dimensional linear system of hypersurfaces of degree $d-1$. This linear system defines a rational map

$$\mathsf{p}_X : |E| \dashrightarrow \mathbb{P}(E).$$

It follows from (1.12) that the map is given by assigning to a point a the linear polar $P_{a^{d-1}}(X)$. We call the map p the *polar map* defined by the hypersurface X. In coordinates, the polar map is given by

$$[x_0, \dots, x_n] \mapsto [\frac{\partial f}{\partial t_0}, \dots, \frac{\partial f}{\partial t_n}].$$

Recall that a hyperplane $H_a = V(\sum a_i \xi_i)$ in the dual projective space $(\mathbb{P}^n)^\vee$ is the point $a = [a_0, \dots, a_n] \in \mathbb{P}^n$. The preimage of the hyperplane H_a under p_X is the polar $P_a(X) = V(\sum a_i \frac{\partial f}{\partial t_i})$.

If X is nonsingular, the polar map is a regular map given by polynomials of degree $d-1$. Since it is a composition of the Veronese map and a projection, it is a finite map of degree $(d-1)^n$.

Proposition 1.2.1 *Assume X is nonsingular. The ramification divisor of the polar map is equal to* $\mathrm{He}(X)$.

Proof Note that, for any finite map $\phi : X \to Y$ of nonsingular varieties, the ramification divisor $\mathrm{Ram}(\phi)$ is defined locally by the determinant of the linear map of locally free sheaves $\phi^*(\Omega^1_Y) \to \Omega^1_X$. The image of $\mathrm{Ram}(\phi)$ in Y is called the *branch divisor*. Both of the divisors may be nonreduced. We have the *Hurwitz formula*

$$K_X = \phi^*(K_Y) + \mathrm{Ram}(\phi). \tag{1.38}$$

The map ϕ is étale outside $\mathrm{Ram}(\phi)$, i.e., for any point $x \in X$ the homomorphism of local ring $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ defines an isomorphism of their formal completions. In particular, the preimage $\phi^{-1}(Z)$ of a nonsingular subvariety $Z \subset Y$ is nonsingular outside the support of $\mathrm{Ram}(\phi)$. Applying this to the polar map we see that the singular points of $P_a(X) = p_X^{-1}(H_a)$ are contained in the ramification locus $\mathrm{Ram}(p_X)$ of the polar map. On the other hand, we know that the set of singular points of first polars is the Hessian $\mathrm{He}(X)$. This shows that $\mathrm{He}(X) \subset \mathrm{Ram}(\mathsf{p}_X)$. Applying the Hurwitz formula for the canonical sheaf

$$K_{\mathbb{P}^n} = p_X^*(K_{(\mathbb{P}^n)^\vee}) + \mathrm{Ram}(\mathsf{p}_X).$$

we obtain that $\deg(\mathrm{Ram}(\mathsf{p}_X)) = (n+1)(d-2) = \deg(\mathrm{He}(X))$. This shows that $\mathrm{He}(X) = \mathrm{Ram}(\mathsf{p}_X)$. □

What is the branch divisor? One can show that the preimage of a hyperplane H_a in $\mathbb{P}(E)$ corresponding to a point $a \in |E|$ is singular if and only if its intersection with the branch divisor is not transversal. This means that the dual hypersurface of the branch divisor is the Steinerian hypersurface. Equivalently, the branch divisor is the dual of the Steinerian hypersurface.

1.2.2 Dual varieties

Recall that the *dual variety* $X^\vee$ of a subvariety X in $\mathbb{P}^n = |E|$ is the closure in the dual projective space $(\mathbb{P}^n)^\vee = |E^\vee|$ of the locus of hyperplanes in $\mathbb{P}^n$ which are tangent to X at some nonsingular point of X.

The dual variety of a hypersurface $X = V(f)$ is the image of X under the rational map given by the first polars. In fact, the point $[\partial_0 f(x), \dots, \partial_n f(x)]$ in $(\mathbb{P}^n)^\vee$ is the hyperplane $V(\sum_{i=0}^n \partial_i f(x) t_i)$ in $\mathbb{P}^n$ which is tangent to X at the point x.

The following result is called the *Reflexivity theorem*. One can find its proof in many modern text-books (e.g. [239], [278], [562], [607]).

Theorem 1.2.2 (Reflexivity Theorem)

$$(X^\vee)^\vee = X.$$

It follows from any proof in loc. cit. that, for any nonsingular point $y \in X^\vee$ and any nonsingular point $x \in X$,

$$\mathbb{T}_x(X) \subset H_y \Leftrightarrow \mathbb{T}_y(X^\vee) \subset H_x.$$

Here we continue to identify a point a in $|E|$ with a hyperplane H_a in $\mathbb{P}(E)$. The set of all hyperplanes in $(\mathbb{P}^n)^\vee$ containing the linear subspace $\mathbb{T}_y(X^\vee)$ is

the dual linear space of $\mathbb{T}_y(X^\vee)$ in $\mathbb{P}^n$. Thus the fiber of the *duality map* (or *Gauss map*)

$$\gamma : X^{\mathrm{ns}} \to X^\vee, \quad x \mapsto \mathbb{T}_x(X), \tag{1.39}$$

over a nonsingular point $y \in X^\vee$ is an open subset of the projective subspace in $\mathbb{P}^n$ equal to the dual of the tangent space $\mathbb{T}_y(X^\vee)$. Here and later X^{ns} denotes the set of nonsingular points of a variety X. In particular, if $X^\vee$ is a hypersurface, the dual space of $\mathbb{T}_y(X^\vee)$ must be a point, and hence the map γ is birational.

Let us apply this to the case when X is a nonsingular hypersurface. The polar map is a finite map, hence the dual of a nonsingular hypersurface is a hypersurface. The duality map is a birational morphism

$$\mathsf{p}_X|_X : X \to X^\vee.$$

The degree of the dual hypersurface $X^\vee$ (if it is a hypersurface) is called the *class* of X. For example, the class of any plane curve of degree > 1 is well-defined.

Example 1.2.3 Let $\mathsf{D}_d(n)$ be the discriminant hypersurface in $|S^d(E^\vee)|$. We would like to describe explicitly the tangent hyperplane of $\mathsf{D}_d(n)$ at its nonsingular point. Let

$$\tilde{\mathcal{D}}_d(n) = \{(X, x) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n : x \in \mathrm{Sing}(X)\}.$$

Let us see that $\tilde{\mathcal{D}}_d(n)$ is nonsingular and the projection to the first factor

$$\pi : \tilde{\mathcal{D}}_d(n) \to \mathsf{D}_d(n) \tag{1.40}$$

is a resolution of singularities. In particular, π is an isomorphism over the open set $\mathsf{D}_d(n)^{\mathrm{ns}}$ of nonsingular points of $\mathsf{D}_d(n)$.

The fact that $\tilde{\mathcal{D}}_d(n)$ is nonsingular follows easily from considering the projection to $\mathbb{P}^n$. For any point $x \in \mathbb{P}^n$ the fiber of the projection is the projective space of hypersurfaces which have a singular point at x (this amounts to $n+1$ linear conditions on the coefficients). Thus $\tilde{\mathcal{D}}_d(n)$ is a projective bundle over $\mathbb{P}^n$ and hence is nonsingular.

Let us see where π is an isomorphism. Let $A_{\mathbf{i}}, |\mathbf{i}| = d$, be the projective coordinates in $\left|\mathcal{O}_{\mathbb{P}^n}(d)\right| = |S^d(E^\vee)|$ corresponding to the coefficients of a hypersurface of degree d and let $t_0, \dots, t_n$ be projective coordinates in $\mathbb{P}^n$. Then $\tilde{\mathcal{D}}_d(n)$ is given by $n+1$ bihomogeneous equations of bidegree $(1, d-1)$:

$$\sum_{|\mathbf{i}|=d} i_s A_{\mathbf{i}} \mathbf{t}^{\mathbf{i}-e_s} = 0, \quad s = 0, \dots, n. \tag{1.41}$$

Here e_s is the s-th unit vector in $\mathbb{Z}^{n+1}$.

A point $(X, [v_0]) = (V(f), [v_0]) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n$ belongs to $\tilde{\mathcal{D}}_d(n)$ if and only if, replacing $A_{\mathbf{i}}$ with the coefficient of f at $\mathbf{t}^i$ and t_i with the i-th coefficient of v_0, we get the identities.

We identify the tangent space of $|S^d(E^\vee)| \times |E|$ at a point $(X, [v_0])$ with the space $S^d(E^\vee)/\mathbb{C}f \oplus E/\mathbb{C}v_0$. In coordinates, a vector in the tangent space is a pair $(g, [v])$, where $g = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}}\mathbf{t}^{\mathbf{i}}$, $v = (x_0, \dots, x_n)$ are considered modulo pairs $(\lambda f, \mu v_0)$. Differentiating equations (1.41), we see that the tangent space is defined by the $(n+1) \times \binom{n+d}{d}$-matrix

$$\begin{pmatrix} \dots & i_0 x^{\mathbf{i}-e_0} & \dots & \sum_{|\mathbf{i}|=d} i_0 i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_0} & \dots & \sum_{|\mathbf{i}|=d} i_0 i_n A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & i_n x^{\mathbf{i}-e_n} & \dots & \sum_{|\mathbf{i}|=d} i_n i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_0} & \dots & \sum_{|\mathbf{i}|=d} i_n i_n A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_n}, \end{pmatrix}$$

where $x^{\mathbf{i}-e_s} = 0$ if $\mathbf{i} - e_s$ is not a non-negative vector. It is easy to interpret solutions of these equations as pairs (g, v) from the above such that

$$\nabla(g)(v_0) + \mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.42}$$

Since $[v_0]$ is a singular point of $V(f)$, $\nabla(f)([v_0]) = 0$. Also $\mathrm{He}(f)(v_0) \cdot v_0 = 0$, as follows from Theorem 1.1.20. This confirms that pairs $(\lambda f, \mu v_0)$ are always solutions. The tangent map $d\pi$ at the point $(V(f), [v_0])$ is given by the projection $(g, v) \mapsto g$, where (g, v) is a solution of (1.42). Its kernel consists of the pairs $(\lambda f, v)$ modulo pairs $(\lambda f, \mu v_0)$. For such pairs the equations (1.42) give

$$\mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.43}$$

We may assume that $v_0 = (1, 0, \dots, 0)$. Since $[v_0]$ is a singular point of $V(f)$, we can write $f = t_0^{d-2} f_2(t_1, \dots, t_n) + \dots$. Computing the Hessian matrix at the point v_0 we see that it is equal to

$$\begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}, \tag{1.44}$$

where $f_2(t_1, \dots, t_n) = \sum_{0 \le i,j \le n} a_{ij} t_i t_j$. Thus a solution of (1.43), not proportional to v_0 exists if and only if $\det \mathrm{He}(f_2) = 0$. By definition, this means that the singular point of X at x is not an ordinary double point. Thus we obtain that the projection map (1.40) is an isomorphism over the open subset of $\mathsf{D}_d(n)$ representing hypersurfaces with an isolated ordinary singularity.

We can also find the description of the tangent space of $\mathsf{D}_d(n)$ at its point

$X = V(f)$ representing a hypersurface with a unique ordinary singular point x. It follows from calculation of the Hessian matrix in (1.44), that its corank at the ordinary singular point is equal to 1. Since the matrix is symmetric, a vector in its nullspace is orthogonal to the column of the matrix. We know that $\mathrm{He}(f)(v_0) \cdot v_0 = 0$. Thus the dot-product $\nabla(g)(v_0) \cdot v_0$ is equal to zero. By Euler's formula, we obtain $g(v_0) = 0$. The converse is also true. This proves that

$$T(\mathsf{D}_d(n))_X = \{g \in S^d(E^\vee)/\mathbb{C}f : g(x) = 0\}. \tag{1.45}$$

Now we are ready to compute the dual variety of $\mathsf{D}_d(n)$. The condition $g(b) = 0$, where $\mathrm{Sing}(X) = \{b\}$ is equivalent to $D_{b^d}(f) = 0$. Thus the tangent hyperplane, considered as a point in the dual space $|S^d(E)| = |S^d(E^\vee)^\vee|$ corresponds to the envelope $b^d = (\sum_{s=0}^n b_s \partial_i)^d$. The set of such envelopes is the Veronese variety V_d^n, the image of $|E|$ under the Veronese map $\mathsf{v}_d : |E| \to |S^d(E)|$. Thus

$$\mathsf{D}_d(n)^\vee \cong \nu_d(\mathbb{P}^n), \tag{1.46}$$

Of course, it is predictable. Recall that the Veronese variety is embedded naturally in $|\mathcal{O}_{\mathbb{P}^n}(d)|^\vee$. Its hyperplane section can be identified with a hypersurface of degree d in $\mathbb{P}^n$. A tangent hyperplane is a hypersurface with a singular point, i.e. a point in $\mathsf{D}_d(n)$. Thus the dual of V_d^n is isomorphic to $\mathsf{D}_d(n)$, and hence, by duality, the dual of $\mathsf{D}_d(n)$ is isomorphic to V_d^n.

Example 1.2.4 Let $Q = V(q)$ be a nonsingular quadric in $\mathbb{P}^n$. Let $A = (a_{ij})$ be a symmetric matrix defining q. The tangent hyperplane of Q at a point $[x] \in \mathbb{P}^n$ is the hyperplane

$$t_0 \sum_{j=0}^n a_{0j} x_j + \cdots + t_n \sum_{j=0}^n a_{nj} x_j = 0.$$

Thus the vector of coordinates $\mathbf{y} = (y_0, \ldots, y_n)$ of the tangent hyperplane is equal to the vector $A \cdot x$. Since A is invertible, we can write $x = A^{-1} \cdot y$. We have

$$0 = x \cdot A \cdot x = (y \cdot A^{-1}) \cdot A \cdot (A^{-1} \cdot y) = y \cdot A^{-1} \cdot y = 0.$$

Here we treat x or y as a row-matrix or as a column-matrix in order the matrix multiplication makes sense. Since $A^{-1} = \det(A)^{-1}\mathrm{adj}(A)$, we obtain that the dual variety of Q is also a quadric given by the adjugate matrix $\mathrm{adj}(A)$.

The description of the tangent space of the discriminant hypersurface from Example 1.2.3 has the following nice application (see also Remark 1.1.33).

Proposition 1.2.5 *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Suppose a is a nonsingular point of the Steinerian hypersurface* $\mathrm{St}(X)$. *Then* $\mathrm{Sing}(P_a(X))$ *consists of an ordinary singular point b and*

$$\mathbb{T}_a(\mathrm{St}(X)) = P_{b^{d-1}}(X).$$

1.2.3 Plücker formulas

Let $X = V(f)$ be a nonsingular irreducible hypersurface that is not a cone. Fix $n-1$ general points $a_1, \ldots, a_{n-1}$ in $\mathbb{P}^n$. Consider the intersection

$$X \cap P_{a_1}(X) \cap \ldots \cap P_{a_{n-1}}(X) = \{b \in \mathbb{P}^n : a_1, \ldots, a_{n-1} \in \mathbb{T}_b(X)\}.$$

The set of hyperplanes through a general set of $n-1$ points is a line in the dual space. This shows that

$$\deg X^\vee = \#X \cap P_{a_1}(X) \cap \ldots \cap P_{a_{n-1}}(X) = d(d-1)^{n-1}. \tag{1.47}$$

The computation does not apply to singular X since all polars $P_a(X)$ pass through singular points of X. In the case when X has only isolated singularities, the intersection of $n-1$ polars with X contains singular points which correspond to hyperplanes which we excluded from the definition of the dual hypersurface. So we get the following formula

$$\deg(X^\vee) = d(d-1)^{n-1} - \sum_{x \in \mathrm{Sing}(X)} i(X, P_{a_1}(X), \ldots, P_{a_{n-1}}(X))_x. \tag{1.48}$$

To state an explicit formula we need some definition. Let $\underline{\phi} = (\phi_1, \ldots, \phi_k)$ be a set of polynomials in $\mathbb{C}[z_1, \ldots, z_n]$. We assume that the holomorphic map $\mathbb{C}^n \to \mathbb{C}^k$ defined by these polynomials has an isolated critical point at the origin. Let $J(\underline{\phi})$ be the jacobian matrix. The ideal $\mathcal{J}(\underline{\phi})$ in the ring of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$ generated by the maximal minors of the Jacobian matrix is called the *Jacobian ideal* of $\underline{\phi}$. The number

$$\mu(\underline{\phi}) = \dim \mathbb{C}[[z_1, \ldots, z_n]]/\mathcal{J}(\underline{\phi})$$

is called the *Milnor number* of $\underline{\phi}$. Passing to affine coordinates, this definition easily extends to the definition of the Milnor number $\mu(X, x)$ of an isolated singularity of a complete intersection subvariety X in $\mathbb{P}^n$.

We will need the following result of Lê Dũng Tráng [360], Theorem 3.7.1.

Lemma 1.2.6 *Let Z be a complete intersection in $\mathbb{C}^n$ defined by polynomials*

$\phi_1, \ldots, \phi_k$ with isolated singularity at the origin. Let $Z_1 = V(\phi_1, \ldots, \phi_{k-1})$. Then

$$\mu(\phi_1, \ldots, \phi_{k-1}) + \mu(\phi_1, \ldots, \phi_{k-1}, \phi_k)$$

$$= \dim \mathbb{C}[[z_1, \ldots, z_n]]/(\phi_1, \ldots, \phi_{k-1}, \mathcal{J}(\phi_1, \ldots, \phi_k)).$$

Now we can state and prove the *Plücker-Teissier formula* for a hypersurface with isolated singularities:

Theorem 1.2.7 *Let X be a hypersurface in $\mathbb{P}^n$ of degree d. Suppose X has only isolated singularities. For any point $x \in \mathrm{Sing}(X)$, let*

$$e(X, x) = \mu(X, x) + \mu(H \cap X, x),$$

where H is a general hyperplane section of X containing x.Then

$$\deg X^\vee = d(d-1)^{n-1} - \sum_{x \in \mathrm{Sing}(X)} e(X, x).$$

.

Proof We have to show that $e(X, x) = i(X, P_{a_1}(X), \ldots, P_{a_{n-1}}(X))_x$. We may assume that $x = [1, 0, \ldots, 0]$ and choose affine coordinates with $z_i = t_i/t_0$. Let $f(t_0, \ldots, t_n) = t_0^d g(z_1, \ldots, z_n)$. Easy calculations employing the Chain Rule, give the formula for the dehomogenized partial derivatives

$$x_0^{-d} \frac{\partial f}{\partial t_0} = dg + \sum \frac{\partial g}{\partial z_i} z_i,$$
$$x_0^{-d} \frac{\partial f}{\partial t_i} = \frac{\partial g}{\partial z_i},\ i = 1, \ldots, n.$$

Let $H = V(h)$ be a general hyperplane spanned by $n-1$ general points $a_1, \ldots, a_{n-1}$, and $h : \mathbb{C}^n \to \mathbb{C}$ be the projection defined by the linear function $h = \sum \alpha_i z_i$. Let

$$F : \mathbb{C}^n \to \mathbb{C}^2, \quad z = (z_1, \ldots, z_n) \mapsto (g(z), h(z)).$$

Consider the Jacobian determinant of the two functions (f, h)

$$J(g, h) = \begin{pmatrix} \frac{\partial g}{\partial z_1} & \cdots & \frac{\partial g}{\partial z_n} \\ \alpha_1 & \ldots & \alpha_n \end{pmatrix}.$$

The ideal $(g, J(g, h))$ defines the set of critical points of the restriction of the map F to $X \setminus V(t_0)$. We have

$$(g, J(g, h)) = (g, \alpha_i \frac{\partial g}{\partial z_j} - \alpha_j \frac{\partial g}{\partial z_i})_{1 \le i < j \le n},$$

The points $(0,\dots,0,\alpha_j,0,\dots,0,-\alpha_i,0,\dots,0)$ span the hyperplane H. We may assume that these points are our points $a_1,\dots,a_{n-1}$. So, we see that $(g, J(g,h))$ coincides with the ideal in the completion of local ring $\mathcal{O}_{\mathbb{P}^n,x}$ generated by f and the polars $P_{a_i}(f)$. By definition of the index of intersection, we have

$$i(X, P_{a_1}(X),\dots,P_{a_{n-1}}(X))_x = \mu(g,h).$$

It remains for us to apply Lemma 1.2.6, where $Z = V(g)$ and $Z_1 = V(g) \cap V(h)$. □

Example 1.2.8 An isolated singular point x of a hypersurface X in $\mathbb{P}^n$ is called an A_k*-singularity* (or a singular point of type A_k) if the formal completion of $\mathcal{O}_{X,x}$ is isomorphic to $\mathbb{C}[[z_1,\dots,z_n]]/(z_1^{k+1}+z_2^2+\cdots+z_n^2)$. If $k=1$, it is an ordinary quadratic singularity (or a *node*), if $k=2$, it is an *ordinary cusp*. We get

$$\mu(X,x) = k,\ \mu(X\cap H, x) = 1.$$

This gives the Plücker formula for hypersurfaces with s singularities of type $A_{k_1},\dots,A_{k_s}$

$$\deg X^\vee = d(d-1)^{n-1} - (k_1+1) - \cdots - (k_s+1). \tag{1.49}$$

In particularly, when X is a plane curve C with δ nodes and κ ordinary cusps, we get a familiar *Plücker formula*

$$\deg C^\vee = d(d-1) - 2\delta - 3\kappa. \tag{1.50}$$

Note that, in case of plane curves, $\mu(H\cap X, x)$ is always equal to $\mathrm{mult}_x X - 1$, where $\mathrm{mult}_x X$ is the multiplicity of X at x.

$$\deg C^\vee = d(d-1) - \sum_{x\in \mathrm{Sing}(X)} (\mu(X,x) + \mathrm{mult}_x X - 1). \tag{1.51}$$

Note that the dual curve $C^\vee$ of a nonsingular curve C of degree $d>2$ is always singular. This follows from the formula for the genus of a nonsingular plane curve and the fact that C and $C^\vee$ are birationally isomorphic. The polar map $C \to C^\vee$ is equal to the normalization map. A singular point of $C^\vee$ corresponds to a line which is either tangent to C at several points, or is an inflection tangent. We skip a local computation which shows that a line which is an inflection tangent at one point with $\mathrm{ordfl} = 1$ (an *honest inflection tangent*) gives an ordinary cusp of $C^\vee$ and a line which is tangent at two points which are not inflection points (*honest bitangent*) gives a node. Thus we obtain that the number $\check{\delta}$ of nodes of $C^\vee$ is equal to the number of honest bitangents of

C and the number $\check{\kappa}$ of ordinary cusps of $C^\vee$ is equal to the number of honest inflection tangents to $C^\vee$.

Assume that C is nonsingular and $C^\vee$ has no other singular points except ordinary nodes and cusps. We know that the number of inflection points is equal to $3d(d-2)$. Applying Plücker formula (1.50) to $C^\vee$, we get that

$$\check{\delta} = \frac{1}{2}\big(d(d-1)(d(d-1)-1)-d-9d(d-2)\big) = \frac{1}{2}d(d-2)(d^2-9). \quad (1.52)$$

This is the (expected) number of bitangents of a nonsingular plane curve. For example, we expect that a nonsingular plane quartic has 28 bitangents.

We refer for discussions of Plücker formulas to many modern text-books (e.g. [220], [231], [267], [239]). A proof of Plücker-Teissiere formula can be found in [558]. A generalization of the Plücker-Teissier formula to complete intersections in projective space was given by S. Kleiman [334]

1.3 Polar s-hedra

1.3.1 Apolar schemes

We continue to use E to denote a complex vector space of dimension $n+1$. Consider the polarization pairing (1.2)

$$S^d(E^\vee) \times S^k(E) \to S^{d-k}(E^\vee), \quad (f,\psi) \mapsto D_\psi(f).$$

Definition 1.3.1 *$\psi \in S^k(E)$ is called* apolar *to $f \in S^d(E^\vee)$ if $D_\psi(f) = 0$. We extend this definition to hypersurfaces in the obvious way.*

Lemma 1.3.2 *For any $\psi \in S^k(E), \psi' \in S^m(E)$ and $f \in S^d(E^\vee)$,*

$$D_{\psi'}(D_\psi(f)) = D_{\psi\psi'}(f).$$

Proof By linearity and induction on the degree, it suffices to verify the assertion in the case when $\psi = \partial_i$ and $\psi' = \partial_j$. In this case it is obvious. □

Corollary 1.3.3 *Let $f \in S^d(E^\vee)$. Let $\mathrm{AP}_k(f)$ be the subspace of $S^k(E)$ spanned by forms of degree k apolar to f. Then*

$$\mathrm{AP}(f) = \bigoplus_{k=0}^{\infty} AP_k(f)$$

is a homogeneous ideal in the symmetric algebra $S(E)$.

Definition 1.3.4 *The quotient ring*

$$A_f = S(E)/\mathrm{AP}(f)$$

is called the apolar ring *of* f.

The ring A_f inherits the grading of $S(E)$. Since any polynomial $\psi \in S^r(E)$ with $r > d$ is apolar to f, we see that A_f is annihilated by the ideal $\mathfrak{m}_+^{d+1} = (\partial_0, \dots, \partial_n)^{d+1}$. Thus A_f is an Artinian graded local algebra over $\mathbb{C}$. Since the pairing between $S^d(E)$ and $S^d(E^\vee)$ has values in $S^0(E^\vee) = \mathbb{C}$, we see that $\mathrm{AP}_d(f)$ is of codimension 1 in $S^d(E)$. Thus $(A_f)_d$ is a vector space of dimension 1 over $\mathbb{C}$ and coincides with the *socle* of A_f, i.e. the ideal of elements of A_f annihilated by its maximal ideal.

Note that the latter property characterizes Gorenstein graded local Artinian rings, see [208], [313].

Proposition 1.3.5 (F. S. Macaulay) *The correspondence $f \mapsto A_f$ is a bijection between $|S^d(E^\vee)|$ and graded Artinian quotient algebras $S(E)/J$ with 1-dimensional socle.*

Proof Let us show how to reconstruct $\mathbb{C}f$ from $S(E)/J$. The multiplication of d vectors in E composed with the projection to $S^d(E)/J_d$ defines a linear map $S^d(E) \to S^d(E)/J_d \cong \mathbb{C}$. Choosing a basis $(S(E)/J)_d$, we obtain a linear function f on $S^d(E)$. It corresponds to an element of $S^d(E^\vee)$.

□

Recall that any closed non-empty subscheme $Z \subset \mathbb{P}^n$ is defined by a unique saturated homogeneous ideal I_Z in $\mathbb{C}[t_0, \dots, t_n]$. Its locus of zeros in the affine space $\mathbb{A}^{n+1}$ is the affine cone C_Z of Z isomorphic to $\mathrm{Spec}(\mathbb{C}[t_0, \dots, t_n]/I_Z)$.

Definition 1.3.6 *Let $f \in S^d(E^\vee)$. A subscheme $Z \subset |E^\vee| = \mathbb{P}(E)$ is called* apolar to f if its homogeneous ideal I_Z is contained in $\mathrm{AP}(f)$, or, equivalently, $\mathrm{Spec}(A_f)$ is a closed subscheme of the affine cone C_Z of Z.

This definition agrees with the definition of an apolar homogeneous form ψ. A homogeneous form $\psi \in S^k(E)$ is apolar to f if and only if the hypersurface $V(\psi)$ is apolar to $V(f)$.

Consider the natural pairing

$$(A_f)_k \times (A_f)_{d-k} \to (A_f)_d \cong \mathbb{C} \tag{1.53}$$

defined by multiplication of polynomials. It is well defined because of Lemma 1.3.2. The left kernel of this pairing consists of $\psi \in S^k(E) \mod \mathrm{AP}(f) \cap S^k(E)$ such that $D_{\psi\psi'}(f) = 0$ for all $\psi' \in S^{d-k}(E)$. By Lemma 1.3.2, $D_{\psi\psi'}(f) = D_{\psi'}(D_\psi(f)) = 0$ for all $\psi' \in S^{d-k}(E)$. This implies $D_\psi(f) = 0$. Thus $\psi \in \mathrm{AP}(f)$ and hence is zero in A_f. This shows that the pairing (1.53)

is a perfect pairing. This is one of the nice features of a Gorenstein Artinian algebra (see [208], 21.2).

It follows that the Hilbert polynomial

$$H_{A_f}(t) = \sum_{i=0}^{d} \dim(A_f)_i t^i = a_d t^d + \cdots + a_0$$

is a reciprocal monic polynomial, i.e. $a_i = a_{d-i}, a_d = 1$. It is an important invariant of a homogeneous form f.

Example 1.3.7 Let $f = l^d$ be the d-th power of a linear form $l \in E^\vee$. For any $\psi \in S^k(E) = (S^k(E)^\vee)^\vee$ we have

$$D_\psi(l^d) = d(d-1)\cdots(d-k+1)l^{d-k}\psi(l) = d!l^{[d-k]}\psi(l),$$

where we set

$$l^{[i]} = \begin{cases} \frac{1}{i!}l^i & \text{if } k \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Here we view $\psi \in S^d(E)$ as a homogeneous form on $E^\vee$. In coordinates, $l = \sum_{i=0}^n a_i t_i, \psi = \psi(\partial_0, \dots, \partial_n)$ and $\psi(l) = d!\psi(a_0, \dots, a_n)$. Thus we see that $AP_k(f), k \le d$, consists of polynomials of degree k vanishing at l. Assume, for simplicity, that $l = t_0$. The ideal $AP(t_0^d)$ is generated by $\partial_1, \dots, \partial_n, \partial_0^{d+1}$. The Hilbert polynomial is equal to $1 + t + \cdots + t^d$.

1.3.2 Sums of powers

For any point $a \in |E^\vee|$ we continue to denote by H_a the corresponding hyperplane in $|E|$.

Suppose $f \in S^d(E^\vee)$ is equal to a sum of powers of nonzero linear forms

$$f = l_1^d + \cdots + l_s^d. \tag{1.54}$$

This implies that for any $\psi \in S^k(E)$,

$$D_\psi(f) = D_\psi(\sum_{i=1}^s l_i^d) = \sum_{i=1}^s \psi(l_i) l_i^{[d-k]}. \tag{1.55}$$

In particular, taking $d = k$, we obtain that

$$\langle l_1^d, \dots, l_s^d \rangle^\perp_{S^d(E)} = \{\psi \in S^d(E) : \psi(l_i) = 0, i = 1, \dots, s\} = (I_Z)_d,$$

where Z is the closed reduced subscheme of points $\{[l_1], \dots, [l_s]\} \subset |E^\vee|$ corresponding to the linear forms l_i, and I_Z denotes its homogeneous ideal.

This implies that the codimension of the linear span $\langle l_1^d, \dots, l_s^d \rangle$ in $S^d(E^\vee)$

is equal to the dimension of $(I_Z)_d$, hence the forms $l_1^d, \ldots, l_s^d$ are linearly independent if and only if the points $[l_1], \ldots, [l_s]$ impose independent conditions on hypersurfaces of degree d in $\mathbb{P}(E) = |E^\vee|$.

Suppose $f \in \langle l_1^d, \ldots, l_s^d \rangle$, then $(I_Z)_d \subset \mathrm{AP}_d(f)$. Conversely, if this is true, we have

$$f \in \mathrm{AP}_d(f)^\perp \subset (I_Z)_d^\perp = \langle l_1^d, \ldots, l_s^d \rangle.$$

If we additionally assume that $(I_{Z'})_d \not\subset \mathrm{AP}_d(f)$ for any proper subset Z' of Z, we obtain, after replacing the forms $l_i's$ by proportional ones, that

$$f = l_1^d + \cdots + l_s^d.$$

Definition 1.3.8 *A* polar s-hedron *of f is a set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, in $|E|$ such that*

$$f = l_1^d + \cdots + l_s^d,$$

and, considered as points $[l_i]$ in $\mathbb{P}(E)$, the hyperplanes H_i impose independent conditions in the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)|$. A polar s-hedron is called nondegenerate *if the hyperplanes $V(l_i)$ are in general linear position (i.e. no $n+1$ hyperplanes intersect).*

Note that this definition does not depend on the choice of linear forms defining the hyperplanes. Also it does not depend on the choice of the equation defining the hypersurface $V(f)$. We ca also view a polar s-hedron as an unordered set of points in the dual space. In the case $n = 2$, it is often called a *polar s-gon*, although this terminology is somewhat confusing since a polygon comes with an order of its set of vertices.. Also in dimension 2 we cam employ the terminology of *s-laterals*.

The following propositions follow from the above discussion.

Proposition 1.3.9 *Let $f \in S^d(E^\vee)$. Then $Z = \{[l_1], \ldots, [l_s]\}$ is a polar s-hedron of f if and only if the following properties are satisfied*

(i) $I_Z(d) \subset \mathrm{AP}_d(f)$*;*

(ii) $I_{Z'}(d) \not\subset \mathrm{AP}_d(f)$ *for any proper subset Z' of Z.*

Proposition 1.3.10 *A set $Z = \{[l_1], \ldots, [l_s]\}$ is a polar s-hedron of $f \in S^d(E^\vee)$ if and only if Z, considered as a closed subscheme of $|E^\vee|$, is apolar to f but no proper subscheme of Z is apolar to f.*

1.3.3 Generalized polar s-hedra

Proposition 1.3.10 allows one to generalize the definition of a polar s-hedron. A polar s-hedron can be viewed as a reduced closed subscheme Z of $\mathbb{P}(E) = |E|^\vee$ consisting of s points. Obviously,

$$h^0(\mathcal{O}_Z) = \dim H^0(\mathbb{P}(E), \mathcal{O}_Z) = s.$$

More generally, we may consider non-reduced closed subschemes Z of $\mathbb{P}(E)$ of dimension 0 satisfying $h^0(\mathcal{O}_Z) = s$. The set of such subschemes is parameterized by a projective algebraic variety $\mathrm{Hilb}^s(\mathbb{P}(E))$ called the *punctual Hilbert scheme* of $\mathbb{P}(E)$ of 0-cycles of length s.

Any $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ defines the subspace

$$I_Z(d) = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))) \subset H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) = S^d(E).$$

The exact sequence

$$0 \to H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_Z) \quad (1.56)$$

$$\to H^1(\mathbb{P}(E), \mathcal{I}_Z(d)) \to 0$$

shows that the dimension of the subspace

$$\langle Z \rangle_d = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))^\perp) \subset |S^d(E^\vee)| \quad (1.57)$$

is equal to $h^0(\mathcal{O}_Z) - h^1(\mathcal{I}_Z(d)) - 1 = s - 1 - h^1(\mathcal{I}_Z(d))$. If Z is reduced and consists of points $p_1, \dots, p_s$, then $\langle Z \rangle_d = \langle v_d(p_1), \dots, v_d(p_s) \rangle$, where $v_d : \mathbb{P}(E) \to \mathbb{P}(S^d(E))$ is the Veronese map. Hence, $\dim \langle Z \rangle_d = s - 1$ if the points $v_d(p_1), \dots, v_d(p_s)$ are linearly independent. We say that Z is *linearly d-independent* if $\dim \langle Z \rangle_d = s - 1$.

Definition 1.3.11 *A* generalized s-hedron *of $f \in S^d(E^\vee)$ is a linearly d-independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ which is apolar to f.*

Recall that Z is apolar to f if, for each $k \geq 0$,

$$I_Z(k) = H^0(\mathbb{P}(E), \mathcal{I}_Z(k)) \subset \mathrm{AP}_k(f). \quad (1.58)$$

According to this definition, a polar s-hedron is a reduced generalized s-hedron. The following is a generalization of Proposition 1.3.9.

Proposition 1.3.12 *A linearly d-independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ is a generalized polar s-hedron of $f \in S^d(E^\vee)$ if and only if*

$$I_Z(d) \subset \mathrm{AP}_d(f).$$

Proof We have to show that the inclusion in the assertion implies $I_Z(d) \subset \mathrm{AP}_k(f)$ for any $k \leq d$. For any $\psi' \in S^{d-k}(E)$ and any $\psi \in I_Z(k)$, the product $\psi\psi'$ belongs to $I_Z(k)$. Thus $D_{\psi\psi'}(f) = 0$. By the duality, $D_\psi(f) = 0$, i.e. $\psi \in \mathrm{AP}_k(f)$. □

Example 1.3.13 Let $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ be the union of k *fat points* p_k, i.e. at each $p_i \in Z$ the ideal $\mathcal{I}_{Z,p_i}$ is equal to the m_i-th power of the maximal ideal. Obviously,

$$s = \sum_{i=1}^{k} \tbinom{n+m_i-1}{m_i-1}.$$

Then the linear system $|\mathcal{I}_Z(d)|$ consists of hypersurfaces of degree d with points p_i of multiplicity $\geq m_i$. One can show (see [313], Theorem 5.3) that Z is apolar to f if and only if

$$f = l_1^{d-m_1+1} g_1 + \cdots + l_k^{d-m_k+1} g_k,$$

where $p_i = V(l_i)$ and g_i is a homogeneous polynomial of degree $m_i - 1$ or the zero polynomial.

Remark 1.3.14 It is not known whether the set of generalized s-hedra of f is a closed subset of $\mathrm{Hilb}^s(\mathbb{P}(E))$. It is known to be true for $s \leq d+1$ since in this case $\dim I_Z(d) = t := \dim S^d(E) - s$ for all $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ (see [313], p.48). This defines a regular map of $\mathrm{Hilb}^s(\mathbb{P}(E))$ to the Grassmannian $G_{t-1}(|S^d(E)|)$ and the set of generalized s-hedra equal to the preimage of a closed subset consisting of subspaces contained in $\mathrm{AP}_d(f)$. Also we see that $h^1(\mathcal{I}_Z(d)) = 0$, hence Z is always linearly d-independent.

1.3.4 Secant varieties and sums of powers

Consider the *Veronese map* of degree d

$$\nu_d : |E| \to |S^d(E)|, \quad [v] \mapsto [v^d],$$

defined by the complete linear system $|S^d E^\vee|$. The image of this map is the *Veronese variety* V_d^n of dimension n and degree d^n. It is isomorphic to $\mathbb{P}^n$. By choosing a monomial basis $\mathbf{t}^{\mathbf{i}}$ in the linear space of homogeneous polynomials of degree d we obtain that the Veronese variety is isomorphic to the subvariety of $\mathbb{P}^{\binom{n+d}{d}-1}$ given by equations

$$A_{\mathbf{i}} \cdot A_{\mathbf{j}} - A_{\mathbf{k}} A_{\mathbf{m}} = 0, \quad \mathbf{i}+\mathbf{j} = \mathbf{k}+\mathbf{m},$$

where $A_{\mathbf{i}}$ are dual coordinates in the space of polynomials of degree d. The image of $\mathbb{P}^n$ under the map defined by a choice of a basis of the complete

linear system of hypersurfaces of degree d is called a n-dimensional Veronese variety of degree d^n.

One can combine the Veronese mapping and the Segre mapping to define a *Segre-Veronese variety* $\mathsf{V}_{n_1,\dots,n_k}(d_1,\dots,d_k)$. It is equal to the image of the map $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ defined by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1}}(d_1)\boxtimes\cdots\boxtimes\mathcal{O}_{\mathbb{P}^{n_k}}(d_k)|$.

The notion of a polar s-hedron acquires a simple geometric interpretation in terms of the secant varieties of the Veronese variety V_d^n. If a set of points $[l_1],\dots,[l_s]$ in $|E|$ is a polar s-hedron of f, then $[f]\in\langle[l_1^d],\dots,[l_s^d]\rangle$, and hence $[f]$ belongs to the $(s-1)$-secant subspace of V_d^n. Conversely, a general point in this subspace admits a polar s-hedron. Recall that for any irreducible nondegenerate projective variety $X\subset\mathbb{P}^N$ of dimension r its *t-secant variety* $\mathrm{Sec}_t(X)$ is defined to be the Zariski closure of the set of points in $\mathbb{P}^N$ which lie in the linear span of dimension t of some set of $t+1$ linearly independent points in X.

The counting constants easily gives

$$\dim\mathrm{Sec}_t(X)\le\min(r(t+1)+t,N).$$

The subvariety $X\subset\mathbb{P}^N$ is called *t-defective* if the inequality is strict. An example of a 1-defective variety is a Veronese surface in $\mathbb{P}^5$.

A fundamental result about secant varieties is the following Lemma whose modern proof can be found in [607], Chapter II, and in [150]

Lemma 1.3.15 (A. Terracini) *Let $p_1,\dots,p_{t+1}$ be general $t+1$ points in X and p be a general point in their span. Then*

$$\mathbb{T}_p(\mathrm{Sec}_t(X))=\overline{\mathbb{T}_{p_1}(X),\dots,\mathbb{T}_{p_{t+1}}(X)}.$$

The inclusion part

$$\overline{\mathbb{T}_{p_1}(X),\dots,\mathbb{T}_{p_{t+1}}(X)}\subset\mathbb{T}_p(\mathrm{Sec}_t(X))$$

is easy to prove. We assume for simplicity that $t=1$. Then $\mathrm{Sec}_1(X)$ contains the cone $C(p_1,X)$ which is swept out by the lines $\overline{p_1q},q\in X$. Therefore, $\mathbb{T}_p(C(p_1,X))\subset\mathbb{T}_p(\mathrm{Sec}_1(X))$. However, it is easy to see that $\mathbb{T}_p(C(p_1,X))$ contains $\mathbb{T}_{p_1}(X)$.

Corollary 1.3.16 $\mathrm{Sec}_t(X)\ne\mathbb{P}^N$ *if and only if, for any $t+1$ general points of X, there exists a hyperplane section of X singular at these points. In particular, if $N\le r(t+1)+t$, the variety X is t-defective if and only if for any $t+1$ general points of X there exists a hyperplane section of X singular at these points.*

Example 1.3.17 Let $X = \mathsf{V}_d^n \subset \mathbb{P}^{\binom{d+n}{n}-1}$ be a Veronese variety. Assume $n(t+1)+t > \binom{d+n}{n} - 1$. A hyperplane section of X is isomorphic to a hypersurface of degree d in $\mathbb{P}^n$. Thus $\mathrm{Sec}_t(\mathsf{V}_d^n) \neq |S^d(E^\vee)|$ if and only if, for any $t+1$ general points in $\mathbb{P}^n$, there exists a hypersurface of degree d singular at these points.

Consider a Veronese curve $\mathsf{V}_d^1 \subset \mathbb{P}^d$. Assume $2t+1 \geq d$. Since $d < 2t+2$, there are no homogeneous forms of degree d which have $t+1$ multiple roots. Thus the Veronese curve $R_d = v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is not t-degenerate for $t \geq (d-1)/2$.

Take $n = 2$ and $d = 2$. For any two points in $\mathbb{P}^2$ there exists a conic singular at these points, namely the double line through the points. This explains why a Veronese surface V_2^2 is 1-defective.

Another example is $\mathsf{V}_4^2 \subset \mathbb{P}^{14}$ and $t = 4$. The expected dimension of $\mathrm{Sec}_4(X)$ is equal to 14. For any 5 points in $\mathbb{P}^2$ there exists a conic passing through these points. Taking it with multiplicity 2, we obtain a quartic which is singular at these points. This shows that V_4^2 is 4-defective.

The following Corollary of Terracini's Lemma is called the *First Main Theorem on apolarity* in [206]. The authors gave an algebraic proof of this Theorem without using Terracini's Lemma.

Corollary 1.3.18 *A general homogeneous form in $S^d(E^\vee)$ admits a polar s-hedron if and only if there exist linear forms $l_1, \ldots, l_s \in E^\vee$ such that, for any nonzero $\psi \in S^d(E)$, the ideal $AP(\psi) \subset S(E^\vee)$ does not contain $\{l_1^{d-1}, \ldots, l_s^{d-1}\}$.*

Proof A general form in $S^d(E^\vee)$ admits a polar s-hedron if and only if the secant variety $\mathrm{Sec}_{s-1}(\mathsf{V}_d^n)$ is equal to the whole space. This means that the span of the tangent spaces at some points $q_i = V(l_i^d), i = 1, \ldots, s$, is equal to the whole space. By Terracini's Lemma, this is equivalent to that the tangent spaces of the Veronese variety at the points q_i are not contained in a hyperplane defined by some $\psi \in S^d(E) = S^d(E^\vee)^\vee$. It remains for us to use that the tangent space of the Veronese variety at q_i is equal to the projective space of all homogeneous forms $l_i^{d-1}l, l \in E^\vee \setminus \{0\}$ (see Exercise 1.18). Thus, for any nonzero $\psi \in S^d(E)$, it is impossible that $P_{l_i^{d-1}l}(\psi) = 0$ for all l and for all i. But $P_{l_i^{d-1}l}(\psi) = 0$ for all l if and only if $P_{l_i^{d-1}}(\psi) = 0$. This proves the assertion. □

The following fundamental result is due to J. Alexander and A. Hirschowitz [5]. A simplified proof can be found in [53] or [91].

Theorem 1.3.19 *If $d > 2$, the Veronese variety V_d^n is t-defective if and only if*

$$(n, d, t) = (2, 4, 4), (3, 4, 8), (4, 3, 6), (4, 4, 13).$$

In all these cases the secant variety $\mathrm{Sec}_t(\mathsf{V}_d^n)$ is a hypersurface. The Veronese variety V_2^n is t-defective only if $1 \leq t \leq n$. Its t-secant variety is of dimension $n(t+1) - \frac{1}{2}(t-2)(t+1) - 1$.

For the sufficiency of the condition, only the case $(4, 3, 6)$ is not trivial. It asserts that for 7 general points in $\mathbb{P}^3$ there exists a cubic hypersurface which is singular at these points. To see this, we use a well-known fact that any $n+3$ general points in $\mathbb{P}^n$ lie on a Veronese curve of degree n (see, for example, [278], Theorem 1.18). So, we find such a curve R through 7 general points in $\mathbb{P}^4$ and consider the 1-secant variety $\mathrm{Sec}_1(R)$. It is a cubic hypersurface given by the catalecticant invariant of a binary quartic form. It contains the curve R as it singular locus.

Other cases are easy. We have seen already the first two cases. The third case follows from the existence of a quadric through nine general points in $\mathbb{P}^3$. The square of its equation defines a quartic with 9 points. The last case is similar. For any 14 general points there exists a quadric in $\mathbb{P}^4$ containing these points. In the case of quadrics we use that the variety of quadrics of corank r is of codimension $r(r+1)/2$ in the variety of all quadrics.

Obviously, if $\dim \mathrm{Sec}_{s-1}(\mathsf{V}_d^n) < \dim |S^d E^\vee)| = \binom{n+d}{n} - 1$, a general form $f \in S^d(E^\vee)$ cannot be written as a sum of s powers of linear forms. Since $\dim \mathrm{Sec}_{s-1}(\mathsf{V}_d^n) \leq \min\{(n+1)s - 1, \binom{n+d}{n} - 1\}$, the minimal number $s(n, d)$ of powers needed to write f as a sum of powers of linear forms satisfies

$$s(n, d) \geq \left\lceil \frac{1}{n+1} \binom{n+d}{n} \right\rceil. \tag{1.59}$$

If V_d^n is not $(s-1)$-defective, then the equality holds. Applying Theorem 1.3.19, we obtain the following.

Corollary 1.3.20

$$s(n, d) = \left\lceil \frac{1}{n+1} \binom{n+d}{n} \right\rceil$$

unless $(n, d) = (n, 2), (2, 4), (3, 4), (4, 3), (4, 4)$. In these exceptional cases $s(n, d) = n+1, 6, 10, 8, 15$ instead of expected $\lceil \frac{n-1}{2} \rceil, 5, 9, 8, 14$.

Remark 1.3.21 If $d > 2$, in all the exceptional cases listed in the previous corollary, $s(n, d)$ is larger by one than the expected number. The variety of forms of degree d that can be written as the sum of the expected number of

powers of linear forms is a hypersurface in $|\mathcal{O}_{\mathbb{P}^n}(d)|$. In the case $(n,d,t) = (2,4,5)$, the hypersurface is of degree 6 and is given by the catalecticant matrix which we will discuss later in this chapter. The curves parameterized by this hypersurface are Clebsch quartics which we will discuss in Chapter 6. The case $(n,d) = (4,3)$ was studied only recently in [421]. The hypersurface is of degree 15. In the other two cases, the equation expresses that the second partials of the quartic are linearly dependent (see [240], pp. 58-59.)

One can also consider the problem of a representation of several forms $f_1,\dots,f_k \in S^d(E^\vee)$ as a sum of powers of the same set (up to proportionality) of linear forms $l_1,\dots,l_s$. This means that the forms share a common polar s-hedron. For example, a well-known result from linear algebra states that two general quadratic forms q_1,q_2 in k variables can be simultaneously diagonalized. In our terminology this means that they have a common polar k-hedron. More precisely, this is possible if the $\det(q_1+\lambda q_2)$ has $n+1$ distinct roots (we will discuss this later in Chapter 8 while studying del Pezzo surfaces of degree 4).

Suppose

$$f_j = \sum_{i=1}^{s} a_i^{(j)} l_i^d, \quad j = 1,\dots,k. \tag{1.60}$$

We view this as an element $\phi \in U^\vee \otimes S^d(E^\vee)$, where $U = \mathbb{C}^k$. The map ϕ is the sum of s linear maps ϕ of rank 1 with the images spanned by l_i^d. So, we can view each ϕ as a vector in $U^\vee \otimes S^d(E^\vee)$ equal to the image of a vector in $U^\vee \otimes E^\vee$ embedded in $U^\vee \otimes E^\vee$ by $u\otimes l \mapsto u \otimes l^d$. Now, everything becomes clear. We consider the Segre-Veronese embedding

$$|U^\vee| \times |E^\vee| \hookrightarrow |U^\vee| \times |S^d(E^\vee)| \hookrightarrow |U^\vee \otimes S^d(E^\vee)|$$

defined by the linear system of divisors of type $(1,d)$ and view $[\phi]$ as a point in the projective space $|U^\vee \otimes S^d(E^\vee)|$ which lies on the $(s-1)$-secant variety of $\mathsf{V}_{k-1,n}(1,d)$.

For any linear map $\phi \in \mathrm{Hom}(U, S^d(E^\vee))$, consider the linear map

$$\mathfrak{T}_\phi : \mathrm{Hom}(U,E) \to \mathrm{Hom}(\bigwedge^2 U, S^{d-1}(E^\vee)),$$

defined by

$$\mathfrak{T}_\phi(\alpha) : u \wedge v \mapsto D_{\alpha(u)}(\phi(v)) - D_{\alpha(v)}(\phi(u)).$$

We call this map the *Toeplitz map*. Suppose that ϕ is of rank 1 with the image spanned by l^d, then $\mathfrak{T}_\phi$ is of rank equal to $\dim \bigwedge^2 U - 1 = (k-2)(k+1)/2$. If we choose a basis $u_1,\dots,u_k$ in U and coordinates $t_0,\dots,t_n$ in E, then the

image is spanned by $l^{d-1}(a_iu_i - a_ju_j)$, where $l = \sum a_it_i$. This shows that, if ϕ belongs to $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$,

$$\text{rank } \mathfrak{T}_\phi \leq s(k-2)(k+1)/2. \tag{1.61}$$

The expected dimension of $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$ is equal to $s(k+n)-1$. Thus, we expect that $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$ coincides with $|U^\vee \otimes S^d(E^\vee)|$ when

$$s \geq \left\lceil \frac{k}{k+n}\binom{n+d}{n} \right\rceil. \tag{1.62}$$

If this happens, we obtain that a general set of k forms admits a common polar s-hedron. Of course, as in the case $k = 1$, there could be exceptions if the secant variety is $(s-1)$-defective.

Example 1.3.22 Assume $d = 2$ and $k = 3$. In this case the matrix of $\mathfrak{T}_\phi$ is a square matrix of size $3(n+1)$. Let us identify the spaces $U^\vee$ and $\bigwedge^2 U$ by means of the volume form $u_1 \wedge u_2 \wedge u_3 \in \bigwedge^3 U \cong \mathbb{C}$. Also identify $\phi(u_i) \in S^2(E^\vee)$ with a square symmetric matrix A_i of size $n+1$. Then, an easy computation shows that one can represent the linear map $\mathfrak{T}_\phi$ by the skew-symmetric matrix

$$\begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix}. \tag{1.63}$$

Now condition (1.61) for

$$s = \left\lceil \frac{k\binom{n+d}{n}}{k+n} \right\rceil = \left\lceil \frac{3(n+2)(n+1)}{2(n+3)} \right\rceil = \begin{cases} \frac{1}{2}(3n+2) & \text{if } n \text{ is even,} \\ \frac{1}{2}(3n+1) & \text{if } n \text{ is odd} \geq 3, \\ 3 & \text{if } n = 1 \end{cases}$$

becomes equivalent to the condition

$$\Lambda = \mathrm{Pf}\begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix} = 0. \tag{1.64}$$

It is known that the secant variety $\mathrm{Sec}_{s-1}(|U| \times |E|)$ of the Segre-Veronese variety is a hypersurface if $n \geq 3$ is odd and the whole space if n is even (see [548], Lemma 4.4). It implies that, in the odd case, the hypersurface is equal to $V(\Lambda)$. Its degree is equal to $3(n+1)/2$. Of course, in the even case, the pfaffian vanishes identically.

In the case $n = 3$, the pfaffian Λ was introduced by E. Toeplitz [564]. It

is an invariant of the net[2] of quadrics in $\mathbb{P}^3$ that vanishes on the nets with common polar pentahedron. Following [247], we call Λ the *Toeplitz invariant*. Let us write its generators f_1, f_2, f_3 in the form (1.60) with $n = 3$ and $s = \frac{1}{2}(3n+1) = 5$. Since the four linear forms l_i are linearly dependent, we can normalize them by assuming that $l_1 + \cdots + l_5 = 0$ and assume that $l_1, \ldots, l_5$ span a 4-dimensional subspace. Consider a cubic form

$$F = \frac{1}{3}\sum_{i=1}^{5} l_i^3,$$

and find three vectors v_i in $\mathbb{C}^4$ such that

$$(l_1(v_j), \ldots, l_5(v_j)) = (a_1^{(j)}, \ldots, a_5^{(j)}), \quad j = 1, 2, 3.$$

Now we check that $f_j = D_{v_j}(F)$ for $j = 1, 2, 3$. This shows that the net spanned by f_1, f_2, f_3 is a net of polar quadrics of the cubic F. Conversely, we will see later that any general cubic form in 4 variables admits a polar pentahedron. Thus any net of polars of a general cubic surface admits a common polar pentahedron. So, the Toeplitz invariant vanishes on a general net of quadrics in $\mathbb{P}^3$ if and only if the net is realized as a net of polar quadrics of a cubic.

Remark 1.3.23 Let (n, d, k, s) denote the numbers such that we have the strict inequality in (1.62). We call such 4-tuples exceptional. Examples of exceptional 4-tuples are $(n, 2, 3, \frac{1}{2}(3n+1))$ with odd $n \geq 2$. The secant hypersurfaces in these cases are given by the Toeplitz invariant Λ. The case $(3, 2, 3, 5)$ was first discovered by G. Darboux [153].[3] It has been rediscovered and extended to any odd n by G. Ottaviani [420]. There are other two known examples. The case $(2, 3, 2, 5)$ was discovered by F. London [366]. The secant variety is a hypersurface given by the determinant of order 6 of the linear map $\mathfrak{T}_\phi$ (see Exercise 1.30). The examples $(3, 2, 5, 6)$ and $(5, 2, 3, 8)$ were discovered recently by E. Carlini and J. Chipalkatti [64]. The secant hypersurface in the second case is a hypersurface of degree 18 given by the determinant of $\mathfrak{T}_\phi$. There are no exceptional 4-tuples $(n, 2, 2, s)$ [64] and no exceptional 4-tuples (n, d, k, s) for large n (with some explicit bound)[1]. We refer to [96], where the varieties of common polar s-hedra are studied.

Remark 1.3.24 Assume that one of the matrices A_1, A_2, A_3 in (1.63) is in-

[2] We employ classical terminology calling a 1-dimensional (resp. 2-dimensional, resp. 3-dimensional) linear system a *pencil* (resp. a *net*, resp. a *web*).

[3] Darboux also wrongly claimed that the case $(3, 2, 4, 6)$ is exceptional, the mistake was pointed out by Terracini [559] without proof, a proof is in [64].

vertible, say let it be A_2. Then

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & -A_1A_2^{-1} \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -A_2^{-1}A_1 & I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & A_2 \\ 0 & B & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix},$$

where

$$B = A_1A_2^{-1}A_3 - A_3A_2^{-1}A_1.$$

This shows that

$$\text{rank} \begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix} = \text{rank } B + 2n + 2.$$

The condition that rank$B \leq 2$ is known in the theory of vector bundles over the projective plane as *Barth's condition* on the net of quadrics in $\mathbb{P}^n$. It does not depend on the choice of a basis of the net of quadrics spanned by the quadrics with matrices A_1, A_2, A_3. Under Barth's condition, the discriminant curve $\det(z_0A_1+z_1A_2+2A_3) = 0$ of singular quadrics in the net is a *Darboux curve* of degree $n+1$ (see [24]).

1.3.5 The Waring problems

The well-known Waring problem in number theory asks about the smallest number $s(d)$ such that each natural number can be written as a sum of $s(d)$ d-th powers of natural numbers. It also asks in how many ways it can be done. Its polynomial analog asks about the smallest number $s(n,d)$ such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as a sum of s d-th powers of linear forms. Corollary (1.3.20) solves this problem.

Other versions of the Waring problem ask the following questions:

- (W1) Given a homogeneous forms $f \in S^d(E^\vee)$, study the *variety of sums of powers* $\text{VSP}(f,s)^o$, i.e. the subvariety of $\mathbb{P}(E)^{(s)}$ that consists of polar s-hedra of f or, more general, the subvariety $\text{VSP}(f,s)$ of $\text{Hilb}^s(\mathbb{P}(E))$ parameterizing generalized polar s-hedra of f.
- (W2) Given s, find the equations of the closure $\text{PS}(s,d;n)$ in $S^d(E^\vee)$ of the locus of homogeneous forms of degree d which can be written as a sum of s powers of linear forms.

We can also ask similar questions for several forms in $S^d(E^\vee)$.

Note that $\mathrm{PS}(s,d;n)$ is the affine cone over the secant variety $\mathrm{Sec}_{s-1}(\mathsf{V}_d^n)$. In the language of secant varieties, the variety $\mathrm{VSP}(f,s)^o$ is the set of linearly independent sets of s points $p_1,\dots,p_s$ in V_d^n such that $[f]\in\langle p_1,\dots,p_s\rangle$ and does not belong to the span of the proper subset of the set of these points. The variety $\mathrm{VSP}(f,s)$ is the set of linearly independent $Z\in \mathrm{Hilb}^s(\mathbb{P}(E))$ such that $[f]\in\langle Z\rangle$. Note that we have a natural map

$$\mathrm{VSP}(f,s)\to G(s,S^d(E)),\quad Z\mapsto \langle Z\rangle_d,$$

where $G(s,S^d(E))=G_{s-1}(|S^d(E)|)$ is the Grassmannian of s-dimensional subspaces of $S^d(E)$. This map is not injective in general.

Also note that for a general form f, the variety $\mathrm{VSP}(f,s)$ is equal to the closure of $\mathrm{VSP}(f,s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E))$ (see [313], 7.2). It is not true for an arbitrary form f. One can also embed $\mathrm{VSP}(f;s)^o$ in $\mathbb{P}(S^d(E))$ by assigning to $\{l_1,\dots,l_s\}$ the product $l_1\cdots l_s$. Thus we can compactify $\mathrm{VSP}(f,s)^o$ by taking its closure in $\mathbb{P}(S^d(E))$. In general, this closure is not isomorphic to $\mathrm{VSP}(f,s)$.

Remark 1.3.25 If (d,n) is not one of the exceptional cases from Corollary 1.3.20 and $\binom{n+d}{d}=(n+1)s$ for some integer s, then a general form of degree d admits only finitely many polar s-hedra. How many? The known cases are given in the following table.

d	n	s	#	reference
2s-1	1	s	1	J. Sylvester [553]
5	2	7	1	D. Hilbert [294], H. Richmond [468], F. Palatini [426]
7	2	12	5	A. Dixon and T. Stuart[170], K. Ranestad and F.-O. Schreyer [454]
8	2	15	16	K. Ranestad and F.-O. Schreyer [454]
3	3	5	1	J. Sylvester [553]

It seems that among these are the only cases when the number of polar s-hedra of a general form is equal to 1. The evidence for this can be found in papers by M. Mella [380], [381], where it is proven that there are no new cases when $n=2, d\geq 5$ and $n\geq 3$ and n divides $\binom{n+d}{n-1}$.

An explicit description of positive-dimensional varieties of sums of powers $\mathrm{VSP}(f,s)$ is known only in a few cases (d,n,s). We will discuss the cases $(d,n,s)=(2s-1,1,s),(3,3,5)$ later. For other cases see papers [315]

$((d,n,s)=(3,5,10))$, [400], $((d,n,s)=(6,2,10))$, [177] $((d,n,s)=(3,2,4))$ and [454] $((d,n,s)=(3,4,8),(2,3,4),(6,2,10))$,

1.4 Dual homogeneous forms

1.4.1 Catalecticant matrices

Let $f\in S^d(E^\vee)$. Consider the linear map (the *apolarity map*)

$$\mathrm{ap}_f^k : S^kE\to S^{d-k}(E^\vee),\quad \psi\mapsto D_\psi(f). \tag{1.65}$$

Its kernel is the space $\mathrm{AP}_k(f)$ of forms of degree k which are apolar to f.

Assume that $f=\sum_{i=1}^s l_i^d$ for some $l_i\in E^\vee$. It follows from (1.55) that

$$\mathrm{ap}_f^k(S^k(E))\subset\langle l_1^{d-k},\dots,l_s^{d-k}\rangle,$$

and hence

$$\mathrm{rank}(\mathrm{ap}_f^k)\le s. \tag{1.66}$$

If we choose a basis in E and a basis in $E^\vee$, then ap_f^k is given by a matrix of size $\binom{k+n}{k}\times\binom{n+d-k}{d-k}$ whose entries are linear forms in coefficients of f.

Choose a basis $\xi_0,\dots,\xi_n$ in E and the dual basis $t_0,\dots,t_n$ in $E^\vee$. Consider a monomial lexicographically ordered basis in $S^k(E)$ (resp. in $S^{d-k}(E^\vee)$). The matrix of ap_f^k with respect to these bases is called the k-th *catalecticant matrix* of f and is denoted by $\mathrm{Cat}_k(f)$. Its entries $c_{\mathbf{uv}}$ are parameterized by pairs $(\mathbf{u},\mathbf{v})\in\mathbb{N}^{n+1}\times\mathbb{N}^{n+1}$ with $|\mathbf{u}|=d-k$ and $|\mathbf{v}|=k$. If we write

$$f=\sum_{|\mathbf{i}|=d}\binom{d}{\mathbf{i}}a_{\mathbf{i}}\mathbf{t}^{\mathbf{i}},$$

then

$$c_{\mathbf{uv}}=a_{\mathbf{u}+\mathbf{v}}.$$

This follows easily from formula (1.5).

Considering $a_{\mathbf{i}}$ as independent variables $t_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(d,n)$.

Example 1.4.1 Let $n=1$. Write $f=\sum_{i=0}^d\binom{d}{i}a_it_0^{d-i}t_1^i$. Then

$$\mathrm{Cat}_k(f)=\begin{pmatrix}a_0 & a_1 & \dots & a_k\\ a_1 & a_2 & \dots & a_{k+1}\\ \vdots & \vdots & \vdots & \vdots\\ a_{d-k} & a_{d-k+1} & \dots & a_d\end{pmatrix}.$$

A square matrix of this type is called a *circulant matrix*, or a *Hankel matrix*. It follows from (1.66) that $f \in \mathrm{PS}(s,d;1)$ implies that all $(s+1)\times(s+1)$ minors of $\mathrm{Cat}_k(f)$ are equal to zero. Thus we obtain that $\mathrm{Sec}_{s-1}(\mathsf{V}_d^1)$ is contained in the subvariety of $\mathbb{P}^d$ defined by $(s+1)\times(s+1)$-minors of the matrices

$$\mathrm{Cat}_k(d,1) = \begin{pmatrix} t_0 & t_1 & \dots & t_k \\ t_1 & t_2 & \dots & t_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ t_{d-k} & t_{d-k+1} & \dots & t_d \end{pmatrix}.$$

For example, if $s=1$, we obtain that the Veronese curve $\mathsf{V}_d^1 \subset \mathbb{P}^d$ satisfies the equations $t_it_j - t_kt_l = 0$, where $i+j = k+l$. It is well-known that these equations generate the homogeneous ideal of the Veronese curve (see, for example,[278]).

Assume $d=2k$. Then the Hankel matrix is a square matrix of size $k+1$. Its determinant vanishes if and only if f admits a nonzero apolar form of degree k. The set of such f's is a hypersurface in the space of binary forms of degree $2k$. It contains the Zariski open subset of forms which can be written as a sum of k powers of linear forms (see section 1.5.1).

For example, take $k=2$. Then the equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0 \tag{1.67}$$

describes binary quartics

$$f = a_0t_0^4 + 4a_1t_0^3t_1 + 6a_2t_0^2t_1^2 + 4a_3t_0t_1^3 + a_4t_1^4$$

which lie in the Zariski closure of the locus of quartics represented in the form $(\alpha_0t_0+\beta_0t_1)^4 + (\alpha_1t_0+\beta_1t_1)^4$. Note that a quartic of this form has simple roots unless it has a root of multiplicity 4. Thus any binary quartic with simple roots satisfying Equation (1.67) can be represented as a sum of two powers of linear forms.

The determinant (1.67) is an invariant of a binary quartic. The cubic hypersurface in $\mathbb{P}^4$ defined by Equation (1.67) is equal to the 1-secant variety of a rational normal curve R_4 in $\mathbb{P}^4$.

Note that

$$\dim \mathrm{AP}_i(f) = \dim \mathrm{Ker}(\mathrm{ap}_f^i) = \tbinom{n+i}{i} - \mathrm{rank}\ \mathrm{Cat}_i(f).$$

Therefore,

$$\dim(A_f)_i = \mathrm{rank}\ \mathrm{Cat}_i(f),$$

and

$$H_{A_f}(t) = \sum_{i=0}^{d} \text{rank Cat}_i(f) t^i. \tag{1.68}$$

Since the ranks of ap_f^i and its transpose are the same, we obtain

$$\text{rank Cat}_i(f) = \text{rank Cat}_{d-i}(f)$$

confirming that $H_{A_f}(t)$ is a reciprocal monic polynomial.

Suppose $d = 2k$ is even. Then the coefficient at t^k in $H_{A_f}(t)$ is equal to the rank of $\text{Cat}_k(f)$. The matrix $\text{Cat}_k(f)$ is a square matrix of size $\binom{n+k}{k}$. One can show that for a general f, this matrix is invertible. A polynomial f is called *degenerate* if $\det(\text{Cat}_k(f)) = 0$. It is called *nondegenerate* otherwise. Thus, the set of degenerate polynomials is a hypersurface (*catalecticant hypersurface*) given by the equation

$$\det(\text{Cat}_k(2k, n)) = 0. \tag{1.69}$$

The polynomial $\det(\text{Cat}_k(2k, n))$ in variables $t_{\mathbf{i}}$, $|\mathbf{i}| = d$, is called the *catalecticant determinant*.

Example 1.4.2 Let $d = 2$. It is easy to see that the catalecticant polynomial is the discriminant polynomial. Thus a quadratic form is degenerate if and only if it is degenerate in the usual sense. The Hilbert polynomial of a quadratic form f is

$$H_{A_f}(t) = 1 + rt + t^2,$$

where r is the rank of the quadratic form.

Example 1.4.3 Suppose $f = t_0^d + \cdots + t_s^d$, $s \le n$. Then $t_0^i, \ldots, t_s^i$ are linearly independent for any i, and hence $\text{rank Cat}_i(f) = s$ for $0 < i < d$. This shows that

$$H_{A_f}(t) = 1 + s(t + \cdots + t^{d-1}) + t^d.$$

Let $\mathcal{P}$ be the set of reciprocal monic polynomials of degree d. One can stratify the space $S^d(E^\vee)$ by setting, for any $p \in \mathcal{P}$,

$$S^d(E^\vee)_p = \{f \in S^d(E^\vee) : H_{A_f} = p\}.$$

If $f \in \text{PS}(s, d; n)$ we know that

$$\text{rank Cat}_k(f) \le h(s, d, n)_k = \min(s, \tbinom{n+k}{n}, \tbinom{n+d-k}{n}).$$

One can show that, for a general enough f, the equality holds (see [313], p.13). Thus there is a Zariski open subset of $\text{PS}(s, d; n)$ which is contained in the strata $S^d(E^\vee)_p$, where $p = \sum_{i=0}^d h(s, d, n)_i t^i$.

1.4.2 Dual homogeneous forms

In this Chapter we have introduced the notion of a dual quadric. If $Q = V(q)$, where q is a nondegenerate quadratic form, then the dual variety $Q^\vee$ is a quadric defined by the quadratic form $q^\vee$ whose matrix is the adjugate matrix of q. Using the notion of the catalecticant matrix, for any homogeneous form of even degree $f \in S^{2k}(E^\vee)$, in a similar fashion one can define the dual homogeneous form $f^\vee \in S^{2k}(E)$.

Consider the pairing

$$\Omega_f : S^k(E) \times S^k(E) \to \mathbb{C}, \tag{1.70}$$

defined by

$$\Omega_f(\psi_1, \psi_2) = \mathrm{ap}_f^k(\psi_1)(\psi_2) = D_{\psi_2}(\mathrm{ap}_f^k(\psi_1)) = D_{\psi_1\psi_2}(f),$$

where we identify the spaces $S^k(E^\vee)$ and $S^k(E)^\vee$. The pairing can be considered as a symmetric bilinear form on $S^k(E)$. Its matrix with respect to a monomial basis in $S^k(E)$ and its dual monomial basis in $S^k(E^\vee)$ is the catalecticant matrix $\mathrm{Cat}_k(f)$.

Let us identify Ω_f with the associated quadratic form on $S^k(E)$ (the restriction of Ω_f to the diagonal). This defines a linear map

$$\Omega : S^{2k}(E^\vee) \to S^2(S^k(E)^\vee), \quad f \mapsto \Omega_f.$$

There is also a natural left inverse map of Ω

$$P : S^2(S^k(E)^\vee) \to S^{2k}(E^\vee)$$

defined by multiplication $S^k(E^\vee) \times S^k(E^\vee) \to S^{2k}(E^\vee)$. All these maps are $\mathrm{GL}(E)$-equivariant and realize the linear representation $S^{2k}(E^\vee)$ as a direct summand in the representation $S^2(S^k(E^\vee))$.

Definition 1.4.4 *Assume that $f \in S^{2k}(E^\vee)$ is nondegenerate. The dual quadratic form $\Omega_f^\vee$ of Ω_f is called the* dual homogeneous form *of f. We will identify it with the polar bilinest form on $S^k V$.*

Remark 1.4.5 Note that, contrary to the assertion of Theorem 2.3 in [183], $\Omega_f^\vee$ is not equal, in general, to $\Omega_{f^\vee}$ for some $f^\vee \in S^{2k}(V)$. We thank Bart van den Dries for pointing out that the adjugate matrix of the catelecticant matrix is not, in general, a catalecticant matrix as was wrongly asserted in the proof.

Recall that the locus of zeros of a quadratic from $q \in S^2(E^\vee)$ consists of vectors $v \in E$ such that the value of the polarized bilinear form $b_q : E \to E^\vee$ at v vanishes at v. Dually, the set of zeros of $q^\vee \in S^2(E)$ consists of linear functions $l \in E^\vee$ such that the value of $b_{q^\vee} : E^\vee \to E$ at l is equal to zero. The

same is true for the dual form $\Omega_f^\vee$. Its locus of zeros consists of linear forms l such that $\Omega_f^{-1}(l^k) \in S^k(E)$ vanishes on l. The degree k homogeneous form $\Omega_f^{-1}(l^k)$ is classically known as the *anti-polar* of l (with respect to f).

Definition 1.4.6 *Two linear forms $l, m \in E^\vee$ are called* conjugate *with respect to a nondegenerate form $f \in S^{2k}(E^\vee)$ if*

$$\Omega_f^\vee(l^k, m^k) = 0.$$

Proposition 1.4.7 *Suppose f is given by (1.54), where the powers l_i^k are linearly independent in $S^k(E^\vee)$. Then each pair l_i, l_j is conjugate with respect to f.*

Proof Since the powers l_i^k are linearly independent, we may include them in a basis of $S^k(E^\vee)$. Choose the dual basis in $S^k(E)$. Then the catalecticant matrix of f has the upper corner matrix of size s equal to the diagonal matrix. Its adjugate matrix has the same property. This implies that $l_i^k, l_i^j, i \neq j$, are conjugate with respect to $\Omega_f^\vee$. □

1.4.3 The Waring rank of a homogeneous form

Since any quadratic form q can be reduced to a sum of squares, one can define its rank as the smallest number r such that

$$q = l_1^2 + \cdots + l_r^2$$

for some $l_1, \ldots, l_r \in E^\vee$.

Definition 1.4.8 *Let $f \in S^d E^\vee$. Its* Waring rank $\mathrm{wrk}(f)$ *is the smallest number r such that*

$$f = l_1^d + \cdots + l_r^d \tag{1.71}$$

for some linear forms $l_1, \ldots, l_r \in E^\vee$.

The next result follows immediately from the proof of Proposition 1.4.7.

Proposition 1.4.9 *Let Ω_f be the quadratic form on $S^k(E)$ associated to $f \in S^{2k}(E^\vee)$. Then the Waring rank of f is greater than or equal to the rank of Ω_f.*

Let f be a nondegenerate form of even degree $2k$. By Corollary 1.3.20,

$$\mathrm{wrk}(f) = s(2k, n) \geq \left\lceil \frac{1}{n+1} \binom{n+d}{d} \right\rceil,$$

with strict inequality only in the following cases

- $d = 2, \mathrm{wrk}(f) = \text{rank } \Omega_f = n + 1$;
- $n = 2, d = 4, \mathrm{wrk}(f) = \text{rank } \Omega_f = 6$;
- $n = 3, d = 4, \mathrm{wrk}(f) = \text{rank } \Omega_f = 10$;
- $n = 4, d = 4, \mathrm{wrk}(f) = \text{rank } \Omega_f = 15$.

In all non-exceptional cases,

$$\mathrm{wrk}(f) \geq \frac{1}{n+1}\binom{n+2k}{n} = \binom{n+k}{n}\frac{(n+2k)\cdots(n+k)}{2k\cdots(k+1)(n+1)} \geq \text{rank } \Omega_f.$$

In most cases, we have strict inequality.

1.4.4 Mukai's skew-symmetric form

Let $\omega \in \bigwedge^2 E$ be a skew-symmetric bilinear form on $E^\vee$. It admits a unique extension to a Poisson bracket $\{,\}_\omega$ on $S(E^\vee)$ which restricts to a skew-symmetric bilinear form

$$\{,\}_\omega : S^{k+1}(E^\vee) \times S^{k+1}(E^\vee) \to S^{2k}(E^\vee). \tag{1.72}$$

Recall that a *Poisson bracket* on a commutative algebra A is a skew-symmetric bilinear map $A \times A \to A, (a, b) \mapsto \{a, b\}$ such that its left and right partial maps $A \to A$ are derivations.

Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form and $\Omega_f^\vee \in S^2(S^k(E))$ be its dual form. For each ω as above, define $\sigma_{\omega,f} \in \bigwedge^2 S^{k+1}(E)$ by

$$\sigma_{\omega,f}(g, h) = \Omega_f^\vee(\{g, h\}_\omega).$$

Theorem 1.4.10 *Let f be a nondegenerate form in $S^{2k}(E^\vee)$ of Waring rank $N = \text{rank } \Omega_f = \binom{n+k}{n}$. For any $Z = \{[\ell_1], \ldots, [\ell_N]\} \in \mathrm{VSP}(f, N)^o$, let $\langle Z\rangle_{k+1}$ be the linear span of the powers l_i^{k+1} in $S^{k+1}(E^\vee)$. Then*

(i) $\langle Z\rangle_{k+1}$ is isotropic with respect to each form $\sigma_{\omega,f}$;
(ii) $\mathrm{ap}_f^{k-1}(S^{k-1}E) \subset \langle Z\rangle_{k+1}$;
(iii) $\mathrm{ap}_f^{k-1}(S^{k-1}E)$ is contained in the radical of each $\sigma_{\omega,f}$.

Proof To prove the first assertion it is enough to check that, for all i, j, one has $\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = 0$. We have

$$\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = \Omega_f^\vee(\{l_i^{k+1}, l_j^{k+1}\}_\omega) = \Omega_f^\vee(l_i^k, l_j^k)\omega(l_i, l_j).$$

Since ℓ_i^k are linearly independent, by Proposition 1.4.7, $\Omega_f^\vee(l_i^k, l_j^k) = 0$. This checks the first assertion.

For any $\psi \in S^{k-1}(E)$,

$$D_\psi(f) = D_\psi(\sum_{i=1}^N l_i^{2k}) = \sum_{i=1}^N D_\psi(l_i^{2k}) = \tfrac{(2k)!}{(k+1)!} \sum_{i=1}^N D_\psi(l_i^{k-1}) l_i^{k+1}.$$

This shows that $\mathrm{ap}_f^{k-1}(S^{k-1}(E))$ is contained in $\langle Z\rangle_{k+1}$. It remains for us to check that $\sigma_{\omega,f}(D_\psi(f), g) = 0$ for any $\psi \in S^{k-1}(E), g \in S^{k+1}(E^\vee), \omega \in \bigwedge^2 E$. Choose coordinates $t_0, \dots, t_n$ in $E^\vee$ and the dual coordinates $\xi_0, \dots, \xi_n$ in E. The space $\bigwedge^2 E$ is spanned by the forms $\omega_{ij} = \xi_i \wedge \xi_j$. We have

$$\{D_\psi(f), g\}_{\omega_{ij}} = D_{\xi_i}(D_\psi(f)) D_{\xi_j}(g) - D_{\xi_j}(D_\psi(f)) D_{\xi_i}(g)$$

$$= D_{\xi_i\psi}(f) D_{\xi_j}(g) - D_{\xi_j\psi}(f) D_{\xi_i}(g) = D_{\psi\xi_i}(f) D_{\xi_j}(g) - D_{\psi\xi_j}(f) D_{\xi_i}(g).$$

For any $g, h \in S^k(E^\vee)$,

$$\Omega_f^\vee(g, h) = \langle \Omega_f^{-1}(g), h\rangle.$$

Thus

$$\sigma_{\omega_{ij},f}(D_\psi(f), g) = \Omega_f^\vee(D_{\psi\xi_i}(f), D_{\xi_j}(g)) - \Omega_f^\vee(D_{\psi\xi_j}(f), D_{\xi_i}(g))$$

$$= \langle \psi\xi_i, D_{\xi_j}(g)\rangle - \langle \psi\xi_j, D_{\xi_i}(g)\rangle = D_\psi(D_{\xi_i\xi_j}(g) - D_{\xi_j\xi_i}(g)) = D_\psi(0) = 0.$$

□

Since $\mathrm{ap}_f^{k-1}(E)$ is contained in the radical of $\sigma_{\omega,f}$, we have the induced skew-symmetric form on $S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(E)$. By Lemma 1.3.2,

$$S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(E) = \mathrm{AP}_{k+1}(f)^\vee.$$

If no confusion arises we denote the induced form by $\sigma_{\omega,f}$ and call it the *Mukai's skew-form*.

One can also consider the collection of the Mukai skew-forms $\sigma_{\omega,f}$ as a linear map

$$\sigma_f : \bigwedge^2 E \to \bigwedge^2 \mathrm{AP}_{k+1}(f), \quad \omega \mapsto \sigma_{\omega,f},$$

or, its transpose

$${}^t\sigma_f : \bigwedge^2 \mathrm{AP}_{k+1}(f)^\vee \to \bigwedge^2 E^\vee. \tag{1.73}$$

Let $Z = \{[l_1], \dots, [l_s]\} \in \mathrm{VSP}(f, s)^o$ be a polar s-hedron of a nondegenerate form $f \in S^{2k}(E^\vee)$ and, as before, let $\langle Z\rangle_{k+1}$ be the linear span of $(k+1)$-th powers of the linear forms l_i. Let

$$L(Z) = \langle Z\rangle_{k+1}/\mathrm{ap}_f^{k-1}(S^{k-1}(E)). \tag{1.74}$$

It is a subspace of $S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(S^{k-1}(E))$ which we identify with the dual space $\mathrm{AP}_{k+1}(f)^\vee$ of $\mathrm{AP}_{k+1}(f)$.

Now observe that $\langle Z\rangle_{k+1}^\perp$ is equal to $I_Z(k+1)$, where we identify Z with the reduced closed subscheme of the dual projective space $\mathbb{P}(E)$. This allows one to extend the definition of $L(Z)$ to any generalized polar s-hedron $Z \in \mathrm{VSP}(f;s)$:

$$L(Z) = I_Z(k+1)^\perp/\mathrm{ap}_f^{k-1}(S^{k-1}(E)) \subset S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(S^{k-1}(E)).$$

Proposition 1.4.11 *Let f be a nondegenerate homogeneous form of degree $2k$ of Waring rank equal to $N_k = \binom{n+k}{k}$. Let $Z, Z' \in \mathrm{VSP}(f, N_k)$. Then*

$$L(Z) = L(Z') \Longleftrightarrow Z = Z'.$$

Proof It is enough to show that

$$I_Z(k+1) = I_{Z'}(k+1) \Longrightarrow Z = Z'.$$

Suppose $Z \neq Z'$. Choose a subscheme Z_0 of Z of length $N_k - 1$ that is not a subscheme of Z'. Since $\dim I_{Z_0}(k) \geq \dim S^k(E^\vee) - h^0(\mathcal{O}_Z) = \binom{n+k}{k} - N_k + 1 = 1$, we can find a nonzero $\psi \in I_{Z_0}(k)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_0}$ is concentrated at one point x and is annihilated by the maximal ideal $\mathfrak{m}_x$. Thus $\mathfrak{m}_x\mathcal{I}_{Z_0} \subset \mathcal{I}_Z$. Let $\xi \in E$ be a linear form on $E^\vee$ vanishing at x but not vanishing at any proper closed subscheme of Z'. This implies that $\xi\psi \in I_Z(k+1) = I_{Z'}(k+1)$ and hence $\psi \in I_{Z'}(k) \subset \mathrm{AP}_k(f)$ contradicting the nondegeneracy of f. □

Lemma 1.4.12 *Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form of Waring rank $N_k = \binom{k+n}{n}$. For any $Z \in \mathrm{VSP}(f, N_k)^o$,*

$$\dim L(Z) = \tbinom{n+k-1}{n-1}.$$

Proof Counting constants, we see that

$$\dim\langle Z\rangle_{k+1} \geq \dim S^{k+1}(E) - N_k,$$

hence

$$\dim L(Z) = \dim\langle Z\rangle_{k+1}^\perp - \dim \mathrm{ap}_f^{k-1}(S^{k-1}(E)) \leq N_k - \tbinom{n+k-1}{n} = \tbinom{n+k-1}{n-1}.$$

We have to consider the exceptional cases where $\mathrm{wrk}(f) = \mathrm{rank}\,\Omega_f$. The assertion is obvious in the case $k = 1$. The space $L(Z)$ is of expected dimension unless $l_1^2, \ldots, l_{n+1}^2$ are linearly dependent. This implies that f is a quadratic form of rank less than $n+1$, contradicting the assumption.

Assume $n = 2, k = 2$ and $\dim L(Z) > 3$, or, equivalently, $\dim\langle Z\rangle_3 > 4$. Since $\mathrm{AP}_2(f) = \{0\}$, there are no conics passing through Z. In particular,

no four points are collinear. Let C be a conic through the points $[l_1], \dots, [l_5]$ and let x_1, x_2 be two additional points on C such that each irreducible component of C contains at least four points. Since $\dim\langle Z\rangle_3 > 4$, we can find a 2-dimensional linear system of cubics through $[l_1], \dots, [l_5], x_1, x_2$. By Bezout's Theorem, C belongs to the fixed part of the linear system. The residual part is a 2-dimensional linear system of lines through $[l_6]$, an obvious contradiction.

Similar arguments check the assertion in the cases $n = 2, k = 3, 4$. In the remaining case $n = 3, k = 2$, we argue as follows. We have $N_2 = 10$. Assume $L(Z) < 6$, or, equivalently, $\dim\langle Z\rangle_3 > 10$. Since $\mathrm{AP}_2(f) = \{0\}$, no 4 lines are collinear (otherwise we pass a quadric through 3 points on the line and the remaining 6 points, it will contain all ten points). Choose three non-collinear points p_1, p_2, p_3 among the ten points and two general points on each line $\overline{p_i p_j}$ and one general point in the plane containing the three points. Then we can find a 3-dimensional linear system of cubics in $|\langle Z\rangle_3|$ passing through the additional 7 points. It contains the plane through p_1, p_2, p_3. The residual linear system consists of quadrics through the remaining 7 points in Z. Since no four lines are collinear, it is easy to see that the dimension of the linear system of quadrics through 7 points is of dimension 2. This contradiction proves the assertion. □

Corollary 1.4.13 *Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form of Waring rank $N_k = \binom{n+k}{n}$. Let $\mathrm{VSP}(f, N_k)^o$ be the variety of polar polyhedra of f. Then the map $Z \mapsto L(Z)$ is an injective map*

$$\mathrm{VSP}(f, N_k)^o \to G(\tbinom{n+k-1}{n-1}, \mathrm{AP}_{k+1}(f)^\vee).$$

Its image is contained in the subvariety of subspaces isotropic with respect to all Mukai's skew forms $\sigma_{\omega,f}$ on $\mathrm{AP}_{k+1}(f)^\vee$.

Example 1.4.14 Assume $n = 2$. Then $\mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f = \binom{k+2}{2}$ if and only if $k = 1, 2, 3, 4$. In these cases the Corollary applies. We will consider the cases $k = 1$ and $k = 2$ later. If $k = 3$, we obtain that $\mathrm{VSP}(f, 10)^o$ embeds in $G(4, 9)$. Its closure is a K3 surface [400], [454]. If $k = 4$, $\mathrm{VSP}(f, 15)^o$ embeds in $G(5, 15)$. It consists of 16 points [454].

1.4.5 Harmonic polynomials

Let $q \in S^2(E^\vee)$ be a nondegenerate quadratic form on E. For convenience of notation, we identify q with the apolarity map $\mathrm{ap}^1_q : E \to E^\vee$. By the universal property of the symmetric power, the isomorphism $q : E \to E^\vee$ extends to a

linear isomorphism $S^k(q) : S^k(E) \to S^k(E^\vee)$ which defines a symmetric nondegenerate pairing

$$(\ , \ )_k : S^k(E) \times S^k(E) \to \mathbb{C}. \tag{1.75}$$

It is easy to check that, for any $\xi \in S^k(E)$ and $v \in E$,

$$(\xi, v^k) = k!\xi(l_v^k),$$

where $l_v \in E^\vee$ is the linear function $\mathrm{ap}_q^1(v)$.

Let us compare the pairing 1.75 with the pairing Ω_{q^k} from (1.70). Choose a basis $\eta_0, \dots, \eta_n$ in E and the dual basis $t_0, \dots, t_n$ in $E^\vee$ such that $q = \frac{1}{2}(\sum t_i^2)$ so that $q(\eta_i) = t_i$. Then

$$S^k(q)(\boldsymbol{\eta^i}) = \boldsymbol{t^i}.$$

However,

$$\mathrm{ap}_{q^k}^k(\boldsymbol{\eta^i}) = k!\boldsymbol{t^i} + qg,$$

for some $g \in S^{k-2}(E^\vee)$. Thus

$$(S^k(q) - \frac{1}{k!}\mathrm{ap}_{q^k}^k)(S^k(E)) \subset qS^{k-2}(E^\vee).$$

Let

$$\mathcal{H}_q^k(E) = (qS^{k-2}(E^\vee))^\perp \subset S^k(E)$$

be the subspace of *q-harmonic symmetric tensors*. In more convenient language, exchanging the roles of E and $E^\vee$, and replacing q with the dual form $q^\vee \in S^2(E)$, we have

$$\mathcal{H}_q^k(E^\vee) = \mathrm{Ker}(D_{q^\vee} : S^k(E^\vee) \to S^{k-2}(E^\vee)).$$

In the previous choice of coordinates, the operator $D_{q^\vee}$ is the *Laplace operator* $\frac{1}{2}\sum \frac{\partial^2}{\partial t_i^2}$. Restricting $\mathrm{ap}_{q^k}^k$ to the subspace $\mathcal{H}_q^k(E)$, we obtain a nondegenerate symmetric pairing

$$\mathcal{H}_q^k(E) \times \mathcal{H}_q^k(E) \to \mathbb{C}$$

which coincides with the restriction of $\frac{1}{k!}\Omega_{q^k}$ to the same subspace. Changing E to $E^\vee$, we also obtain a symmetric nondegenerate pairing

$$\mathcal{H}_q^k(E^\vee) \times \mathcal{H}_q^k(E^\vee) \to \mathbb{C}$$

which can be defined either by the restriction of the pairing (1.75) or by the quadratic form $\frac{1}{k!}\Omega_{(q^\vee)^k}$. Note that all these pairings are equivariant with respect to the orthogonal group $\mathrm{O}(E, q)$, i.e. can be considered as pairings of the

linear representations of $\mathrm{O}(E,q)$. We have the direct sum decomposition of linear representations

$$S^k(E) = \mathcal{H}^k_q(E) \oplus q^\vee S^{k-2}(E). \tag{1.76}$$

The summand $q^\vee S^{k-2}(E)$ coincides with $\mathrm{ap}^{k-2}_{q^\vee}(S^{k-2}(E^\vee))$. The linear representation $\mathcal{H}^k_q(E)$ is an irreducible representation of $\mathrm{O}(E,q)$ (see [252]).

Next let us see that, in the case when f is a power of a nondegenerate quadratic polynomial, the Mukai form coincides, up to a scalar multiple, with the skew form on the space of harmonic polynomials studied by N. Hitchin in [299] and [300].

The Lie algebra $\mathfrak{o}(E,q)$ of the orthogonal group $\mathrm{O}(E,q)$ is equal to the Lie subalgebra of the Lie algebra $\mathfrak{gl}(E)$ of endomorphisms of E that consists of operators $A : E \to E$ such that the composition $A \circ q^{-1} : E^\vee \to E \to E$ is equal to the negative of its transpose. This defines a linear isomorphism of vector spaces

$$\bigwedge^2 E^\vee \to \mathfrak{o}(E,q), \quad \omega \mapsto \tilde{\omega} = q^{-1} \circ \omega : E \to E^\vee \to E.$$

Now, taking $\omega \in \bigwedge^2 E^\vee$, and identifying $S^{k+1}(E^\vee)/\mathrm{ap}^{k-1}_{q^k}(S^{k-1}(E))$ with $\mathcal{H}^{k+1}_q(E^\vee)$, we obtain the Mukai pairing

$$\sigma_{\omega,q^k} : \mathcal{H}^{k+1}_q(E^\vee) \times \mathcal{H}^{k+1}_q(E^\vee) \to \mathbb{C}$$

on the space of harmonic $k+1$-forms on E.

Proposition 1.4.15 *For any $g, h \in \mathcal{H}^{k+1}_q(E^\vee)$ and any $\omega \in \bigwedge^2 E^\vee$,*

$$\sigma_{\omega,q^k}(g,h) = \frac{(k+1)^2}{k!}(\tilde{\omega} \cdot g, h)_{k+1},$$

where $(\,,\,)_{k+1} : S^{k+1}(E^\vee) \times S^{k+1}(E^\vee) \to \mathbb{C}$ is the symmetric pairing defined by $S^{k+1}(q^{-1})$.

Proof It is known that the space $\mathcal{H}^{k+1}_q(E^\vee)$ is spanned by the forms $q(v)^{k+1}$, where v is an isotropic vector for q, i.e. $[v] \in V(q)$ (see [252], Proposition 5.2.6). So, it is suffices to check the assertion when $g = q(v)^{k+1}$ and $h = q(w)^{k+1}$ for some isotropic vectors $v, w \in E$. Choose a basis $(\xi_0, \dots, \xi_n)$ in E and the dual basis $t_0, \dots, t_n$ in $E^\vee$ as in the beginning of this subsection. An element $u \in \mathfrak{o}(E,q)$ can be written in the form $\sum a_{ij} t_i \frac{\partial}{\partial t_j}$ for some skew-symmetric matrix (a_{ij}). We identify (a_{ij}) with the skew 2-form $\omega \in \bigwedge^2 E$. We can also write $g = (\alpha \cdot t)^{k+1}$ and $h = (\beta \cdot t)^{k+1}$, where we use the dot-product notation for the sums $\sum \alpha_i t_i$. We have

$$(u \cdot g, h)_{k+1} = (k+1)!\Big(\sum a_{ij} t_i \frac{\partial}{\partial t_j}(\alpha \cdot t)^{k+1}\Big)(\beta) = (k+1)!(k+1)(\alpha \cdot \beta)^k \omega(\alpha \cdot t, \beta \cdot t).$$

The computations from the proof of Theorem 1.4.10, show that

$$\sigma_{\omega,q^k}(g,h) = \Omega^{\vee}_{q^k}((\alpha\cdot t)^k,(\beta\cdot t)^k)\omega(\alpha\cdot t,\beta\cdot t).$$

It is easy to see that $\Omega^{\vee}_{q^k}$ coincides with $\Omega_{(q^{\vee})^k}$ on the subspace of harmonic polynomials. We have

$$\Omega_{(q^{\vee})^k}\big((\alpha\cdot t)^k,(\beta\cdot t)^k\big) = D_{(\alpha\cdot t)^k}(\tfrac{1}{2}\sum \xi_i^2)^k((\beta\cdot t)^k)$$

$$= k!D_{(\alpha\cdot\xi)^k}((\beta\cdot t)^k) = (k!)^2(\alpha\cdot\beta)^k.$$

This checks the assertion.

□

Computing the catalecticant matrix of q^k we find that q^k is a nondegenerate form of degree $2k$. Applying Corollary 1.4.13, we obtain that in the cases listed in Corollary 1.3.20, there is an injective map

$$\mathrm{VSP}(q^k,\tbinom{n+k}{n}) \to G(\tbinom{n+k-1}{n-1},\mathcal{H}^{k+1}_q(E^{\vee})). \tag{1.77}$$

Its image is contained in the subvariety of subspaces isotropic with respect to the skew-symmetric forms $(g,h)\mapsto(u\cdot g,k)_{k+1}, u\in\mathfrak{o}(E,q)$.

The following Proposition gives a basis in the space of harmonic polynomials (see [388]). We assume that $(E,q) = (\mathbb{C}^{n+1},\frac{1}{2}\sum t_i^2)$.

Proposition 1.4.16 *For any set of non-negative integers $(b_0,\dots,b_n)$ such that $b_i\le 1$ and $\sum b_i = k$, let*

$$H^k_{b_0,\dots,b_n} = \sum(-1)^{[a_0/2]}\frac{k![a_0/2]!}{\prod_{i=0}^n a_i!\prod_{i=1}^n(\frac{b_i-a_i}{2})!}\prod_{i=0}^n t_i^{a_i},$$

where the summation is taken over the set of all sequences of non-negative integers $(a_0,\dots,a_n)$ such that

- $a_i\equiv b_i \mod 2, i=0,\dots,n,$
- $\sum_{i=0}^n a_i = k,$
- $a_i\le b_i, i=1,\dots,n.$

Then the polynomials $H^k_{b_0,\dots,b_n}$ form a basis of the space $\mathcal{H}^k_q(\mathbb{C}^{n+1})$.

For any polynomial $f\in\mathbb{C}[t_0,\dots,t_n]$ one can find the projection Hf to the subspace of harmonic polynomials. The following formula is taken from [591].

$$Hf = f - \sum_{s=1}^{[k/2]} (-1)^{s+1} \frac{q^s \Delta^s f}{2^s s!(n-3+2k)(n-5+2k)\cdots(n-2s-1+2k)}, \tag{1.78}$$

where $\Delta = \sum \frac{\partial^2}{\partial t_i^2}$ is the Laplace operator.

Example 1.4.17 Let $n = 2$ so that $\dim E = 3$. The space of harmonic polynomials $\mathcal{H}_q^k(E^\vee)$ is of dimension $\binom{k+2}{2} - \binom{k}{2} = 2k+1$. Since the dimension is odd, the skew form σ_{ω,q^k} is degenerate. It follows from Proposition 1.4.15 that its radical is equal to the subspace of harmonic polynomials g such that $\tilde{\omega} \cdot g = 0$ (recall that $\tilde{\omega}$ denotes the element of $\mathfrak{o}(E,q)$ corresponding to $\omega \in \bigwedge^2 E$). In coordinates, a vector $u = (u_0, u_1, u_2) \in \mathbb{C}^3$ corresponds to the skew-symmetric matrix

$$\begin{pmatrix} 0 & u_0 & u_1 \\ -u_0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}$$

representing an endomorphism of E, or an element of $\bigwedge^2 E$. The Lie bracket is the cross-product of vectors. The action of a vector u on $f \in \mathbb{C}[t_0, t_1, t_2]$ is given by

$$u \cdot f = \sum_{i,j,k=0}^{2} \epsilon_{ijk} t_i u_j \frac{\partial f}{\partial t_k},$$

where $\epsilon_{i,j,k} = 0$ is totally skew-symmetic with values equal to $0, 1, -1$.

For any $v \in E$, let us consider the linear form $l_v = q(v) \in E^\vee$. We know that $q(v)^k \in \mathcal{H}_q^k(E^\vee)$ if $[v] \in V(q)$. If $[v] \notin V(q)$, then we can consider the projection f_v of $(l_v)^k$ to $\mathcal{H}_q^k(E^\vee)$. By (1.78), we get

$$f_v = l_v^k + \sum_{s=1}^{[k/2]} (-1)^s \frac{k(k-1)\cdots(k-2s+1)}{2^s s!(2k-1)\cdots(2k-2s+1)} q(v)^s q^s l_v^{k-2s}. \tag{1.79}$$

We have

$$u \cdot l_v = l_{u \times v}.$$

Since $f \mapsto u \cdot f$ is a derivation of $\mathrm{Sym}(E^\vee)$ and $u \cdot q = 0$, we obtain

$$u \cdot f_v = l_{u\times v}\big(k l_v^{k-1} + \sum_{s=1}^{[k/2]} (-1)^s \frac{k(k-1)\cdots(k-2s+1)(k-2s) q(v)^s l_v^{k-2s-2}}{2^s s!(2k-1)\cdots(2k-2s+1)}\big). \tag{1.80}$$

This implies that the harmonic polynomial f_u satisfies $u \cdot f_u = 0$ and hence

belongs to the radical of the skew form σ_{u,q^k}. The Lie algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2)$ and its irreducible representation on the space of degree k harmonic polynomials is isomorphic to the representation of $\mathfrak{sl}(2)$ on the space of binary forms of degree $2k$. It is easy to see that the space of binary forms invariant under a nonzero element of $\mathfrak{sl}(2)$ is one-dimensional. This implies that the harmonic polynomial f_u spans the radical of σ_{u,q^k} on $\mathcal{H}^k_q(E^\vee)$.

Let $f \in H^k(E^\vee)$ be a nonzero harmonic polynomial of degree k. The orthogonal complement $f^\perp$ of f with respect to $(\ ,\ )_k : \mathcal{H}^k_q(E^\vee) \times \mathcal{H}^k_q(E^\vee) \to \mathbb{C}$ is of dimension $2k$. The restriction of the skew-symmetric form σ_{u,q^k} to $f^\perp$ is degenerate if and only if $f_u \in f^\perp$, i.e. $(f_u, f)_k = (l^k_u, f) = f(u) = 0$. Here we used that the decomposition (1.76) is an orthogonal decomposition with respect to $(\ ,\ )_k$. Let Pf be the pfaffian of the skew form σ_{u,q^k} on $f^\perp$. It is equal to zero if and only if the form is degenerate. By the above, it occurs if and only if $f(u) = 0$. Comparing the degrees, this gives

$$V(f) = V(\mathrm{Pf}).$$

So, every harmonic polynomial can be expressed in a canonical way as a pfaffian of a skew-symmetric matrix with entries linear forms, a result due to N. Hitchin [301].

1.5 First examples

1.5.1 Binary forms

Let U be a 2-dimensional linear space and $f \in S^d(U^\vee) \setminus \{0\}$. The hypersurface $X = V(f)$ can be identified with a positive divisor $\mathrm{div}(f) = \sum m_i x_i$ of degree d on $|U| \cong \mathbb{P}^1$. Since $\bigwedge^2 U \cong \mathbb{C}$, we have a natural isomorphism $U \to U^\vee$ of linear representations of $\mathrm{SL}(U)$. It defines a natural isomorphism between the projective line $|U|$ and its dual projective line $\mathbb{P}(U)$. In coordinates, a point $a = [a_0, a_1]$ is mapped to the hyperplane $V(a_1 t_0 - a_0 t_1)$ whose zero set is equal to the point a. If X is reduced (i.e. f has no multiple roots), then, under the identification of $|U|$ and $\mathbb{P}(U)$, X coincides with its dual $X^\vee$. In general, $X^\vee$ consists of simple roots of f. Note that this is consistent with the Plückeri-Teissier formula. The degrees of the Hessian and the Steinerian coincide, although they are different if $d > 3$. Assume that X is reduced. The partial derivatives of f define the polar map $g : |U| \to |U|$ of degree $d - 1$. The ramification divisor $\mathrm{He}(X)$ consists of $2d - 4$ points and it is mapped bijectively onto the branch divisor $\mathrm{St}(X)$.

Example 1.5.1 We leave the case $d = 2$ to the reader. Consider the case $d = 3$. In coordinates

$$f = a_0t_0^3 + 3a_1t_0^2t_1 + 3a_2t_0t_1^2 + a_3t_1^3.$$

All invariants are powers of the *discriminant invariant*

$$\Delta = a_0^2a_3^2 + 4a_0a_2^3 + 4a_1^3a_3 - 6a_0a_1a_2a_3 - 3a_1^2a_2^2. \tag{1.81}$$

whose symbolic expression is $(12)^2(13)(24)(34)^2$ (see [580], p. 244). The Hessian covariant

$$H = (a_0a_2 - a_1^2)t_0^2 + (a_0a_3 - a_1a_2)t_0t_1 + (a_1a_3 - a_2^2)t_1^2.$$

Its symbolic expression is $(ab)a_xb_y$. There is also a cubic covariant

$$J = J(f, H) = \det \begin{pmatrix} t_0^3 & 3t_0^2t_1 & 3t_0t_1^2 & t_1^3 \\ a_2 & -2a_1 & a_0 & 0 \\ a_3 & -a_2 & -a_1 & a_0 \\ 0 & -a_3 & -2a_2 & a_1 \end{pmatrix}$$

with symbolic expression $(ab)^2(ac)^2b_xc_x^2$. The covariants f, H and J form a complete system of covariants, i.e. generate the module of covariants over the algebra of invariants.

Example 1.5.2 Consider the case $d = 4$. In coordinates,

$$f = a_0t_0^4 + 4a_1t_0^3t_1 + 6a_2t_0^2t_1^2 + 4a_3t_0t_1^3 + a_4t_1^4.$$

There are two basic invariants S and T on the space of quartic binary forms. Their symbolic expression are $S = (12)^4$ and $T = (12)^2(13)^2(23)^2$. Explicitly,

$$\begin{aligned} S &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ T &= a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3. \end{aligned} \tag{1.82}$$

Note that T coincides with the determinant of the catalecticant matrix of f. Each invariant is a polynomial in S and T. For example, the discriminant invariant is equal to

$$\Delta = S^3 - 27T^2.$$

The Hessian $\mathrm{He}(X) = V(H)$ and the Steinerian $S(X) = V(K)$ are both of degree 4. We have

$$H = (a_0a_2 - a_1^2)t_0^4 + 2(a_0a_3 - a_1a_2)t_0^3t_1 + (a_0a_4 + 2a_1a_3 - 3a_2^2)t_0^2t_1^2$$

$$+2(a_1a_4 - a_2a_3)t_0t_1^3 + (a_2a_4 - a_3^2)t_1^4.$$

and

$$K = \Delta((a_0t_0+a_1t_1)x^3+3(a_1t_0+a_2t_1)x^2y+3(a_2t_0+a_3t_1)xy^2+(a_3t_0+a_4t_1)y^3).$$

Observe that the coefficients of H (resp. K) are of degree 2 (resp. 4) in coefficients of f. There is also a covariant $J = J(f, H)$ of degree 6 and the module of covariants is generated by f, H, J over $\mathbb{C}[S, T]$. In particular, $K = \alpha Tf + \beta SH$, for some constants α and β. By taking f in the form

$$f = t_0^4 + 6mt_0^2t_1^2 + t_1^4, \tag{1.83}$$

and comparing the coefficients we find

$$2K = -3Tf + 2SH. \tag{1.84}$$

Under identification $|U| = \mathbb{P}(U)$, a generalized k-hedron Z of $f \in S^d(U^\vee)$ is the zero divisor of a form $g \in S^k(U)$ which is apolar to f. Since

$$H^1(|E|, \mathcal{I}_Z(d)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-k)) = 0, \quad k \geq d+1,$$

any Z is automatically linearly independent. Identifying a point $[g] \in |S^k(U)|$ with the zero divisor $\mathrm{div}(g)$, we obtain

Theorem 1.5.3 *Assume $n = 1$. Then*

$$\mathrm{VSP}(f; k) = |\mathrm{AP}_k(f)|.$$

Note that the kernel of the map

$$S^k(U) \to S^{d-k}(U^\vee), \quad \psi \mapsto D_\psi(f)$$

is of dimension $\geq \dim S^k(U) - \dim S^{d-k}(U^\vee) = k+1-(d-k+1) = 2k-d$. Thus $D_\psi(f) = 0$ for some nonzero $\psi \in S^k(U)$, whenever $2k > d$. This shows that f has always generalized polar k-hedron for $k > d/2$. If d is even, a binary form has an apolar $d/2$-form if and only if $\det \mathrm{Cat}_{d/2}(f) = 0$. This is a divisor in the space of all binary d-forms.

Example 1.5.4 Take $d = 3$. Assume that f admits a polar 2-hedron . Then

$$f = (a_1t_0 + b_1t_1)^3 + (a_2t_0 + b_2t_1)^3.$$

It is clear that f has 3 distinct roots. Thus, if $f = (a_1t_0 + b_1t_1)^2(a_2t_0 + b_2t_1)$ has a double root, it does not admit a polar 2-hedron. However, it admits a generalized 2-hedron defined by the divisor $2p$, where $p = (b_1, -a_1)$. In the secant variety interpretation, we know that any point in $|S^3(E^\vee)|$ either lies on a unique secant or on a unique tangent line of the rational cubic curve. The space $\mathrm{AP}_2(f)$ is always one-dimensional. It is generated either by a binary quadric $(-b_1\xi_0 + a_1\xi_1)(-b_2\xi_0 + a_2\xi_1)$, or by $(-b_1\xi_0 + a_1\xi_1)^2$.

Thus $\mathrm{VSP}(f,2)^o$ consists of one point or empty but $\mathrm{VSP}(f,2)$ always consists of one point. This example shows that $\mathrm{VSP}(f,2) \neq \overline{\mathrm{VSP}(f,2)}^o$ in general.

1.5.2 Quadrics

It follows from Example 1.3.17 that $\mathrm{Sec}_t(\mathsf{V}_2^n) \neq |S^2(E^\vee)|$ if and only if there exists a quadric with $t+1$ singular points in general position. Since the singular locus of a quadric $V(q)$ is a linear subspace of dimension equal to $\mathrm{corank}(q)-1$, we obtain that $\mathrm{Sec}_n(\mathsf{V}_2^n) = |S^2(E^\vee)|$, hence any general quadratic form can be written as a sum of $n+1$ squares of linear forms $l_0,\dots,l_n$. Of course, linear algebra gives more. Any quadratic form of rank $n+1$ can be reduced to sum of squares of the coordinate functions. Assume that $q = t_0^2+\cdots+t_n^2$. Suppose we also have $q = l_0^2+\cdots+l_n^2$. Then the linear transformation $t_i \mapsto l_i$ preserves q and hence is an orthogonal transformation. Since polar polyhedra of q and λq are the same, we see that the projective orthogonal group $\mathrm{PO}(n+1)$ acts transitively on the set $\mathrm{VSP}(f,n+1)^o$ of polar $(n+1)$-hedra of q. The stabilizer group G of the coordinate polar polyhedron is generated by permutations of coordinates and diagonal orthogonal matrices. It is isomorphic to the semi-direct product $2^n \rtimes \mathfrak{S}_{n+1}$ (the Weyl group of root systems of types B_n, D_n), where we use the notation 2^n for the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^n$. Thus we obtain

Theorem 1.5.5 *Let q be a quadratic form in $n+1$ variables of rank $n+1$. Then*

$$\mathrm{VSP}(q,n+1)^o \cong \mathrm{PO}(n+1)/2^n \rtimes \mathfrak{S}_{n+1}.$$

The dimension of $\mathrm{VSP}(q,n+1)^o$ *is equal to* $\frac{1}{2}n(n+1)$.

Example 1.5.6 Take $n=1$. Using the Veronese map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, we consider a nonsingular quadric $Q = V(q)$ as a point p in $\mathbb{P}^2$ not lying on the conic $C = V(t_0t_2 - t_1^2)$. A polar 2-gon of q is a pair of distinct points p_1, p_2 on C such that $p \in \langle p_1, p_2\rangle$. The set of polar 2-gons can be identified with the pencil of lines through p with the two tangent lines to C deleted. Thus $W(q,2)^o = \mathbb{P}^1 \setminus \{0,\infty\} = \mathbb{C}^*$. There are two generalized 2-gons $2p_0$ and $2p_\infty$ defined by the tangent lines. Each of them gives the representation of q as l_1l_2, where $V(l_i)$ are the tangents. We have $\mathrm{VSP}(f,2) = \overline{\mathrm{VSP}(f,2)}^o \cong \mathbb{P}^1$.

Let $q \in S^2(E^\vee)$ be a nondegenerate quadratic form. We have an injective map (1.77)

$$\mathrm{VSP}(q,n+1)^o \to G(n, \mathcal{H}_q^2(E)) \cong G(n, \tbinom{n+2}{2} - 1). \tag{1.85}$$

Its image is contained in the subvariety $G(n, \mathcal{H}^2_q(E))_\sigma$ of subspaces isotropic with respect to the Mukai skew forms.

Recall that the Grassmann variety $G(m, W)$ of linear m-dimensional subspaces of a linear space W of dimension N carries the natural rank n vector bundle $\mathcal{S}$, the *universal subbundle*. Its fiber over a point $L \in G(m, W)$ is equal to L. It is a subbundle of the trivial bundle $W_{G(m,W)}$ associated to the vector space W. We have a natural exact sequence

$$0 \to \mathcal{S} \to W_{G(m,W)} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the *universal quotient bundle*, whose fiber over L is equal to W/L. By restriction, we can view the Mukai form $\sigma_q : \bigwedge^2 E \to \bigwedge^2 \mathcal{H}^2_q(E^\vee)$ as a section of the vector bundle $\bigwedge^2 \mathcal{S}^\vee \otimes \bigwedge^2 E^\vee$. The image of $\mathrm{VSP}(q, n+1)$ is contained in the zero locus of a section of this bundle defined by σ_q. Since the rank of the vector bundle is equal to $\binom{n}{2}\binom{n+1}{2}$, we expect that the dimension of its zero locus is equal to

$$\dim G(n, \tbinom{n+2}{2} - 1) - \binom{n}{2}\binom{n+1}{2} = n(\tbinom{n+2}{2} - 1 - n) - \binom{n}{2}\binom{n+1}{2}.$$

Unfortunately, this number is ≤ 0 for $n > 2$, so the expected dimension is wrong. However, when $n = 2$, we obtain that the expected dimension is equal to $3 = \dim \mathrm{VSP}(q, 3)$. We can view $\sigma_{\omega,q}$ as a hyperplane in the Plücker embedding of $G(2, \mathcal{H}^2_q(E)) \cong G(2, 5)$. So, $\mathrm{VSP}(q, 3)$ embeds into the intersection of 3 hyperplane sections of $G(2, 5)$.

Theorem 1.5.7 *Let q be a nondegenerate quadratic form on a 3-dimensional vector space E. Then the image of* $\mathrm{VSP}(q, 3)$ *in $G(2, \mathcal{H}^2_q(E))$, embedded in the Plücker space, is a smooth irreducible 3-fold equal to the intersection of $G(2, \mathcal{H}^2_q(E))$ with a linear space of codimension 3.*

Proof We have $\dim \mathcal{H}^2_q(E) = 5$, so $G(2, \mathcal{H}^2_q(E)) \cong G(2, 5)$ is of dimension 6. Hyperplanes in the Plücker space are elements of the space $|\bigwedge^2 \mathcal{H}^2_q(E)^\vee|$. Note that the functions $s_{q,\omega}$ are linearly independent. In fact, a basis ξ_0, ξ_1, ξ_2 in E gives a basis $\omega_{01} = \xi_0 \wedge \xi_1, \omega_{02} = \xi_0 \wedge \xi_2, \omega_{12} = \xi_1 \wedge \xi_2$ in $\bigwedge^2 E$. Thus the space of sections $s_{q,\omega}$ is spanned by 3 sections s_{01}, s_{02}, s_{12} corresponding to the forms ω_{ij}. Without loss of generality, we may assume that $q = t_0^2 + t_1^2 + t_2^2$. If we take $a = t_0 t_1 + t_2^2, b = -t_0^2 + t_1^2 + t_2^2$, we see that $s_{01}(a, b) \neq 0, s_{12}(a, b) = 0, s_{02}(a, b) = 0$. Thus a linear dependence between the functions s_{ij} implies the linear dependence between two functions. It is easy to see that no two functions are proportional. So our 3 functions $s_{ij}, 0 \leq i < j \leq 2$ span a 3-dimensional subspace of $\bigwedge^2 \mathcal{H}^2_q(E^\vee)$

and hence define a codimension 3 projective subspace L in the Plücker space $|\bigwedge^2 \mathcal{H}^2_q(E)|$. The image of $\mathrm{VSP}(q,3)$ under the map (1.85) is contained in the intersection $G(2,E)\cap L$. This is a 3-dimensional subvariety of $G(2,\mathcal{H}^2_q(E))$, and hence contains $\mu(\mathrm{VSP}(q,3))$ as an irreducible component. We skip an argument, based on counting constants, which proves that the subspace L belongs to an open Zariski subset of codimension 3 subspaces of $\bigwedge^2 \mathcal{H}^2_q(E)$ for which the intersection $L\cap G(2,\mathcal{H}^2_q(E))$ is smooth and irreducible (see [183]). □

It follows from the adjunction formula and the known degree of $G(2,5)$ that the closure of $\mathrm{VSP}(q,3)^o$ in $G(2,\mathcal{H}^2_q(E))$ is a smooth Fano variety of degree 5. We will discuss it again in the next chapter.

Remark 1.5.8 One can also consider the varieties $\mathrm{VSP}(q,s)$ for $s > n+1$. For example, we have

$$t_0^2 - t_2^2 = \tfrac{1}{2}(t_0+t_1)^2 + \tfrac{1}{2}(t_0-t_1)^2 - \tfrac{1}{2}(t_1+t_2)^2 - \tfrac{1}{2}(t_1^2-t_2)^2,$$
$$t_0^2+t_1^2+t_2^2 = (t_0+t_2)^2+(t_0+t_1)^2+(t_1+t_2)^2-(t_0+t_1+t_2)^2.$$

This shows that $\mathrm{VSP}(q,n+2), \mathrm{VSP}(q,n+3)$ are not empty for any nondegenerate quadric Q in $\mathbb{P}^n, n\geq 2$.

Exercises

1.1 Suppose X is a plane curve and $x\in X$ is its ordinary double point. Show that the pair consisting of the tangent line of $P_a(X)$ at x and the line $\overline{ax}$ is harmonically conjugate (see section 2.1.2) to the pair of tangents to the branches of X at x in the pencil of lines through x. If x is an ordinary cusp, then show that the polar line of $P_a(X)$ at x is equal to the cuspidal tangent of X at x.

1.2 Show that a line contained in a hypersurface X belongs to all polars of X with respect to any point on this line.

1.3 Find the multiplicity of the intersection of a plane curve C with its Hessian at an ordinary double point and at an ordinary cusp of C. Show that the Hessian has a triple point at the cusp.

1.4 Suppose a hypersurface X in $\mathbb{P}^n$ has a singular point x of multiplicity $m>1$. Prove that $\mathrm{He}(X)$ has this point as a point of multiplicity $\geq (n+1)m-2n$.

1.5 Suppose a hyperplane is tangent to a hypersurface X along a closed subvariety Y of codimension 1. Show that Y is contained in $\mathrm{He}(X)$.

1.6 Suppose f is the product of d distinct linear forms $l_i(t_0,\dots,t_n)$. Let A be the matrix of size $(n+1)\times d$ whose i-th column is formed by the coefficients of l_i (defined, of course up to proportionality). Let Δ_I be the maximal minor of A corresponding to a subset I of $[1,\dots,d]$ and f_I be the product of linear forms $l_i, i\notin I$. Show that

$$\mathrm{He}(f) = (-1)^n(d-1)f^{n-1}\sum_I \Delta_I^2 f_I^2$$

([396], p. 660).

1.7 Find an example of a reduced hypersurface whose Hessian surface is nowhere reduced.

1.8 Show that the locus of points on the plane where the first polars of a plane curve X are tangent to each other is the Hessian of X and the set of common tangents is the Cayleyan curve .

1.9 Show that each inflection tangent of a plane curve X, considered as a point in the dual plane, lies on the Cayleyan of X.

1.10 Show that the class of the Steinerian $\mathrm{St}(X)$ of a plane curve X of degree d is equal to $3(d-1)(d-2)$ but its dual is not equal to $\mathrm{Cay}(X)$.

1.11 Let $\mathsf{D}_{m,n} \subset \mathbb{P}^{mn-1}$ be the image in the projective space of the variety of $m \times n$ matrices of rank $\leq \min\{m,n\} - 1$.

$$\tilde{\mathcal{D}}_{m,n} = \{(A,x) \in \mathbb{P}^{mn-1} \times \mathbb{P}^n : A \cdot x = 0\}$$

is a resolution of singularities of $\mathsf{D}_{m,n}$. Find the dual variety of $\mathsf{D}_{m,n}$. **1.12** Find the dual variety of the Segre variety $s(\mathbb{P}^n \times \mathbb{P}^n) \subset \mathbb{P}^{n^2+2n}$.

1.13 Let X be the union of k nonsingular conics in general position. Show that $X^\vee$ is also the union of k nonsingular conics in general position.

1.14 Let X has only δ ordinary nodes and κ ordinary cusps as singularities. Assume that the dual curve $X^\vee$ has also only $\check{\delta}$ ordinary nodes and $\check{\kappa}$ ordinary cusps as singularities. Find $\check{\delta}$ and $\check{\kappa}$ in terms of d, δ, κ.

1.15 Give an example of a self-dual (i.e. $X^\vee \cong X$) plane curve of degree > 2.

1.16 Show that the Jacobian of a net of plane curves has a double point at each simple base point unless the net contains a curve with a triple point at the base point [214].

1.17 Let $|L|$ be a general n-dimensional linear system of quadrics in $\mathbb{P}^n$ and $|L|^\perp$ be the $(\binom{n+2}{2} - n - 2)$-dimensional subspace of apolar quadric in the dual space. Show that the variety of reducible quadrics in $|L|^\perp$ is isomorphic to the Reye variety of $|L|$ and has the same degree.

1.18 Show that the embedded tangent space of the Veronese variety V_d^n at a point represented by the form l^d is equal to the projectivization of the linear space of homogeneous polynomials of degree d of the form $l^{d-1}m$.

1.19 Using the following steps, show that V_3^4 is 6-defective by proving that for 7 general points p_i in $\mathbb{P}^4$ there is a cubic hypersurface with singular points at the p_i's.

(i) Show that there exists a Veronese curve R_4 of degree 4 through the seven points.
(ii) Show that the secant variety of R_4 is a cubic hypersurface which is singular along R_4.

1.20 Let q be a nondegenerate quadratic form in $n+1$ variables. Show that $\mathrm{VSP}(q, n+1)^o$ embedded in $G(n,E)$ is contained in the linear subspace of codimension n.

1.21 Compute the catalecticant matrix $\mathrm{Cat}_2(f)$, where f is a homogeneous form of degree 4 in 3 variables.

1.22 Let $f \in S^{2k}(E^\vee)$ and Ω_f be the corresponding quadratic form on $S^k(E)$. Show that the quadric $V(\Omega_f)$ in $|S^k(E)|$ is characterized by the following two properties:

- Its preimage under the Veronese map $\nu_k : |E| \to |S^k(E)|$ is equal to $V(f)$;

- Ω_f is apolar to any quadric in $|S^k(E^\vee)|$ which contains the image of the Veronese map $|E^\vee| = \mathbb{P}(E) \to |S^k(E^\vee)| = \mathbb{P}(S^k(E))$.

1.23 Let C_k be the locus in $|S^{2k}(E^\vee)|$ of hypersurfaces $V(f)$ such that $\det \mathrm{Cat}_k(f) = 0$. Show that C_k is a rational variety. [Hint: Consider the rational map $C_k \dashrightarrow |E|$) which assigns to $V(f)$ the point defined by the subspace $\mathrm{AP}_k(f)$ and study its fibres].

1.24 Give an example of a polar 4-gon of the cubic $t_0t_1t_2 = 0$.

1.25 Find all binary forms of degree d for which $\mathrm{VSP}(f, 2)^o = \emptyset$.

1.26 Let f be a form of degree d in $n+1$ variables. Show that $\mathrm{VSP}(f, \binom{n+d}{d})^o$ is an irreducible variety of dimension $n\binom{n+d}{d}$.

1.27 Describe the variety $\mathrm{VSP}(f, 4)$, where f is a nondegenerate quadratic form in 3 variables.

1.28 Show that a smooth point y of a hypersurface X belongs to the intersection of the polar hypersurfaces $P_x(X)$ and $P_{x^2}(X)$ if and only if the line connecting x and y intersects X at the point y with multiplicity ≥ 3.

1.29 Show that the vertices of two polar tetrahedra of a nonsingular quadric in $\mathbb{P}^3$ are base points of a net of quadrics. Conversely, the set of 8 base points of a general net of quadrics can be divided in any way into two sets, each of two sets is the set of vertices of a polar tetrahedron of the same quadric[537].

1.30 Suppose two cubic plane curves $V(f)$ and $V(g)$ admit a common polar pentagon. Show that the determinant of the 6×6-matrix $[\mathrm{Cat}_1(f)\mathrm{Cat}_1(g)]$ vanishes [224].

Historical Notes

Although some aspects of the theory of polarity for conics were known to mathematicians of Ancient Greece, the theory originates in projective geometry, in the works of G. Desargues, G. Monge and J. Poncelet. For Desargues the polar of a conic was a generalization of the diameter of a circle (when the pole is taken at infinity). He referred to a polar line as a"transversale de l'ordonnance". According to the historical accounts found in [214], vol. II, and [134], p. 60, the name "polaire" was introduced by J. Gergonne. Apparently, the polars of curves of higher degree appear first in the works of E. Bobillier [45] and then, with introduction of projective coordinates, in the works of J. Plücker [447]. They were the first to realize the duality property of polars: if a point x belongs to the s-th polar of a point y with respect to a curve of degree d, then y belongs to the $(d-s)$-th polar of x with respect to the same curve. Many properties of polar curves were stated in a purely geometric way by J. Steiner [542], as was customary for him, with no proofs. Good historical accounts can be found in [41] and [432], p.279.

The Hessian and the Steinerian curves with their relations to the theory of polars were first studied by J. Steiner [542] who called them *conjugate Kerncurven*. The current name for the Hessian curve was coined by J. Sylvester

[554] in honor of O. Hesse who was the first to study the Hessian of a ternary cubic [288] under the name *der Determinante* of the form. The current name of the Steinerian curve goes back to G. Salmon [492] and L. Cremona [141]. The Cayleyan curve was introduced by A. Cayley in [72] who called it the *pippiana*. The current name was proposed by L. Cremona. Most of the popular classical text-books in analytic geometry contain an exposition of the polarity theory (e.g. [113], [214], [492]).

The theory of dual varieties, generalization of Plücker formulae to arbitrary dimension is still a popular subject of modern algebraic geometry. It is well-documented in modern literature and for this reason this topic is barely touched here.

The theory of apolarity was a very popular topic of classical algebraic geometry. It originates from the works of Rosanes [478] who called apolar forms of the same degree *conjugate forms* and Reye [462]. who introduced the term "apolar". The condition of polarity $D_\psi(f) = 0$ was viewed as vanishing of the simultaneous bilinear invariant of a form f of degree d and a form ψ of class d. It was called the *harmonizant*.. We refer for survey of classical results to [432] and to a modern exposition of some of these results to [183] which we followed here.

The Waring problem for homogeneous forms originates from a more general problem of finding a canonical form for a homogeneous form. Sylvester's result about reducing a cubic form in four variables to the sum of 5 powers of linear forms is one of the earliest examples of solution of the Waring problem. We will discuss this later in the book. F. Palatini was the first who recognized the problem as a problem about the secant variety of the Veronese variety [424], [425] and as a problem of the existence of envelopes with a given number of singular points (in less general form the relationship was found earlier by J. E. Campbell [60]). The Alexander-Hirschowitz Theorem was claimed by J. Bronowski [57] in 1933, but citing C. Ciliberto [100], he had only a plausibility argument. The case $n = 2$ was first established by F. Palatini [425] and the case $n = 3$ was solved by A. Terracini [560]. Terracini was the first to recognize the exceptional case of cubic hypersurfaces in $\mathbb{P}^4$ [559]. The original proof of Terracini's Lemma can be found in [561]. We also refer to [240] for a good modern survey of the problem. A good historical account and in depth theory of the Waring problems and the varieties associated to it can be found in the book of A. Iarrobino and V. Kanev [313].

The fact that a general plane quintic admits a unique polar 7-gon was first mentioned by D. Hilbert in his letter to C. Hermite [294]. The proofs were given later by Palatini [427] and H. Richmond [468], [470]

In earlier editions of his book [493] G. Salmon mistakenly applied counting

constants to assert that three general quadrics in $\mathbb{P}^3$ admit a common polar pentahedron. G. Darboux [153] was the fist to show that the counting of constants is wrong. W. Frahm [224] proved that the net of quadrics generated by three quadrics with a common polar pentahedron must be a net of polars of a cubic surface and also has the property that its discriminant curve is a Lüroth quartic, a plane quartic which admits an inscribed pentagon. In [564] E.Toeplitz (the father of Otto Toeplitz) introduced the invariant Λ of three quadric surfaces whose vanishing is necessary and sufficient for the existence of a common polar pentahedron. The fact that two general plane cubics do not admit a common polar pentagon was first discovered by F. London [366]. The Waring Problem continues to attract attention of contemporary mathematicians. Some references to the modern literature can found in this chapter.

2

Conics and quadric surfaces

2.1 Self-polar triangles

2.1.1 Veronese quartic surfaces

Let $\mathbb{P}^2 = |E|$ and $|S^2(E^\vee)| \cong \mathbb{P}^5$ be the space of conics in $\mathbb{P}^2$. Recall, for this special case, the geometry of the *Veronese quartic surface* V_2^2, the image of the Veronese map

$$\mathsf{v}_2 : |E^\vee| \to |S^2(E^\vee)|, \quad [l] \mapsto [l^2].$$

If we view $S^2(E^\vee)$ as the dual space of $S^2(E)$, then the Veronese surface parameterizes hyperplanes H_l in $S^2(E)$ of conics passing through the point $[l]$ in the dual plane $|E^\vee|$. The Veronese map v_2 is given by the complete linear system $|\mathcal{O}_{|E^\vee|}(2)| = |S^2(E)|$. Thus the preimage of a hyperplane in $|S^2(E^\vee)|$ is a conic in the plane $|E^\vee|$. The conic is singular if and only if the hyperplane is tangent to the Veronese surface. There are two possibilities, either the singular conic C is the union of two distinct lines (a line-pair), or it is equal to a double line. In the first case the hyperplane is tangent to the surface at a single point. The point is the image of the singular point $[l]$ of the conic. In the second case, the hyperplane is tangent to the Veronese surface along a curve R equal to the image of the line C_{red} under the restriction of the Veronese map. It follows that the curve R is a conic cut out on the Veronese surface by a plane. We see in this way that the *dual variety of the Veronese surface* is isomorphic to the discriminant cubic hypersurface $\mathsf{D}_2(2)$ parameterizing singular conics.

The tangent plane to the Veronese surface at a point $[l]^2$ is the intersection of hyperplanes which cut out a conic in $|E^\vee|$ with singular point $[l]$. The plane of conics in $|E|$ apolar to such conics is the plane of reducible conics with one component equal to the line $V(l)$.

Since any quadratic form of rank 2 in E can be written as a sum of quadratic forms of rank 1, the secant variety $\mathrm{Sec}_1(\mathsf{V}_2^2)$ coincides with $\mathsf{D}_2(2)$. Also, it

coincides with the *tangential variety* $\mathrm{Tan}(\mathsf{V}_2^2)$, the union of tangent planes $\mathbb{T}_x(\mathsf{V}_2^2), \mathsf{x} \in \mathsf{V}_2^2$. It is singular along the Veronese surface.

Choosing a basis in E we can identify the space $S^2(E^\vee)$ with the space of symmetric 3×3-matrices. The Veronese surface V_2^2 in $|S^2(E^\vee)|$ is identified with matrices of rank 1. Its equations are given by 2×2-minors. The variety of matrices of rank ≤ 2 is the cubic hypersurface whose equation is given by the determinant.

Let us look at a possible projection of V_2^2 to $\mathbb{P}^4$. It is given by a linear subsystem $|V|$ of $|S^2(E)|$. Let K be the apolar conic to all conics from $|V|$. It is a point $\mathfrak{o}$ in the dual space $|S^2(E^\vee)|$ equal to the center of the projection. The conic K could be nonsingular, a line-pair, or a double line. In the first two cases $\mathfrak{o} \notin \mathsf{V}_2^2$. The image of the projection is a quartic surface in $\mathbb{P}^4$, called a *projected Veronese surface*. If K is nonsingular, $\mathfrak{o}$ does not lie on $\mathrm{Sec}_1(\mathsf{V}_2^2)$, hence the projected Veronese surface is a nonsingular quartic surface in $\mathbb{P}^4 = \mathbb{P}(V)$. If K is a line-pair, then $\mathfrak{o}$ lies on a tangent plane of V_2^2 at some point $[l^2]$. Hence it lies on the plane spanning a conic contained in V_2^2. The restriction of the projection map to this conic is of degree 2, and its image is a double line on the projected Veronese surface. Two ramification points are mapped to two *pinch points* of the surface. Finally, $\mathfrak{o}$ could be on V_2^2. The image of the projection is a cubic surface S in $\mathbb{P}^4$. All conics on V_2^2 containing $\mathfrak{o}$ are projected to lines on S. So, S is a nonsingular cubic scroll in $\mathbb{P}^4$ isomorphic to the blow-up of V_2^2, hence of $\mathbb{P}^2$, at one point. In our future notation for rational normal scrolls (see 8.1.1), it is the scroll $S_{1,4}$.

Let us now project V_2^2 further to $\mathbb{P}^3$. This time, the linear system $|V|$ defining the projection is of dimension 3. Its apolar linear system is a pencil, a line ℓ in $|S^2(E^\vee)|$. Suppose the apolar pencil does not intersect V_2^2. In this case the pencil of conics does not contain a double line, hence contains exactly three line-pairs. The three line-pairs correspond to the intersection of ℓ with the cubic hypersurface $\mathrm{Sec}_1(\mathsf{V}_2^2)$. As we saw in above, this implies that the image S of the projection is a quartic surface with three double lines. These lines are concurrent. In fact, a pencil of plane sections of S containing one of the lines has residual conics singular at the points of intersection with the other two lines. Since the surface is irreducible, this implies that the other two lines intersect the first one. Changing the order of the lines, we obtain that each pair of lines intersect. This is possible only if they are concurrent (otherwise they are coplanar, and plane containing the lines intersect the quartic surface along a cubic taken with multiplicity 2).

The projection of a Veronese surface from a line not intersecting V_2^2 is called a *Steiner quartic*. Choose coordinates t_0, t_1, t_2, t_3 such that the equations of the singular lines are $t_1 = t_2 = 0, t_1 = t_3 = 0$ and $t_2 = t_3 = 0$. Then the equation

of a Steiner surface can be reduced to the form $t_0t_1t_2t_3 + g_4 = 0$. By taking the partial derivatives at the point $[1, 0, 0, 0]$ and general points of the singular lines, we find that g_4 is a linear combination of the monomial $t_1^2t_2^2, t_1^2t_3^2, t_2^2t_3^2$. Finally, by scaling the coordinates, we reduce the equation to the form

$$t_0t_1t_2t_3 + t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2 = 0. \tag{2.1}$$

An explicit birational map from $\mathbb{P}^2$ onto the surface is given by

$$[y_0, y_1, y_2] \mapsto [(-y_0+y_1+y_2)^2, (y_0-y_1+y_2)^2, (y_0+y_1-y_2)^2, (y_0+y_1+y_2)^2].$$

Next, we assume that the center of the projection is line ℓ intersecting V_2^2. In this case the image of the projection is a cubic scroll, the projection of the rational normal scroll $S_{1,4}$ to $\mathbb{P}^3$. There are two possibilities, the pencil of conics defined by ℓ has two singular members, or one singular member, a double line. This gives two possible cubic scrolls. We will give their equations in the next Chapter.

Replacing E with $|E^\vee|$ we can define the Veronese surface in $|S^2(E)|$, the image of the plane $|E|$ under the map given by the complete linear system of conics. We leave it to the reader to "dualize" the statements from above.

2.1.2 Polar lines

Let C be a nonsingular conic. For any point $a \in \mathbb{P}^2$, the first polar $P_a(C)$ is a line, the *polar line* of a. For any line ℓ there exists a unique point a such that $P_a(C) = l$. The point a is called the *pole* of ℓ. The point a considered as a line in the dual plane is the polar line of the point ℓ with respect to the dual conic $\check{C}$.

Borrowing terminology from the Euclidean geometry, we call three non-collinear lines in $\mathbb{P}^2$ a *triangle*. The lines themselves will be called the *sides* of the triangle. The three intersection points of pairs of sides are called the *vertices* of the triangle.

A set of three non-collinear lines ℓ_1, ℓ_2, ℓ_3 is called a *self-polar triangle* with respect to C if each ℓ_i is the polar line of C with respect to the opposite vertex. It is easy to see that it suffices that only two sides are polar to the opposite vertices.

Proposition 2.1.1 *Three lines $\ell_i = V(l_i)$ form a self-polar triangle for a conic $C = V(q)$ if and only if they form a polar triangle of C.*

Proof Let $\ell_i \cap \ell_j = [v_{ij}]$. If $q = l_1^2 + l_2^2 + l_3^2$, then $D_{v_{ij}}(q) = 2l_k$, where $k \neq i, j$. Thus a polar triangle of C is a self-conjugate triangle. Conversely, if $V(D_{v_{ij}}(q)) = \ell_k$, then $D_{v_{ik}v_{ij}}(q) = D_{v_{jk}v_{ij}}(q) = 0$. This shows that the

conic C is apolar to the linear system of conics spanned by the reducible conics $\ell_i + \ell_j$. It coincides with the linear system of conics through the three points ℓ_1, ℓ_2, ℓ_3 in the dual plane. Applying Proposition 1.3.10, we obtain that the self-conjugate triangle is a polar triangle.

Of course, we can prove the converse by computation. Let

$$2q = a_{00}t_0^2 + a_{11}t_1^2 + a_{22}t_2^2 + 2a_{01}t_0t_1 + 2a_{02}t_0t_2 + 2a_{12}t_1t_2 = 0.$$

Choose projective coordinates in $\mathbb{P}^2$ such that $\ell_i = V(t_i)$. Then

$$\begin{aligned} P_{[1,0,0]}(X) = \ell_1 = V(\frac{\partial q}{\partial t_0}) = V(a_{00}t_0 + a_{01}t_1 + a_{02}t_2), \\ P_{[0,1,0]}(X) = \ell_2 = V(\frac{\partial q}{\partial t_1}) = V(a_{11}t_1 + a_{01}t_0 + a_{12}t_2), \\ P_{[0,0,1]}(X) = \ell_2 = V(\frac{\partial q}{\partial t_2}) = V(a_{22}t_2 + a_{02}t_0 + a_{12}t_1) \end{aligned} \tag{2.2}$$

implies that $q = \frac{1}{2}(t_0^2 + t_1^2 + t_2^2)$. □

Remark 2.1.2 Similarly one can define a self-polar $(n+1)$-hedron of a quadric in $\mathbb{P}^n$ and prove that it coincides with its polar $(n+1)$-hedron. The proof of the existence of such $(n+1)$-hedron was the classical equivalent of the theorem from linear algebra about reduction of a quadratic form to principal axes.

Let $Q = V(q)$ and $Q' = V(q')$ be two quadrics in a projective space $\mathbb{P}^1$. We say that Q and Q' are *harmonically conjugate* if the dual quadric of Q is apolar to Q'. In other words, if $D_{q^\vee}(q') = 0$. In coordinates, if

$$q = \alpha t_0^2 + 2\beta t_0t_1 + \gamma t_1^2, \quad q' = \alpha' t_0^2 + 2\beta' t_0t_1 + \gamma' t_1^2.$$

then $q^\vee = \gamma\eta_0^2 - 2\beta\eta_0\eta_1 + \alpha\eta_1^2$, and the condition becomes

$$-2\beta\beta' + \alpha\gamma' + \alpha'\gamma = 0. \tag{2.3}$$

It shows that the relation is symmetric (one can extend it to quadrics in higher-dimensional spaces but it will not be symmetric).

Of course, a quadric in $\mathbb{P}^1$ can be identified with a set of two points in $\mathbb{P}^1$, or one point with multiplicity 2. This leads to the classical definition of *harmonically conjugate* $\{a, b\}$ and $\{c, d\}$ in $\mathbb{P}^1$. We will see later many other equivalent definitions of this relation.

Let $\mathbb{P}^1 = |U|$, where $\dim U = 2$. Since $\dim \bigwedge^2 U = 1$, we can identify $|E|$ with $\mathbb{P}(E)$. Explicitly, a point with coordinates $[a, b]$ is identified with a point $[-b, a]$ in the dual coordinates. Under this identification, the dual quadric $q^\vee$

vanishes at the zeros of q. Thus, (2.3) is equivalent to the polarity condition

$$D_{cd}(q) = D_{ab}(q') = 0, \tag{2.4}$$

where $V(q) = \{a, b\}, V(q') = \{c, d\}$.

Proposition 2.1.3 *Let ℓ_1, ℓ_2, ℓ_3 be a triangle with vertices $a = \ell_1 \cap \ell_2, b = \ell_1 \cap \ell_3$ and $c = \ell_2 \cap \ell_3$. Then the triangle is a self-polar triangle of a conic C if and only if $a \in P_b(C) \cap P_c(C)$ and the pairs of points $C \cap \ell_3$ and (b, c) are harmonically conjugate.*

Proof Consider the pair $C \cap \ell_3$ as a quadric q in ℓ_3. We have $c \in P_b(C)$, thus $D_{bc}(q) = 0$. Restricting to ℓ_3 and by using (2.4), we see that the pairs b, c and $C \cap \ell_3$ are harmonically conjugate. Conversely, if $D_{bc}(q) = 0$, the polar line $P_b(C)$ contains a and intersects ℓ_3 at c, hence coincides with $\overline{ac}$. Similarly, $P_c(C) = \overline{ab}$. □

Any triangle in $\mathbb{P}^2$ defines the dual triangle in the dual plane $(\mathbb{P}^2)^\vee$. Its sides are the pencils of lines with the base point of one of the vertices.

Corollary 2.1.4 *The dual of a self-polar triangle of a conic C is a self-polar triangle of the dual conic $\check{C}$.*

2.1.3 The variety of self-polar triangles

Here, by more elementary methods, we will discuss a compactification of the variety $\mathrm{VSP}(q, 3)$ of polar triangles of a nondegenerate quadratic form in three variables.

Let C be a nonsingular conic. The group of projective transformations of $\mathbb{P}^2$ leaving C invariant is isomorphic to the projective complex orthogonal group

$$\mathrm{PO}(3) = \mathrm{O}(3)/(\pm I_3) \cong \mathrm{SO}(3).$$

It is also isomorphic to the group $\mathrm{PSL}(2)$ via the Veronese map

$$\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2, \quad [t_0, t_1] \mapsto [t_0^2, t_0 t_1, t_1^2].$$

Obviously, PO_3 acts transitively on the set of self-polar triangles of C. We may assume that $C = V(\sum t_i^2)$. The stabilizer subgroup of the self-polar triangle defined by the coordinate lines is equal to the subgroup generated by permutation matrices and orthogonal diagonal matrices. It is easy to see that it is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3 \cong \mathfrak{S}_4$. Thus we obtain the following.

Theorem 2.1.5 *The set of self-polar triangles of a nonsingular conic has a structure of a homogeneous space SO_3/Γ, where Γ is a finite subgroup isomorphic to $\mathfrak{S}_4$.*

A natural compactification of the variety of self-conjugate triangles of a non-degenerate conic q is the variety $\mathrm{VSP}(q, 3)$ which we discussed in the previous chapter. In Theorem 1.5.7, we have shown that it is isomorphic to the intersection of the Grassmannian $G(3, 5)$ with a linear subspace of codimension 3. Let us see this construction in another way, independent of the theory developed in the previous chapter.

Let $V = \mathsf{V}_2^2$ be a Veronese surface in $\mathbb{P}^5$. We view $\mathbb{P}^5$ as the projective space of conics in $\mathbb{P}^2$ and V_2^2 as its subvariety of double lines. A trisecant plane of V is spanned by three linearly independent double lines. A conic $C \in \mathbb{P}^5$ belongs to this plane if and only if the corresponding three lines form a self-polar triangle of C. Thus the set of self-polar triangles of C can be identified with the set of trisecant planes of the Veronese surface which contain C. The latter will also include *degenerate self-polar triangles* corresponding to the case when the trisecant plane is tangent to the Veronese surface at some point. Projecting from C to $\mathbb{P}^4$ we will identify the set of self-polar triangles (maybe degenerate) with the set of trisecant lines of the projected Veronese surface V_4. This is a closed subvariety of the Grassmann variety $G_1(\mathbb{P}^4)$ of lines in $\mathbb{P}^4$.

Let E be a linear space of odd dimension $2k+1$ and let $G(2, E) := G_1(|E|)$ be the Grassmannian of lines in $|E|$. Consider its Plücker embedding $\bigwedge^2 : G(2, E) \hookrightarrow G_1(\bigwedge^2 E) = |\bigwedge^2 E|$. Any nonzero $\omega \in (\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$ defines a hyperplane H_ω in $|\bigwedge^2 E|$. Consider ω as a linear map $\alpha_\omega : E \to E^\vee$ defined by $\alpha_\omega(v)(w) = \omega(v, w)$. The map α_ω is skew-symmetric in the sense that its transpose map coincides with $-\alpha_\omega$. Thus its determinant is equal to zero, and $\mathrm{Ker}(\alpha_\omega) \neq \{0\}$. Let v_0 be a nonzero element of the kernel. Then for any $v \in E$ we have $\omega(v_0, v) = \alpha_\omega(v)(v_0) = 0$. This shows that ω vanishes on all decomposable 2-vectors $v_0 \wedge v$. This implies that the intersection of the hyperplane H_ω with $G(2, E)$ contains all lines which intersect the linear subspace $C_\omega = |\mathrm{Ker}(\alpha_\omega)| \subset |E|$ which we call the *pole* of the hyperplane H_ω.

Now recall the following result from linear algebra (see Exercise 2.1). Let A be a skew-symmetric matrix of odd size $2k + 1$. Its principal submatrices A_i of size $2k$ (obtained by deleting the i-th row and the i-th column) are skew-symmetric matrices of even size. Let Pf_i be the pfaffians of A_i (i.e. $\det(A_i) = \mathrm{Pf}_i^2$). Assume that $\mathrm{rank}(A) = 2k$, or, equivalently, not all Pf_i vanish. Then the system of linear equations $A \cdot x = 0$ has 1-dimensional null-space generated by the vector $(a_1, \ldots, a_{2k+1})$, where $a_i = (-1)^{i+1}\mathrm{Pf}_i$.

Let us go back to Grassmannians. Suppose we have an $s+1$-dimensional subspace W in $\bigwedge^2 E^\vee$ spanned by $\omega_0, \ldots, \omega_s$. Suppose that, for any $\omega \in W$, we have rank $\alpha_\omega = 2k$, or, equivalently, the pole C_ω of H_ω is a point. It follows from the theory of determinant varieties that the subvariety

$$\{\mathbb{C}\omega \in |\bigwedge^2 E^\vee| : \text{corank } \alpha_\omega \geq i\}$$

is of codimension $\binom{i}{2}$ in $|\bigwedge^2 E^\vee|$ (see [279], [341]). Thus, if $s < 4$, a general W will satisfy the assumption. Consider a regular map $\Phi : |W| \to |E|$ defined by $\omega \mapsto C_\omega$. If we take $\omega = t_0\omega_0 + \cdots + t_s\omega_s$ so that $t = (t_0, \ldots, t_s)$ are projective coordinate functions in $|W|$, we obtain that Φ is given by $2k+1$ principal pfaffians of the matrix A_t defining ω.

We shall apply the preceding to the case when $\dim E = 5$. Take a general 3-dimensional subspace W of $\bigwedge^2 E^\vee$. The map $\Phi : |W| \to |E| \cong \mathbb{P}^4$ is defined by homogeneous polynomials of degree 2. Its image is a projected Veronese surface S. Any trisecant line of S passes through 3 points on S which are the poles of elements w_1, w_2, w_3 from W. These elements are linearly independent, otherwise their poles lie on the conic image of a line under Φ. But no trisecant line can be contained in a conic plane section of S. We consider $\omega \in W$ as a hyperplane in the Plücker space $|\bigwedge^2 E|$. Thus, any trisecant line is contained in all hyperplanes defined by W. Now, we are ready to prove the following.

Theorem 2.1.6 *Let $\bar{X}$ be the closure in $G_1(\mathbb{P}^4)$ of the locus of trisecant lines of a projected Veronese surface. Then $\bar{X}$ is equal to the intersection of $G_1(\mathbb{P}^4)$ with three linearly independent hyperplanes. In particular, $\bar{X}$ is a Fano 3-fold of degree 5 with canonical sheaf $\omega_{\bar{X}} \cong \mathcal{O}_{\bar{X}}(-2)$.*

Proof As we observed in above, the locus of poles of a general 3-dimensional linear space W of hyperplanes in the Plücker space is a projected Veronese surface V and its trisecant variety is contained in $Y = \cap_{w\in W} H_w \cap G_1(\mathbb{P}^4)$. So, its closure $\bar{X}$ is also contained in Y. On the other hand, we know that $\bar{X}$ is irreducible and 3-dimensional (it contains an open subset isomorphic to the homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$). By Bertini's Theorem the intersection of $G_1(\mathbb{P}^4)$ with a general linear space of codimension 3 is an irreducible 3-dimensional variety. This proves that $Y = \bar{X}$. By another Bertini's Theorem, Y is smooth. The rest is the standard computation of the canonical class of the Grassmann variety and the adjunction formula. It is known that the canonical class of the Grassmannian $G = G_m(\mathbb{P}^n)$ of m-dimensional subspaces of $\mathbb{P}^n$ is equal to

$$K_G = \mathcal{O}_G(-n-1). \tag{2.5}$$

By the adjunction formula, the canonical class of $\bar{X} = G_1(\mathbb{P}^4) \cap H_1 \cap H_2 \cap H_3$ is equal to $\mathcal{O}_{\bar{X}}(-2)$. □

Corollary 2.1.7 *The homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$ admits a smooth compactification $\bar{X}$ isomorphic to the intersection of $G_1(\mathbb{P}^4)$, embedded via Plücker in $\mathbb{P}^9$, with a linear subspace of codimension 3. The boundary $\bar{X} \setminus X$ is an anticanonical divisor cut out by a hypersurface of degree 2.*

Proof The only unproven assertion is one about the boundary. To check this, we use that the 3-dimensional group $G = \mathrm{SL}(2)$ acts transitively on a 3-dimensional variety X minus the boundary. For any point $x \in X$, consider the map $\mu_x : G \to X, g \mapsto g \cdot x$. Its fiber over the point x is the isotropy subgroup G_x of x. The differential of this map defines a linear map $\mathfrak{g} = T_e(G) \to T_x(X)$. When we let x vary in X, we get a map of vector bundles

$$\phi : \mathfrak{g}_X = \mathfrak{g} \times X \to T(X).$$

Now take the determinant of this map

$$\bigwedge^3 \phi = \bigwedge^3 \mathfrak{g} \times X \to \bigwedge^3 T(X) = K_X^\vee,$$

where K_X is the canonical line bundle of X.

□

Remark 2.1.8 There is another construction of the variety $\mathrm{VSP}(q, 3)$ due to S. Mukai and H. Umemura [397]. Let V_6 be the space of homogeneous binary forms $f(t_0, t_1)$ of degree 6. The group $\mathrm{SL}(2)$ has a natural linear representation in V_6 via linear change of variables. Let $f = t_0 t_1 (t_0^4 - t_1^4)$. The zeros of this polynomials are the vertices of a regular octahedron inscribed in $S^2 = \mathbb{P}^1(\mathbb{C})$. The stabilizer subgroup of f in $\mathrm{SL}(2)$ is isomorphic to the binary octahedron group $\Gamma \cong \mathfrak{S}_4$. Consider the projective linear representation of $\mathrm{SL}(2)$ in $|V_6| \cong \mathbb{P}^5$. In the loc. cit. it is proven that the closure $\bar{X}$ of this orbit in $|V_6|$ is smooth and $B = \bar{X} \setminus X$ is the union of the orbits of $[t_0^5 t_1]$ and $[t_0^6]$. The first orbit is of dimension 2. Its isotropy subgroup is isomorphic to the multiplicative group $\mathbb{C}^*$. The second orbit is 1-dimensional and is contained in the closure of the first one. The isotropy subgroup is isomorphic to the subgroup of upper triangular matrices. One can also show that B is equal to the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under a $\mathrm{SL}(2)$-equivariant map given by a linear system of curves of bidegree $(5, 1)$. Thus B is of degree 10, hence is cut out by a quadric. The image of the second orbit is a smooth rational curve in B and is equal to the singular locus of B. The fact that the two varieties are isomorphic follows from the theory

of Fano 3-folds. It can be shown that there is a unique Fano threefold V with $\mathrm{Pic}(V) = \mathbb{Z}\frac{1}{2}K_V$ and $K_V^3 = 40$.

2.1.4 Conjugate triangles

Let $C = V(f)$ be a nonsingular conic. Given a triangle with sides ℓ_1, ℓ_2, ℓ_3, the poles of the sides are the vertices of the triangle which is called the *conjugate triangle*. Its sides are the polar lines of the vertices of the original triangle. It is clear that this defines a duality in the set of triangles. Clearly, a triangle is *self-conjugate* if and only if it is a self-polar triangle.

The following is an example of conjugate triangles. Let ℓ_1, ℓ_2, ℓ_3 be three tangents to C at the points p_1, p_2, p_3, respectively. They form a triangle which can be viewed as a *circumscribed triangle*. It follows from Theorem 1.1.5 that the conjugate triangle has vertices p_1, p_2, p_3. It can be viewed as an *inscribed triangle*. The lines $\ell_1' = \overline{p_2p_3}, \ell_2' = \overline{p_1p_3}, \ell_3' = \overline{p_1p_2}$ are polar lines with respect to the vertices q_1, q_2, q_3 of the circumscribed triangle (see the picture).

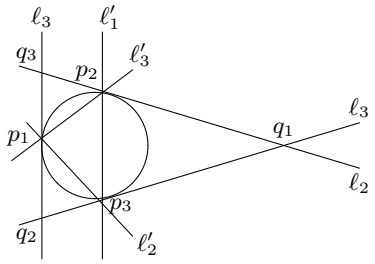


Figure 2.1 Special conjugate triangles

In general, let a side ℓ_i of a triangle Δ intersect the conic C at p_i and p_i'. Then the vertices of the conjugate triangle are the intersection points of the tangent of C at the points p_i, p_i'.

Two lines in $\mathbb{P}^2$ are called *conjugate* with respect to C if the pole of one of the lines belongs to the other line. It is a reflexive relation on the set of lines. Obviously, two triangles are conjugate if and only if each of the sides of the first triangle is conjugate to a side of the second triangle.

Recall the basic notion of projective geometry, the *perspectivity*. Two triangles are called *perspective* from a line (resp. from a point) if there exists a bijection between their sets of sides (resp. vertices) such that the intersection points of the corresponding sides (resp. the lines joining the corresponding points) lie on the same line (resp. intersect at one point). The line is called the *line of perspectivity* or *perspectrix*, and the point is called the *center of per-*

spectivity or *perspector*. The *Desargues Theorem* asserts that the properties of being perspective from a line or from a point are equivalent.

Theorem 2.1.9 (M. Chasles) *Two conjugate triangles with no common vertex are perspective.*

Proof Choose coordinates such that the sides ℓ_1, ℓ_2, ℓ_3 of the first triangle are $t_0 = 0, t_1 = 0, t_2 = 0$, respectively. Then the vertices of the first triangle $\ell_2 \cap \ell_3 = p_1 = [1,0,0], \ell_1 \cap \ell_3 = p_2 = [0,1,0]$ and $\ell_1 \cap \ell_3 = p_3 = [0,0,1]$. Let

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \tag{2.6}$$

be the symmetric matrix defining the conic. Then the polar lines ℓ_i' of the point p_i is given by the equation $\alpha t_0 + \beta t_1 + \gamma t_2 = 0$, where (α, β, γ) is the i-th column of A. The vertices of the conjugate triangle are $\ell_1 \cap \ell_1' = (0, c, -b), \ell_2 \cap \ell_2' = (e, 0, -b)$ and $\ell_3 \cap \ell_3' = (e, -c, -0)$. The condition that the points are on a line is the vanishing of the determinant

$$\det \begin{pmatrix} 0 & c & -b \\ e & 0 & -b \\ e & -c & 0 \end{pmatrix}.$$

Computing the determinant, we verify that it indeed vanishes. □

Now let us consider the following problem. Given two triangles $\{\ell_1, \ell_2, \ell_3\}$ and $\{\ell_1', \ell_2', \ell_3'\}$ without common sides, find a conic C such that the triangles are conjugate to each other with respect to C.

Since $\dim \bigwedge^3 E = 1$, we can define a natural isomorphism $|\bigwedge^2 E^\vee| \to |E|$. Explicitly, it sends the line $[l \wedge l']$ to the intersection point $[l] \cap [l']$. Suppose the two triangles are conjugate with respect to a conic C. Let $|E| \to |E^\vee|$ be the isomorphism defined by the conic. The composition $|\bigwedge^2 E^\vee| \to |E| \to |E^\vee|$ must send $\ell_i \wedge \ell_j$ to ℓ_k'. Let $\ell_i = [l_i], \ell_i' = [l_i']$. Choose coordinates t_0, t_1, t_2 in E and let X, Y be the 3×3-matrices with j-row equal to coordinates of ℓ_i and ℓ_i', respectively. Of course, these matrices are defined by the triangles only up to scaling the columns. It is clear that the k-column of the inverse matrix X^{-1} can be taken for the coordinates of the point $\ell_i \cap \ell_j$ (here $i \neq j \neq k$). Now we are looking for a symmetric matrix A such that $AX^{-1} = {}^tY$. The converse is also true. If we find such a matrix, the rows of X and Y would represent two conjugate triangles with respect to the conic defined by the matrix A. Fix some coordinates of the sides of the two triangles to fix the matrices X, Y. Then we

are looking for a diagonal invertible matrix D such that

$$QA = {}^tYDX \quad \text{is a symmetric matrix.} \tag{2.7}$$

There are three linear conditions $a_{ij} = a_{ji}$ for a matrix $A = (a_{ij})$ to be symmetric. So we have three equations and we also have three unknowns, the entries of the matrix D. The condition for the existence of a solution must be given in terms of a determinant whose entries depend on the coordinates of the sides of the triangles. We identify l_i and l'_i with vectors in $\mathbb{C}^3$ and use the dot-product in $\mathbb{C}^3$ to get the following three equations with unknowns $\lambda_1, \lambda_2, \lambda_3$

$$\begin{aligned}
\lambda_1 l_1 \cdot l'_2 - \lambda_2 l_2 \cdot l'_1 &= 0 \\
\lambda_1 l_1 \cdot l'_3 - \lambda_3 l_3 \cdot l'_1 &= 0 \\
\lambda_2 l_2 \cdot l'_3 - \lambda_3 l_3 \cdot l'_2 &= 0.
\end{aligned}$$

The matrix of the coefficients of the system of linear equations is equal to

$$M = \begin{pmatrix} l_1 \cdot l'_2 & -l_2 \cdot l'_1 & 0 \\ l_1 \cdot l'_3 & 0 & -l_3 \cdot l'_1 \\ 0 & l_2 \cdot l'_3 & -l_3 \cdot l'_2 \end{pmatrix}.$$

The necessary condition is that

$$\det M = (l_3 \cdot l'_1)(l_1 \cdot l'_2)(l_2 \cdot l'_3) - (l_2 \cdot l'_1)(l_1 \cdot l'_3)(l_3 \cdot l'_2) = 0. \tag{2.8}$$

We also need a solution with nonzero coordinates. It is easy to check (for example, by taking coordinates where X or Y is the identity matrix), that the existence of a solution with a zero coordinate implies that the triangles have a common vertex. This contradicts our assumption.

Note that condition (2.7) is invariant with respect the action of $\mathrm{GL}(E)$ since any $G \in \mathrm{GL}(E)$ transforms X, Y to GX, GY, and hence transforms A to ${}^tGAG$ which is still symmetric. Taking $l_1 = t_0, l_2 = t_1, l_3 = t_2$, we easily check that condition (2.8) is equivalent to the condition that the two triangles with sides defined by l_1, l_2, l_3 and l'_1, l'_2, l'_3 are perspective from a line. Thus we obtain the following.

Corollary 2.1.10 *Two triangles with no common side are conjugate triangles with respect to some conic if and only if they are perspective triangles.*

Taking the inverse of the matrix A from (2.7), we obtain that $X^{-1}D^{-1}{}^tB^{-1}$ is symmetric. It is easy to see that the j-th column of X^{-1} can be taken for the coordinates of the side of the triangle opposite the vertex defined by the j-th column of X. This shows that the dual triangles are conjugate with respect to the dual quadric defined by the matrix A^{-1}. This proves Desargues' Theorem, we used before.

Theorem 2.1.11 (G. Desargues) *Two triangles are perspective from a point if and only if they are perspective from a line.*

Let C be a nonsingular conic and $\mathfrak{o}$ be a point in the plane but not in C. The projection from $\mathfrak{o}$ defines an involution $\tau_{\mathfrak{o}}$ on C with two fixed points equal to the set $P_{\mathfrak{o}}(C) \cap C$. This involution can be extended to the whole plane such that $\mathfrak{o}$ and the polar line $P_{\mathfrak{o}}$ is its set of fixed points. To show this, we may assume C is the conic $V(t_0t_2 - t_1^2)$, image of the Veronese map $\nu_2 : \mathbb{P}^1 \to C, [u_0, u_1] \mapsto [u_0^2, u_0u_1, u_1^2]$. We identify a point $x = [x_0, x_1, x_2]$ in the plane with a symmetric matrix

$$X = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}$$

so that the conic is given by the equation $\det X = 0$. Consider the action of $G \in SL(2)$ on $\mathbb{P}^2$ which sends X to ${}^tGXG$. This defines an isomorphism from $\mathrm{PSL}(2)$ to the subgroup of $\mathrm{PGL}(3)$ leaving the conic C invariant. In this way, any automorphism of C extends to a projective transformation of the plane leaving C invariant. Any nontrivial element of finite order in $\mathrm{PGL}(3)$ is represented by a diagonalizable matrix, and hence its set of fixed points consists of either a line plus a point, or 3 isolated points. The first case occurs when there are two equal eigenvalues, the second one when all eigenvalues are distinct. In particular, an involution belongs to the first case. It follows from the definition of the involution τ that the two intersection points of $P_{\mathfrak{o}}(C)$ with C are fixed under the extended involution $\tilde{\tau}$. So, the point $\mathfrak{o}$, being the intersection of the tangents to C at these points, is fixed. Thus the set of fixed points of the extended involution $\tilde{\tau}$ is equal to the union of the line $P_{\mathfrak{o}}(C)$ and the point $\mathfrak{o}$.

As an application, we get a proof of the following Pascal's Theorem from projective geometry.

Theorem 2.1.12 *Let $p_1, \ldots, p_6$ be the set of vertices of a hexagon inscribed in a nonsingular conic C. Then the intersection points of the opposite sides $\overline{p_ip_{i+1}} \cap \overline{p_{i+3}p_{i+4}}$, where i is taken modulo 3, are collinear.*

Proof A projective transformation of $\mathbb{P}^1$ is uniquely determined by the images of three distinct points. Consider the transformation of the conic C (identified with $\mathbb{P}^1$ by a Veronese map) which transforms p_i to $p_{i+3}, i = 1, 2, 3$. This transformation extends to a projective transformation τ of the whole plane leaving C invariant. Under this transformation, the pairs of the opposite sides $\overline{p_ip_{i+3}}$ are left invariant, thus their intersection point is fixed. A projective transformation with three fixed points on a line, fixes the line pointwise. So, all three intersection points lie on a line. □

The line joining the intersection points of opposite sides of a hexagon is called the *Pascal line*. Changing the order of the points, we get 60 Pascal lines associated with six points on a conic.

One can see that the triangle Δ_1 with sides $\overline{p_1p_2}, \overline{p_1p_6}, \overline{p_2p_3}$ and the triangle Δ_2 with sides $\overline{p_4p_5}, \overline{p_3p_4}, \overline{p_5p_6}$ are in perspective from the Pascal line. Hence they are perspective from the pole of the Pascal line with respect to the conic. Note that not all vertices of the triangles are on the conic.

Dually, we obtain *Brianchon's Theorem*.

Theorem 2.1.13 *Let $p_1, \dots, p_6$ be the set of vertices of a hexagon whose sides touch a nonsingular conic C. Then the diagonals $\overline{p_ip_{i+3}}, i = 1, 2, 3$ intersect at one point.*

We leave it to the reader to find two perspective triangles in this situation.

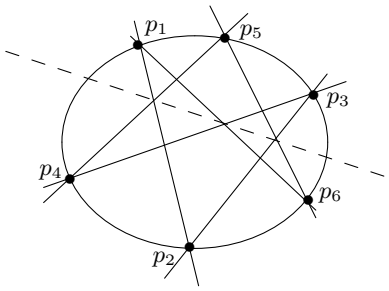


Figure 2.2 Pascal's Theorem

We view a triangle as a point in $(\mathbb{P}^2)^3$. Thus the set of ordered pairs of conjugate triangles is an open subset of the hypersurface in $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3 = (\mathbb{P}^2)^6$ defined by Equation (2.8). The equation is multilinear and is invariant with respect to the projective group $\mathrm{PGL}(3)$ acting diagonally, with respect to the cyclic group of order 3 acting diagonally on the product $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3$, and with respect to the switch of the factors in the product $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3$. It is known from the invariant theory that the determinant of the matrix M, considered as a section of the sheaf $H^0((\mathbb{P}^2)^6, \mathcal{O}_{\mathbb{P}^2}^{\boxtimes 6})$ must be a linear combination of the products of the maximal minors (ijk) of the matrix whose columns are the six vectors $l_1, l_1', l_2, l_2', l_3, l_3'$ such that each columns occurs in the product once. We use that $\det M = 0$ expresses the condition that the intersection points $\ell_i \cap \ell_i'$ are collinear.

Fix a basis in $\Lambda^3(E)$ to define a natural isomorphism

$$\bigwedge^2(\bigwedge^2 E) \to E,\ (v_1 \wedge v_2, w_1 \wedge w_2) \mapsto (v_1 \wedge v_2 \wedge w_1)w_2 - (v_1 \wedge v_2 \wedge w_2)w_1.$$

This corresponds to the familiar identity for the vector product of 3-vectors

$$(v_1 \times v_2) \times (w_1 \wedge w_2) = (v_1 \times v_2 \times w_1)w_2 - (v_1 \times v_2 \times w_2)w_1.$$

If we apply this formula to $E^\vee$ instead of E, we obtain that the line spanned by the points $\ell_1 \cap \ell_1'$ and $\ell_2 \cap \ell_2'$ has equation $\det(l_1, l_1', l_2)l_2' - \det(l_1, l_1', l_2')l_2 = 0$. The condition that this line also passes through the intersection point $\ell_3 \cap \ell_3'$ is

$$\det(l_3, l_3', \det(l_1, l_1', l_2)l_2' - \det(l_1, l_1', l_2')l_2)$$

$$= \det(l_1, l_1', l_2)\det(l_3, l_3', l_2') - \det(l_1, l_1', l_2')\det(l_3, l_3', l_2) = 0.$$

This shows that the determinant in (2.8) can be written in symbolic form as

$$(12, 34, 56) := (123)(456) - (124)(356). \tag{2.9}$$

Remark 2.1.14 Let $X = (\mathbb{P}^2)^{[3]}$ be the Hilbert scheme of $\mathbb{P}^2$ of 0-cycles of degree 3. It is a minimal resolution of singularities of the 3d symmetric product of $\mathbb{P}^2$. Consider the open subset of X formed by unordered sets of 3 non-collinear points. We may view a point of U as a triangle. Thus any nonsingular conic C defines an automorphism g_C of U of order 2. Its set of fixed points is equal to the variety of self-polar triangles of C. The automorphism of U can be viewed as a birational automorphism of X.

One can also give a moduli-theoretical interpretation of the 3-dimensional GIT-quotient of the variety X modulo the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ leaving the conic C invariant. Consider the intersection of the sides of the triangle with vertices a, b, c. They define three pairs of points on the conic. Assume that the six points are distinct. The double cover of the conic branched over six distinct points is a hyperelliptic curve B of genus 2. The three pairs define 3 torsion divisor classes which generate a maximal isotropic subspace in the group of 2-torsion points in the Jacobian variety of the curve B (see Chapter 5). This gives a point in the moduli space of principally polarized abelian surfaces together with a choice of a maximal isotropic subspace of 2-torsion points. It is isomorphic to the quotient of the Siegel space $\mathcal{H}_2$ modulo the group $\Gamma_0(2)$ of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z})$ such that $C \equiv 0 \mod 2$.

2.2 Poncelet relation

2.2.1 Darboux's Theorem

Let C be a conic, and let $T = \{\ell_1, \ell_2, \ell_3\}$ be a circumscribed triangle. A conic C' which has T as an inscribed triangle is called the *Poncelet related conic*.

Since passing through a point impose one condition, we have ∞^2 Poncelet related conics corresponding to a fixed triangle T. Varying T, we expect to get ∞^5 conics, so that any conic is Poncelet related to C with respect to some triangle. But surprisingly this is wrong! Darboux's Theorem asserts that there is a pencil of divisors $p_1 + p_2 + p_3$ such that the triangles T with sides tangent to C at the points p_1, p_2, p_3 define the same Poncelet related conic.

We shall prove it here. In fact, we shall prove a more general result, in which triangles are replaced with n-polygons. An *n-polygon* P in $\mathbb{P}^2$ is an ordered set of $n \geq 3$ points $(p_1, \ldots, p_n)$ in $\mathbb{P}^2$ such that no three points p_i, p_{i+1}, p_{i+2} are collinear. The points p_i are the *vertices* of P, the lines $\overline{p_i, p_{i+1}}$ are called the *sides* of P (here $p_{n+1} = p_1$). The number of n-gons with the same set of vertices is equal to $n!/2n = (n-1)!/2$.

We say that P circumscribes a nonsingular conic C if each side is tangent to C. Given any ordered set $(q_1, \ldots, q_n)$ of n points on C, let ℓ_i be the tangent lines to C at the points q_i. Then they are the sides of the n-gon P with vertices $p_i = \ell_i \cap \ell_{i+1}, i = 1, \ldots, n$ ($\ell_{n+1} = \ell_1$). The n-gon P circumscribes C. This gives a one-to-one correspondence between n-gons circumscribing C and ordered sets of n points on C.

Let $P = (p_1, \ldots, p_n)$ be a n-gon that circumscribes a nonsingular conic C. A conic S is called *Poncelet n-related* to C with respect to P if all points p_i lie on C.

Let us start with any two conics C and S. We choose a point p_1 on S and a tangent ℓ_1 to C passing through p_1. It intersects S at another point p_2. We repeat this construction. If the process stops after n steps (i.e. we are not getting new points p_i), we get an inscribed n-gon in S which circumscribes C. In this case S is Poncelet related to C. The *Darboux Theorem* which we will prove later says that, if the process stops, we can construct infinitely many n-gons with this property starting from an arbitrary point on S.

Consider the following correspondence on $C \times S$:

$$R = \{(x, y) \in C \times S : \overline{xy} \text{ is tangent to } C \text{ at } x\}.$$

Since, for any $x \in C$ the tangent to C at x intersects S at two points, and, for any $y \in S$ there are two tangents to C passing through y, so we get that R is of bidegree $(2, 2)$. This means if we identify C, S with $\mathbb{P}^1$, then R is a curve of bidegree $(2, 2)$. As is well-known R is a curve of arithmetic genus 1.

Lemma 2.2.1 *The curve R is nonsingular if and only if the conics C and S intersect at four distinct points. In this case, R is isomorphic to the double cover of C (or S) ramified over the four intersection points.*

Proof Consider the projection map $\pi_S : R \to S$. This is a map of degree 2.

A branch point $y \in S$ is a point such that there only one tangent to C passing through y. Obviously, this is possible only if $y \in C$. It is easy to see that R is nonsingular if and only if the double cover $\pi_S : R \to S \cong \mathbb{P}^1$ has four branch points. This proves the assertion. □

Note that, if R is nonsingular, the second projection map $\pi_C : R \to C$ must also have 4 branch points. A point $x \in C$ is a branch point if and only if the tangent of C at x is tangent to S. So we obtain that two conics intersect transversally if and only if there are four different common tangents.

Take a point $(x[0], y[0]) \in R$ and let $(x[1], y[1]) \in R$ be defined as follows: $y[1]$ is the second point on S on the tangent to $x[0]$, $x[1]$ is the point on C different from $x[0]$ at which a line through $y[1]$ is tangent to C. This defines a map $\tau_{C,S} : R \to R$. This map has no fixed points on R and hence, if we fix a group law on R, is a translation map t_a with respect to a point a. Obviously, we get an n-gon if and only if t_a is of order n, i.e. the order of a in the group law is n. As soon as this happens we can use the automorphism for constructing n-gons starting from an arbitrary point $(x[0], y[0])$. This is Darboux's Theorem which we have mentioned in above.

Theorem 2.2.2 (G. Darboux) *Let C and S be two nondegenerate conics intersecting transversally. Then C and S are Poncelet n-related if and only if the automorphism $\tau_{C,S}$ of the associated elliptic curve R is of order n. If C and S are Poncelet n related, then starting from any point $x \in C$ and any point $y \in S$ there exists an n-gon with a vertex at y and one side tangent to C at y which circumscribes C and inscribed in S.*

In order to check explicitly whether two conics are Poncelet related one needs to recognize when the automorphism $\tau_{C,S}$ is of finite order. Let us choose projective coordinates such that C is the Veronese conic $t_0t_2 - t_1^2 = 0$, the image of $\mathbb{P}^1$ under the map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$. By using a projective transformation leaving C invariant we may assume that the four intersection points p_1, p_2, p_3, p_4 of C and S are the images of the points $0, 1, \infty, a$. Then R is isomorphic to the elliptic curve given by the affine equation

$$y^2 = x(x-1)(x-a).$$

The conic S belongs to the pencil of conics with base points $p_1, \ldots, p_4$:

$$(t_0t_2 - t_1^2) + \lambda t_1(at_0 - (1+a)t_1 + t_2) = 0.$$

We choose the zero point in the group law on R to be the point $(x[0], y[0]) = (p_4, p_4) \in C \times S$. Then the automorphism $\tau_{C,S}$ sends this point to $(x[1], y[1])$,

where

$$y[1] = (\lambda a, \lambda(1+a)+1, 0), \quad x[1] = ((a+1)^2\lambda^2, 2a(1+a)\lambda, 4a^2).$$

Thus $x[1]$ is the image of the point $(1, \frac{2a}{(a+1)\lambda}) \in \mathbb{P}^1$ under the Veronese map. The point $y[1]$ corresponds to one of the two roots of the equation

$$y^2 = \frac{2a}{(a+1)\lambda}(\frac{2a}{(a+1)\lambda} - 1)(\frac{2a}{(a+1)\lambda} - a).$$

So we need a criterion characterizing points $(x, \pm\sqrt{x(x-1)(x-a)})$ of finite order. Note that different choice of the sign corresponds to the involution $x \mapsto -x$ of the elliptic curve. So, the order of the points corresponding to two different choices of the sign are the same. We have the following result of A. Cayley.

Theorem 2.2.3 (A. Cayley) *Let R be an elliptic curve with affine equation*

$$y^2 = g(x),$$

where $g(x)$ is a cubic polynomial with three distinct nonzero roots. Let $y = \sum_{i=0}^{\infty} c_i x^i$ be the formal power Taylor expansion of y in terms of the local parameter x at the point $p = (0, \sqrt{g(0)})$. Then p is of order $n \geq 3$ if and only if

$$\begin{vmatrix} c_2 & c_3 & \dots & c_{k+1} \\ c_3 & c_4 & \dots & c_{k+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{k+1} & c_{k+2} & \dots & c_{2k} \end{vmatrix} = 0, \quad n = 2k+1,$$

$$\begin{vmatrix} c_3 & c_4 & \dots & c_{k+1} \\ c_4 & c_5 & \dots & c_{k+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{k+1} & c_{k+2} & \dots & c_{2k-1} \end{vmatrix} = 0, \quad n = 2k.$$

Proof Let ∞ be the point at infinity of the affine curve $y^2 - g(x) = 0$. The rational function x (resp. y) has pole of order 2 (resp. 3) at ∞. If $n = 2k+1$, the rational functions $1, x, \dots, x^k, y, xy, \dots, x^{k-1}y$ form a basis of the linear space $H^0(C, \mathcal{O}_C(n\infty))$. If $n = 2k$, the same is true for the functions $1, x, \dots, x^k, y, xy, \dots, x^{k-2}y$. A point $p = (0, c_0)$ is a n-torsion point if and only if there is a linear combination of these functions which vanishes at this point with order n. Since x is a local parameter at the point p, we can expand y in a formal power series $y = \sum_{k=0}^{\infty} c_k x^k$. Let us assume $n = 2k+1$, the

other case is treated similarly. We need to find some numbers $(a_0, \ldots, a_{2k})$ such that, after plugging in the formal power series,

$$a_0 + a_1 x + \ldots + a_k x^k + a_{k+1} y + \ldots + a_{2k} x^{k-1} y$$

is divisible by x^{2k+1}. This gives a system of n linear equations

$$a_i + a_{k+1} c_i + \cdots + a_{k+1+i} c_0 = 0,\ i = 0, \ldots, k,$$
$$a_{2k} c_{2+i} + a_{2k-1} c_{3+i} + \cdots + a_{k+1} c_{k+1+i} = 0,\ i = 0, \ldots, k-1.$$

The first $k+1$ equations allow us to eliminate $a_0, \ldots, a_k$. The last k equations have a solution for $(a_{k+1}, \ldots, a_{2k})$ if and only if the first determinant in the assertion of the Theorem vanishes.

□

To apply the Proposition we have to take

$$\alpha = \frac{2a}{(a+1)\lambda}, \quad \beta = 1 + \frac{2a}{(a+1)\lambda}, \quad \gamma = a + \frac{2a}{(a+1)\lambda}.$$

Let us consider the variety $\mathcal{P}_n$

Let us look at the quotient of $\mathcal{P}_n$ by PSL(3). Consider the rational map $\beta : \mathbb{P}^5 \times \mathbb{P}^5 \to (\mathbb{P}^2)^{(4)}$ which assigns to (C, S) the point set $C \cap S$. The fiber of β over a subset B of four points in general linear position is isomorphic to an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the pencil of conics with base point B. Since we can always transform such B to the set of points $\{[1,0,0],[0,1,0],[0,0,1],[1,1,1]\}$, the group PSL(3) acts transitively on the open subset of such 4-point sets. Its stabilizer is isomorphic to the permutation group $\mathfrak{S}_4$ generated by the following matrices:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The orbit space $\mathcal{P}_n/\mathrm{PSL}(3)$ is isomorphic to a curve in an open subset of $\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_4$, where $\mathfrak{S}_4$ acts diagonally. By considering one of the projection maps, we obtain that $\mathcal{P}_n/\mathrm{PSL}(3)$ is an open subset of a cover of $\mathbb{P}^1$ of degree N equal to the number of Poncelet n-related conics in a given pencil of conics with 4 distinct base points with respect to a fixed conic from the pencil. This number was computed by F. Gerbardi [243] and is equal to $\frac{1}{2}T(n)$. A modern account of Gerbardi's result is given in [26]. A smooth compactification of $\mathcal{P}_n/\mathrm{PSL}(3)$ is the modular curve $X^0(n)$ that parameterizes the isomorphism classes of the pairs (R, e), where R is an elliptic curve and e is a point of order n in R.

Proposition 2.2.4 *Let C and S be two nonsingular conics. Consider each n-gon inscribed in C as a subset of its vertices, and also as a positive divisor of degree n on C. The closure of the set of n-gons inscribed in C and circumscribing S is either empty, or a g^1_n, i.e. a linear pencil of divisors of degree n.*

Proof First observe that two polygons inscribed in C and circumscribing S which share a common vertex must coincide. In fact, the two sides passing through the vertex in each polygon must be the two tangents of S passing through the vertex. They intersect C at another two common vertices. Continuing in this way, we see that the two polygons have the same set of vertices. Now consider the Veronese embedding v_n of $C \cong \mathbb{P}^1$ in $\mathbb{P}^n$. An effective divisor of degree n is a plane section of the Veronese curve $\mathsf{V}^1_n = \nu_n(\mathbb{P}^1)$. Thus the set of effective divisors of degree n on C can be identified with the dual projective space $(\mathbb{P}^n)^\vee$. A hyperplane in $(\mathbb{P}^n)^\vee$ is the set of hyperplanes in $\mathbb{P}^n$ which pass through a fixed point in $\mathbb{P}^n$. The degree of an irreducible curve $X \subset (\mathbb{P}^n)^\vee$ of divisors is equal to the cardinality of the set of divisors containing a fixed general point of V^1_n. In our case it is equal to 1. □

2.2.2 Poncelet curves and vector bundles

Let C and S be two Poncelet n-related conics in the plane $\mathbb{P}^2 = |E|$. Recall that this means that there exist n points $p_1, \ldots, p_n$ on C such that the tangent lines $\ell_i = \mathbb{T}_{p_i}(C)$ meet on S. One can drop the condition that S is a conic. We say that a plane curve S of degree $n-1$ is *Poncelet related* to the conic C if there exist n points, as above, such that the tangents to C at these points meet on S.

We shall prove an analog of Darboux's Theorem for Poncelet related curves of degree larger than 2. First, we have to remind some constructions in the theory of vector bundles over the projective plane.

Let $\mathbb{P}^1 = |U|$ for some vector space U of dimension 2 and let $\mathbb{P}^2 = |V|$ for some vector space V of dimension 3. A closed embedding $\mathsf{v} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ has the image isomorphic to a nonsingular conic, a Veronese curve. This defines an isomorphism

$$E^\vee = H^0(|E|, \mathcal{O}_{|E|}(1)) \cong H^0(|U|, \mathcal{O}_{|U|}(2)) = S^2(U^\vee).$$

Its transpose defines an isomorphism $E \cong S^2(U)$. This gives a bijective correspondence between nonsingular conics and linear isomorphisms $E \to S^2(U)$. Also, since $\dim \bigwedge^2 U = 1$, a choice of a basis in $\bigwedge^2 U$ defines a linear isomorphism $U \cong U^\vee$. This gives an isomorphism of projective spaces $|U| \cong |U|^\vee$

that does not depend on a choice of a basis in $\bigwedge^2 U$. Thus a choice of a nonsingular conic in $|E|$ also defines an isomorphism $|E^\vee| \to |S^2(U)|$ which must be given by a nonsingular conic in $|E^\vee|$. This is of course the dual conic.

Fix an isomorphism $\mathbb{P}^2 \cong |S^2(U)|$ defined by a choice of a conic C in $\mathbb{P}^2$. Consider the multiplication map $S^2(U) \otimes S^{n-2}(U) \to S^n(U)$. It defines a rank 2 vector bundle $\mathcal{S}_{n,C}$ on $\mathbb{P}^2$ whose fiber at the point $x = [q] \in |S^2(U)|$ is equal to the quotient space $S^n(U)/qS^{n-2}(U)$. One easily sees that it admits a resolution of the form

$$0 \to S^{n-2}(U)(-1) \to S^n(U) \to \mathcal{S}_{n,C} \to 0, \tag{2.10}$$

where we identify a vector space V with the vector bundle π^*V, where π is the structure map to the point. The vector bundle $\mathcal{S}_{n,C}$ is called the *Schwarzenberger vector bundle* associated to the conic C.

$$(S^n(U)/qS^{n-2}(U))^\vee = \{f \in S^n(U^\vee) : D_q(f) = 0\}. \tag{2.11}$$

Embedding $|U^\vee|$ in $|S^n(U^\vee)|$ by means of the Veronese map, we will identify the divisor of zeros of q with a divisor $V(q)$ of degree 2 on the Veronese curve $R_n \subset |S^n(U^\vee)|$, or, equivalently, with a 1-secant of R_n. A hyperplane containing this divisor is equal to $V(qg)$ for some $g \in S^{n-2}(U)$. Thus the linear space (2.11) can be identified with the projective span of $V(q)$. In other words, the fibres of the dual projective bundle $\mathcal{S}^\vee_{n,C}$ are equal to the secants of the Veronese curve R_n.

It follows from (2.10) that the vector bundle $\mathcal{S}_{n,C}$ has the first Chern class of degree $n-1$ and the second Chern class is equal to $n(n-1)/2$. Thus we expect that a general section of $\mathcal{S}_{n,C}$ has $n(n-1)/2$ zeros. We identify the space of sections of $\mathcal{S}_{n,C}$ with the vector space $S^n(U)$. A point $[s] \in |S^n(U)|$ can be viewed as a hyperplane H_s in $|S^n(U^\vee)|$. Its zeros are the secants of R_n contained in H_s. Since H_s intersects R_n at n points $p_1, \dots, p_n$, any 1-secant $\overline{p_ip_j}$ is a 1-secant contained in H_s. The number of such 1-secants is equal to $n(n-1)/2$.

Recall that we can identify the conic with $|U|$ by means of the Veronese map $\nu_2 : |U| \to |S^2(U)|$. Similarly, the dual conic $C^\vee$ is identified with $|U^\vee|$. By using the Veronese map $\mathsf{v}_n : |U^\vee| \to |S^n(U^\vee)|$, we can identify $C^\vee$ with R_n. Now a point on R_n is a tangent line on the original conic C, hence n points $p_1, \dots, p_n$ from the above are the sides ℓ_i of an n-gon circumscribing C. A secant $\overline{p_ip_j}$ from the above is a point in $\mathbb{P}^2$ equal to the intersection point $q_{ij} = \ell_i \cap \ell_j$. And the $n(n-1)/2$ points q_{ij} represent the zeros of a section s of the Schwarzenberger bundle $\mathcal{S}_{n,C}$.

For any two linearly independent sections s_1, s_2, their determinant $s_1 \wedge s_2$ is a section of $\bigwedge^2 \mathcal{S}_{n,C}$ and hence its divisor of zeros belongs to the linear

system $|\mathcal{O}_{\mathbb{P}^2}(n-1)|$. When we consider the pencil $\langle s_1, s_2\rangle$ spanned by the two sections, the determinant of each member $s = \lambda s_1 + \mu s_2$ has the zeros on the same curve $V(s_1 \wedge s_2)$ of degree $m-1$.

Let us summarize this discussion by stating and proving the following generalization of Darboux's Theorem.

Theorem 2.2.5 *Let C be a nonsingular conic in $\mathbb{P}^2$ and let $\mathcal{S}_{n,C}$ be the associated Scwarzenberger rank 2 vector bundle over $\mathbb{P}^2$. Then n-gons circumscribing C are parameterized by $|H^0(\mathcal{S}_{n,C})|$. The vertices of the polygon Π_s defined by a section s correspond to the subscheme $Z(s)$ of zeros of the section s. A curve of degree $n-1$ passing through the vertices corresponds to a pencil of a sections of $\mathcal{S}_{n,C}$ containing s and is equal to the determinant of a basis of the pencil.*

Proof A section s with the subscheme of zeros $Z(s)$ with ideal sheaf $\mathcal{I}_{Z(s)}$ defines the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \stackrel{s}{\to} \mathcal{S}_{n,C} \to \mathcal{I}_Z(n-1) \to 0.$$

A section of $\mathcal{I}_Z(n-1)$ is a plane curve of degree $n-1$ passing through $Z(s)$. The image of a section t of $\mathcal{S}_{n,C}$ in $H^0(\mathcal{I}_Z(n-1))$ is the discriminant curve $s \wedge t$. Any curve defined by an element from $H^0(\mathcal{I}_Z(n-1))$ passes through the vertices of the n-gon Π_s and is uniquely determined by a pencil of sections containing s. □

One can explicitly write the equation of a Poncelet curve as follows. First we choose a basis ξ_0, ξ_1 of the space U and the basis $(\xi_0^d, \xi_0^{d-1}\xi_1, \ldots, \xi_1^d)$ of the space $S^d(U)$. The dual basis in $S^n(U^\vee)$ is $(\binom{d}{i} t_0^{d-i} t_1^i)_{0\le i\le d}$. Now the coordinates in the plane $|S^2(U)|$ are $t_0^2, 2t_0t_1, t_2^2$, so a point in the plane is a binary conic $Q = a\xi_0^2 + 2b\xi_0\xi_1 + c\xi_1^2$. For a fixed $x = [Q] \in |S^2(U)|$, the matrix of the multiplication map $S^{n-2}(U) \to S^n(U), G \mapsto QG$ is

$$K(x) = \begin{pmatrix} a & & & & \\ 2b & a & & & \\ c & 2b & \ddots & & \\ & c & \ddots & \ddots & \\ & & \ddots & \ddots & a \\ & & & \ddots & 2b \\ & & & & c \end{pmatrix}.$$

A section of $\mathcal{S}_{n,C}$ is given by $f = \sum_{i=0}^n c_i \xi_0^{n-i}\xi_1^i \in S^n(U)$. Its zeros is the

set of points x such that the vector $\mathbf{c}$ of the coefficients belongs to the column subspace of the matrix $K(x)$. Now we vary f in a pencil of binary forms whose coefficient vector $\mathbf{c}$ belongs to the nullspace of some matrix A of size $(n-1) \times (n+1)$ and rank $n-1$. The determinant of this pencil of sections is the curve in the plane defined by the degree $n-1$ polynomial equation in $x = [a, b, c]$

$$\det\big(K(x) \cdot A\big) = 0.$$

Note that the conic C in our choice of coordinates is $V(t_1^2 - t_0 t_2)$.

Remark 2.2.6 Recall that a section of $\mathcal{S}_{n,C}$ defines a n-gon in the plane $|S^2(U)|$ corresponding to the hyperplane section $H_s \cap R_n$. Its vertices is the scheme of zeros $Z(s)$ of the section s. Let $\pi : X(s) \to \mathbb{P}^2$ be the blow-up of $Z(s)$. For a general s, the linear system of Poncelet curves through $Z(s)$ embeds the surface $X(s)$ in $|S^n(U^\vee)|$ with the image equal to $H_s \cap \mathrm{Sec}_1(R_n)$. The exceptional curves of the blow-up are mapped onto the secants of R_n which are contained in H_s. These are the secants $\overline{p_i p_j}$, where $H_s \cap R_n = \{p_1, \ldots, p_n\}$. The linear system defining the embedding is the proper transform of the linear system of curves of degree $n-1$ passing through $\frac{1}{2}n(n-1)$ points of $Z(s)$. This implies that the embedded surface $X(s)$ has the degree equal to $(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$. This is also the degree of the secant variety $\mathrm{Sec}_1(R_n)$. For example, take $n = 4$ to get that the secant variety of R_4 is a cubic hypersurface in $\mathbb{P}^4$ whose hyperplane sections are cubic surfaces isomorphic to the blow-up of the six vertices of a complete quadrilateral.

2.2.3 Complex circles

Fix two points in the plane and consider the linear system of conics passing through the two points. It maps the plane to $\mathbb{P}^3$ with the image equal to a nonsingular quadric $Q = V(q)$. Thus we may identify each conic from the linear system with a hyperplane in $\mathbb{P}^3$, or using the polarity defined by Q, with a point. When the two points are the points $[0, 1, \pm i]$ in the real projective plane with the line at infinity $t_0 = 0$, a real conic becomes a circle, and we obtain that the geometry of circles can be translated into the orthogonal geometry of real 3-dimensional projective space. In coordinates, the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is given by

$$[t_0, t_1, t_2] \mapsto [x_0, x_1, x_2, x_3] = [t_1^2 + t_2^2, t_0 t_1, t_0 t_2, t_0^2].$$

Its image is the quadric

$$Q = V(x_0 x_3 - x_1^2 - x_2^2).$$

Explicitly, a point $[v] = [\alpha_0, \alpha_1, \alpha_2, \alpha_3] \in \mathbb{P}^3$ defines the *complex circle*

$$S(v) : \alpha_0(t_1^2 + t_2^2) - 2t_0(\alpha_1 t_1 + \alpha_2 t_2) + \alpha_3 t_0^2 = 0. \tag{2.12}$$

By definition, its center is the point $c = [\alpha_0, \alpha_1, \alpha_2]$, its radius square R^2 is defined by the formula

$$\alpha_0^2 R^2 = \alpha_1^2 + \alpha_2^2 - \alpha_0\alpha_3 = q(\alpha). \tag{2.13}$$

Let us express the property that two circles are tangent to each other. It applies to complex circles as well.

Proposition 2.2.7 *Let $[v], [w]$ be two points in $\mathbb{P}^3$, and let $S(v), S(w)$ be two complex circles corresponding to planes in $\mathbb{P}^3$ which are polar to the points with respect to the quadric $Q = V(q)$. Then the two circles touch each other if and only if*

$$(v, v)(w, w) - (v, w)^2 = 0, \tag{2.14}$$

where (v, w) denotes the bilinear form associated to the quadratic form q.

Proof Let $\ell = V(\lambda v + \mu w)$ be the line spanned by the points $[v]$ and $[w]$. Via polarity, it corresponds to a pencil of planes in $\mathbb{P}^3$. The preimages of two planes are tangent if and only if the pencil contains a plane tangent to the quadric Q. Dually this means that the line ℓ is tangent to Q. This is equivalent to the binary form

$$q(\lambda v + \mu w) = \lambda^2(v, v) + 2(v, w)\lambda\mu + \mu^2(w, w)$$

having has a double root. Of course, this happens if and only if (2.14) holds. □

Note that relation (2.14) is of degree 2 in v and w. If we identify the space of circles with $\mathbb{P}^3$, this implies that the pairs of touching complex circles is a hypersurface in $\mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(2, 2)$. It is easy to see that the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$ is the double locus of the hypersurface.

Fix two complex irreducible circles $S = S(v)$ and $S' = S(w)$ and consider the variety R of complex circles $S(x)$ touching S and S'. It is equal to the quartic curve, the intersection of two quadratic cones Q_S and $Q_{S'}$ of conics touching S and S',

$$(v, v)(x, x) - (v, x)^2 = (w, w)(x, x) - (w, x)^2 = 0.$$

Since the singular points of these cones $[v]$ and $[w]$ satisfy these equations, the quartic curve has two singular points. In fact, it is the union of two conics given by the equations

$$\sqrt{(v, v)}(w, x) \pm \sqrt{(w, w)}(v, x) = 0.$$

The two conics intersect at the points $[x]$ such that $(x,x) = 0$ and $(v,x) = (v,w) = 0$. The first condition means that $[x]$ is the *null-circle*, i.e. $\alpha_0^2 R^2 = 0$ in (2.13). It is the union of two lines connecting one if the two intersection points of S and S' outside the line at infinity $t_0 = 0$ with the two intersection points at infinity. In the case when S and S' touch each other the whole pencil generated by S and S' becomes a component of the quartic curve entering with multiplicity 2. So, the two cones Q_S and $Q_{S'}$ touch each other along the line spanned by S and S'.

Theorem 2.2.8 (J. Steiner) *Suppose, after m steps, S_m is equal to S_1. Then, starting from arbitrary conic S_1' touching S and S', we get a sequence of concis $S_1', \ldots, S_m' = S_1'$ tangent to S and S' with S_k' tangent to S_{k-1}'.*

Proof Let R be one of the conic components of the variety of complex circles touching S and S'. Let

$$X = \{(S_1, S_2) \in R \times R : S_1 \text{touches } S_2\}.$$

It is a curve of bidegree $(4,4)$ on $R \times R \cong \mathbb{P}^1 \times \mathbb{P}^1$. The fiber of its projection to the first factor over a point represented by a conic S_1 consists of three points. One them is at the diagonal and enters with multiplicity 2. This implies that X consists of the diagonal taken with multiplicity 2 and the residual curve F of bidegree $(2,2)$. The fiber of the first projection $X \to R$ over S_1 consists of complex circles which touch S and S_1 and also touch S' and S_1. It consists of the intersection of two quartic curves, each has a double line as component. The double lines are represented by the pencil generated by S and S_1 and the pencil generated by S' and S_1. The only case when the fiber consists of one point is when S_1 is one of the two null-lines touching S and S' at their intersection point not at infinity. In this case the quadric Q_{S_1} of circles touching S_1 is the double plane of circles passing through the singular point of S_1. Thus we see that the residual curve F has only two branch points for each of the two projections $X \to R$. Since its arithmetic genus is equal to 1, it must consist of two irreducible curves of bidegree $(1,1)$ intersecting at two points a, b. If we fix one of the components F_1, then the map $(S_1, S_2) \mapsto (S_2, S_3)$ is the automorphism of $F_1 \setminus \{a,b\} \cong \mathbb{C}^*$. The sequence $S_1, S_2, S_3, \ldots$ terminates if and only if this automorphism is of finite order m. As soon as it is, we can start from any S_1 and obtain a finite sequence $(S_1, \ldots, S_m = S_1)$. □

Remark 2.2.9 We followed the proof from [26]. When S and S' are concentric real circles, the assertion is evident. The general case of real conics can be reduced to this case (see [222], [500]). Poncelet's and Steiner's Theorems are examples of a *porism*, which can be loosely stated as follows. If one can find

one object satisfying a certain special property then there are infinitely many such objects. There are some other poristic statements for complex circles: Emch' Theorem and the *zig-zag theorem* discussed in [26].

2.3 Quadric surfaces

2.3.1 Polar properties of quadrics

Many of the polar properties of conics admit extension to nonsingular quadrics in higher-dimensional $\mathbb{P}^n$. For example, a *self-polar* $(n+1)$*-hedron* is defined as a collection of $n+1$ ordered hyperplanes $V(l_i)$ in general linear position such that the pole of each plane $V(l_i)$ is equal to the intersection point of the remaining hyperplanes. Similarly to the case of conics, one proves that a self-polar $(n+1)$-hedron is the same as a polar $(n+1)$-hedron of the quadric.

The definition of the conjugate $(n+1)$-hedra is a straightforward extension of the definition of conjugate triangles. We say that two simplexes Σ and Σ' are *mutually polar* with respect to a quadric Q if the poles of the facets of T' are vertices of T. This implies that the images of k-dimensional faces of T under the polarity defined by Q are the opposite $(n-k)$-dimensional facets of Σ'. The condition (2.7) extends to any dimension. However, it does not translate to a single equation on the coefficients of the linear forms defining the polyhedra. This time we have a system of $n(n+1)/2$ linear equations with $n+1$ unknowns and the condition becomes the rank condition.

We adopt the terminology of convex geometry to call the set of $n+1$ linearly independent hyperplanes a *simplex*. The intersection of a subset of k hyperplanes will be called an $(n-k)$-dimensional *face*. If $k=n$, this is a *vertex* , if $k=n-1$, this is an *edge*, if $n=0$ this is a *facet* .

The notion of perspectivity of triangles extends to quadrics of any dimension. We say that two simplexes are *perspective* from a point $\mathfrak{o}$ if there is a bijection between the sets of vertices such that the lines joining the corresponding vertices pass through the point $\mathfrak{o}$. We say that the two simplexes are perspective from a hyperplane if this hyperplane contains the intersections of corresponding facets. We have also an extension of Desargues' Theorem.

Theorem 2.3.1 (G. Desargues) *Two simplexes are perspective from a point if and only if they are perspective from a hyperplane.*

Proof Without loss of generality, we may assume that the first simplex Σ is the coordinate simplex with vertices $p_i=[e_i]$ and it is perspective from the point $\mathfrak{o}=[e]=[1,\dots,1]$. Let $q_i=[v_i]$ be the vertices of the second simplex

Σ_2. Then we have $v_i = e + \lambda_i e_i$ for some scalars λ_i. After subtracting, we obtain $v_i - v_j = \lambda_i e_i - \lambda_j e_j$. Thus any two edges $\overline{p_i p_j}$ and $\overline{q_i q_j}$ meet at a point r_{ij} which lies on the hyperplane $H = V(\sum_{i=0}^{n} \frac{1}{\lambda_i} t_i)$. Since the intersection of the facet of Σ_1 opposite the point p_k with the facet of Σ_2 opposite the point q_k contains all points r_{ij} with $i, j \neq k$, and they span the intersection, we get that the two simplexes are perspective from H. The converse assertion follows by duality. □

Remark 2.3.2 As remarked [527], p.252, the previous assertion is a true space generalization of the classical Desargues's Theorem. Other generalization applies to two space triangles and asserts that the perspectivity from a point implies that the intersection points of the corresponding sides (which automatically intersect) are collinear.

Let $b_q : E \to E^\vee$ be an isomorphism defined by a nonsingular quadric $Q = V(q)$. For any linear subspace L of E, the subspace $b_q(L)^\perp$ of E is called the *polar* of L with respect to Q. It is clear that the dimensions of a subspace and its polar subspace add up to the dimension of $|E|$. Two subspaces Λ and Λ' of the same dimension are called *conjugate* if the polar subspace of Λ intersects Λ'.

These classical definitions can be rephrased in terms of standard definitions of multilinear algebra. Let Λ (resp. Λ') be spanned by $[v_1], \ldots, [v_k]$ (resp. $[w_1], \ldots, [w_k]$). For any two vectors $v, w \in E$, let $(v, w)_q$ denote the value of the polar bilinear form b_q of q on $(v, w)_q$.

Lemma 2.3.3 *Λ and Λ' are conjugate with respect to Q if and only if*

$$\det \begin{pmatrix} (v_1, w_1)_q & (v_2, w_1)_q & \ldots & (v_k, w_1)_q \\ (v_1, w_2)_q & (v_2, w_2)_q & \ldots & (v_k, w_2)_q \\ \vdots & \vdots & \vdots & \vdots \\ (v_1, w_k)_q & (v_2, w_k)_q & \ldots & (v_k, w_k)_q \end{pmatrix} = 0.$$

Proof Let $b_q : E \to E^\vee$ be the linear isomorphism defined by the polar bilinear form of q. The linear funtions $b_q(v_1), \ldots, b_q(v_k)$ form a basis of a k-dimensional subspace L of $E^\vee$ whose dual $L^\perp$ is an $(n-k)$-dimensional subspace of E. It is easy to see that the spans of $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ have a common nonzero vector if and only if $L^\perp$ intersects nontrivially the latter span. The condition for this is that, under the natural identification $\bigwedge^k(E^\vee)$ and $\bigwedge^k(E)^\vee$, we have

$$b_q(v_1) \wedge \ldots \wedge b_q(v_k)(w_1 \wedge \ldots \wedge w_k) = \det((v_i, w_j)_q) = 0.$$

□

It follows from the Lemma that the relation to be conjugate is symmetric.

From now on, until the end of this section, we assume that $n = 3$.

A tetrahedron in $\mathbb{P}^3$ with conjugate opposite edges is called *self-conjugate*. It is clear that a polar tetrahedron of Q is self-conjugate, but the converse is not true.

Let T be a tetrahedron with vertices $p_1 = [v_1], \ldots, p_4 = [v_4]$. Suppose that two pairs of opposite edges are conjugate with respect to some quadric Q. Then T is self-conjugate (see [552], B. III, p. 135, or [537], 7.381). The proof is immediate. Suppose the two conjugate pairs of edges are $(\overline{p_1p_2}, \overline{p_3p_4})$ and $(\overline{p_1p_3}, \overline{p_2p_4})$. For brevity, let us denote $(v_i, v_j)_q$ by (ij). Then $(13)(24) - (14)(23) = 0$, and $(12)(34) - (14)(23) = 0$ imply, after subtraction, $(13)(24) - (12)(34) = 0$. This means that the remaining pair $(\overline{p_1p_3}, \overline{p_2p_3})$ is conjugate.

We know that two conjugate triangles are perspective. In the case of quadrics we have a weaker property expressed on the following Chasles' Theorem.

Theorem 2.3.4 *[M. Chasles] Let T and T' be two mutually polar tetrahedra with respect to a quadric Q. Suppose no two opposite edges of T are conjugate. Then the lines joining the corresponding vertices belong to the same ruling of lines of some nonsingular quadric Q'.*

Proof Let p_1, p_2, p_3, p_4 be the vertices of T and let q_1, q_2, q_3, q_4 be the vertices of T'. In the following, $\{i, j, k, l\} = \{1, 2, 3, 4\}$. By definition, q_l is a pole of the plane spanned by p_i, p_j, p_k and the matching between the vertices is $p_i \mapsto q_i$. Suppose the edge $\overline{p_ip_j}$ is not conjugate to the opposite edge $\overline{p_kp_l}$. This means that it does not intersect the edge $\overline{q_iq_j}$. This implies that the lines $\overline{p_iq_i}$ and $\overline{p_jq_j}$ do not intersect. By symmetry of the conjugacy relation, we also obtain that the lines $\overline{p_kq_k}$ and $\overline{p_lq_l}$ do not intersect. Together this implies that we may assume that the first three lines $\ell_i = \overline{p_iq_i}$ are not coplanar.

Without loss of generality, we may assume that the first tetrahedron T is the coordinate tetrahedron. Let $A = (a_{ij})$ be a symmetric matrix defining the quadric Q and let $C = \mathrm{adj}(A) = (c_{ij})$ be the adjugate matrix defining the dual quadric. The coordinates of facets of T are columns of $A = (a_{ij})$. The coordinates of the intersection point of three facets defined by three columns A_i, A_j, A_k of A are equal to the column C_m of C, where $m \neq i, j, k$. Thus a general point on the line generated by the point $[1, 0, 0, 0]$ has coordinates $[\lambda, \mu c_{12}, c_{13}, c_{14}]$, and similar for other three lines. Recall that by Steiner's construction (see [267], p. 528) one can generate a nonsingular quadric by two projectively equivalent pencils of planes through two skew lines. The quadric is the union of the intersection of the corresponding planes. Apply this construction to the pencil of planes through the first two lines. They projectively

matched by the condition that the corresponding planes in the pencils contain the same point $[c_{31}, c_{32}, \lambda, c_{41}]$ on the third line. The two planes from each pencil are defined by the equations

$$\det \begin{pmatrix} t_0 & t_1 & t_2 & t_3 \\ 1 & 0 & 0 & 0 \\ \lambda c_{11} & c_{12} & c_{13} & c_{14} \\ c_{31} & c_{32} & \lambda & c_{34} \end{pmatrix}$$

$$= t_1 c_{13} c_{34} + t_2(c_{14} c_{32} - c_{12} c_{34}) - t_3 c_{13} c_{32} + \lambda(t_3 c_{12} - t_1 c_{14}) = 0,$$

$$\det \begin{pmatrix} t_0 & t_1 & t_2 & t_3 \\ 0 & 1 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & \lambda & c_{34} \end{pmatrix}$$

$$= t_0 c_{23} c_{34} + t_2(c_{24} c_{31} - c_{21} c_{34}) - t_3 c_{23} c_{31} + \lambda(t_3 c_{21} - t_1 c_{24}) = 0,$$

Eliminating λ, we find the equation of the quadric

$$(c_{12}c_{34} - c_{24}c_{13})(c_{23}t_0t_3 + c_{14}t_1t_2) + (c_{13}c_{24} - c_{14}c_{23})(c_{12}t_2t_3 + c_{34}t_0t_1)$$

$$+(c_{14}c_{23} - c_{12}c_{34})(c_{13}t_1t_3 + c_{24}t_0t_2) = 0.$$

By definition, the quadric contains the first three lines. It is immediately checked that a general point $[c_{41}, c_{42}, c_{43}, \lambda]$ on the fourth line lies on the quadric. □

The following result follows from the beginning of the proof.

Proposition 2.3.5 *Let T and T' be two mutually polar tetrahedra. Assume that T (and hence T') is self-conjugate. Then T and T' are in perspective from the intersection points of the lines joining the corresponding vertices and perspective from the polar plane of this point.*

One can think that the covariant quadric Q' constructed in the proof of Chasles' Theorem 2.3.4 degenerates to a quadratic cone. Counting parameters, it is easy to see that the pairs of perspective tetrahedra depend on the same number 19 of parameters as pairs of tetrahedra mutually polar with respect to some quadric. It is claimed in [21], v. 3, p.45, that any two perspective tetrahedra are, in fact, mutually polar with respect to some quadric. Note that the polarity condition imposes three conditions, and the self-conjugacy condition imposes two additional conditions. This agrees with counting constants $(5 = 24 - 19)$.

One can apply the previous construction to the problem of writing a quadratic

form q as a sum of five squares of linear forms. Suppose we have two self-conjugate tetrahedra T and T' with respect to a quadric Q that are also mutually polar with respect to Q. By Proposition 2.3.5, they are in perspective. Choose coordinates such that T is the coordinate tetrahedron and let $A = (a_{ij})_{0\le i,j\le 3}$ be a symmetric matrix defining Q. We know that the equations of facets H_i of T' are $V(\sum_{j=0}^{3} a_{ij}t_j)$. Since T is self-conjugate, the intersection lines $H_0 \cap H_1$ meet the coordinate lines $t_0 = t_1 = 0$. This means that the equations $a_{20}t_2 + a_{30}t_3 = 0$ and $a_{21}t_2 + a_{31}t_3 = 0$ have a nonzero solution, i.e. $a_{20}a_{31} = a_{21}a_{30}$. Similarly, we get that $a_{10}a_{32} = a_{30}a_{12}$ and $a_{01}a_{32} = a_{02}a_{31}$. Using the symmetry of the matrix, this implies that the six products are equal. Hence $a_{03}a_{13}/a_{12} = a_{23}a_{03}/a_{02} = a_{03}a_{13}/a_{01}$ are all equal to some number α. Then the equation of the quadrics can be written as a sum of five squares

$$\sum_{i=0}^{3} a_{ii}t_i^2 + 2\sum_{0\le i<j\le 3} a_{ij}t_it_j$$

$$= \sum_{i=0}^{2}(a_{ii} - \alpha a_{i3})t_i^2 + (a_{33} - \alpha)t_3^2 + \alpha^{-1}(\sum_{i=0}^{2} a_{i3}t_i + \alpha t_3)^2 = 0.$$

Here we assume that A is general enough. The center of perspective of the two tetrahedra is the pole of the plane $V(a_{03}t_0 + a_{13}t_1 + a_{23}t_2 + \alpha t_3)$.

The pentad of points consisting of the vertices of a self-conjugate tetrahedron with regard to a quadric Q and the center of the perspectivity $\mathfrak{o}$ of the tetrahedron and its polar tetrahedron form a *self-conjugate pentad* (and pentahedron in the dual space). This means that the pole of each plane spanned by three vertices lies on the opposite edge. As follows from above, the pentad of points defined by a self-conjugate tetrahedron defines a polar polyhedron of Q consisting of the polar planes of the pentad.

Proposition 2.3.6 *Let $H_i = V(l_i), i = 1,\dots,5$, form a nondegenerate polar pentahedron of a quadric $Q = V(q)$. Let $p_1,\dots,p_5$ be the poles of the planes $V(l_i)$ with respect to Q. Then the pentad $p_1,\dots,p_5$ is self-conjugate and is a polar polyhedron of the dual quadric.*

Proof Let x_i be the pole of H_i with respect to Q. Then the pole of the plane spanned by x_i, x_j, x_k is the point $x_{ijk} = H_i \cap H_j \cap H_k$. We may assume that $q = \sum_{i=0}^{4} l_i^2$. Then $P_{x_{ijk}}(Q)$ belongs to the pencil $\mathcal{P}$ generated by the remaining two planes H_r, H_s. When we vary a point along the edge $\overline{x_r x_s}$ the polar plane of the point belongs to the pencil $\mathcal{P}$. For one of the points, the polar

plane will be equal to the plane $P_{x_{ijk}}(Q)$, hence this points coincide with x_{ijk}. By definition, the pentad is self-conjugate.

The second assertion can be checked by straightforward computation. Since the polar pentahedron is nondegenerate, we can choose coordinates such that the polar pentahedron of Q is to equal to the union of the coordinate tetrahedron and the plane $V(\sum t_i)$. We can write

$$2q = \sum_{i=0}^{3} \lambda_i t_i^2 + (\sum_{i=0}^{3} t_i)^2$$

for some nonzero scalars λ_i. For any $v = (a_0, a_1, a_2, a_3) \in \mathbb{C}^4$, we have

$$D_v(q) = \sum_{i=0}^{3} (a + \lambda_i a_i) t_i$$

where $a = \sum_{i=0}^{3} a_i$. Let $\xi_i = a + \lambda_i a_i$ be considered as coordinates in the dual space. We can express a_i in terms of ξ_i by solving a system of linear equations with matrix

$$\begin{pmatrix} \lambda_0 & 1 & 1 & 1 \\ 1 & \lambda_1 & 1 & 1 \\ 1 & 1 & \lambda_2 & 1 \\ 1 & 1 & 1 & \lambda_3 \end{pmatrix}.$$

Write $a_j = L_j(\xi_0, \dots, \xi_3) = \sum_{j=0}^{3} c_{ij}\xi_j$, where (c_{ij}) is the inverse matrix. Let $v_j^* = (c_{0j}, c_{1j}, c_{2j}, c_{3j})$. The dual quadric consists of points $(\xi_0, \xi_1, \xi_2, \xi_3)$ such that $q(a_0, a_1, a_2, a_3) = 0$. This gives the equation of the dual quadric

$$Q^\vee = V(\sum_{i=0}^{3} \lambda_i L_i(\xi_0, \xi_1, \xi_2, \xi_3)^2 + \big(\sum_{i=0}^{3} L_i(\xi_0, \xi_1, \xi_2, \xi_3)\big)^2).$$

So, we see that the dual quadric has the polar polyhedron defined by the planes $V(L_i), V(\sum L_i)$. We have

$$D_{v_j^*}(q) = \sum_{i=0}^{3} (\lambda_i a_i + a) c_{ij} t_i = t_j,\ j = 0, 1, 2, 3,$$

hence $D_{\sum v_j^*}(q) = \sum t_j$. This checks that the points of the pentad are poles of the planes of the polar pentahedron of Q. □

Remark 2.3.7 Let $\Pi_1, \dots, \Pi_N$ be sets of m-hedra in $\mathbb{P}^n, n > 1$, with no common elements. Suppose that these polyhedra, considered as hypersurfaces in $\mathbb{P}^n$ of degree m (the unions of their hyperplanes), belong to the same pencil. It is easy to see that this is equivalent to that the first two m-hedra Π_1, Π_2

are perspective from each hyperplane of $\Pi_3, \dots, \Pi_k$. The open problem is as follows.

What is the maximal possible number $N(n, m)$ of such polyhedra?

By taking a general hyperplane, we get $N(n, m) \leq N(2, m)$. It is known that $N(2, m) \geq 3$ and $N(2, 2), N(2, 3) = 4$. It was proven by J. Stipins [547] (see also [606]) that $N(2, m) \leq 4$ for all m and it is conjectured that $N(2, m) = 3$ for $m \neq 3$.

In the next chapter we will consider the case $n = 2, m = 3, N = 4$. In the case $n = 3, m = 4, N = 3$, the three tetrahedra are called *desmic desmic* (the name is due to C. Stephanos [546]). The configuration of the 12 planes forming three desmic tetrahedra has a beautiful geometry (see, for example, [378], [379]). A general member of the pencil generated by three desmic tetrahedra is a *desmic quartic surface*. It has 12 singular points and represents a special embedding of a Kummer surface of the product of two isomorphic elliptic curves. We refer to [308] for some modern treatment of desmic quartic surfaces.

2.3.2 Invariants of a pair of quadrics

Let $Q_1 = V(f)$ and $Q_2 = V(g)$ be two quadrics in $\mathbb{P}^n$ (not necessarily nonsingular). Consider the pencil $V(t_0 f + t_1 g)$ of quadrics spanned by C and S. The zeros of the discriminant equation $D = \mathrm{discr}(t_0 f + t_1 g) = 0$ correspond to singular quadrics in the pencil. In coordinates, if f, g are defined by symmetric matrices $A = (a_{ij}), B = (b_{ij})$, respectively, then $D = \det(t_0 A + t_1 B)$ is a homogeneous polynomial of degree $\leq n+1$. Choosing different system of coordinates replaces A, B by $Q^T AQ, Q^T BQ$, where Q is an invertible matrix. This replaces D with $\det(Q)^2 D$. Thus the coefficients of D are invariants on the space of pairs of quadratic forms on $\mathbb{C}^{n+1}$ with respect to the action of the group $\mathrm{SL}(n+1)$. To compute D explicitly, we use the following formula for the determinant of the sum of two $m \times m$ matrices $X + Y$:

$$\det(X + Y) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \Delta_{i_1, \dots, i_k}, \tag{2.15}$$

where $\Delta_{i_1, \dots, i_k}$ is the determinant of the matrix obtained from X by replacing the columns $X_{i_1}, \dots, X_{i_k}$ with the columns $Y_{i_1}, \dots, Y_{i_k}$. Applying this formula to our case, we get

$$D = \Theta_0 t_0^{n+1} + \sum_{i=1}^{n} \Theta_i t_0^{n+1-i} t^i + \Theta_{n+1} t_n^{n+1} \tag{2.16}$$

where $\Theta_0 = \det A, \Theta_{n+1} = \det B$, and

$$\Theta_k = \sum_{1 \le i_1 < \ldots < i_k \le n+1} \det(A_1 \ldots B_{i_1} \ldots B_{i_k} \ldots A_{n+1}),$$

where $A = [A_1 \ldots A_{n+1}], B = [B_1 \ldots B_{n+1}]$. We immediately recognize the geometric meanings of vanishing of the first and the last coefficients of D. The coefficient Θ_0 (resp. Θ_{n+1}) vanishes if and only if Q_1 (resp. Q_2) is a singular conic.

Proposition 2.3.8 *Let Q_1 and Q_2 be two general quadrics. The following conditions are equivalent.*

(i) $\Theta_1 = 0$;
(ii) Q_2 is apolar to the dual quadric $Q_1^\vee$;
(iii) Q_1 admits a polar simplex with vertices on Q_2.

Proof First note that

$$\Theta_1 = \mathrm{Tr}(B\mathrm{adj}(A)). \tag{2.17}$$

Now $\mathrm{adj}(A)$ is the matrix defining $Q_1^\vee$ and the equivalence of (i) and (ii) becomes clear.

Since Θ_i are invariants of (Q_1, Q_2), we may assume that $Q_1 = V(\sum_{i=0}^n t_i^2)$. Suppose (iii) holds. Since the orthogonal group of C acts transitively on the set of polar simplexes of Q_1, we may assume that the coordinate simplex is inscribed in Q_2. Then the points $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$, must be on Q_2. Hence

$$Q_2 = V(\sum_{0 \le i < j \le n} a_{ij} t_i t_j),$$

and the condition (i) is verified.

Now suppose (i) holds. Choose coordinates such that $Q_1 = V(\alpha_i t_i^2)$. Start from any point on Q_2 but not on Q_1, and choose a projective transformation that leaves Q_1 invariant and sends the point to the point $p_1 = [1, 0, \ldots, 0]$. The quadric Q_2 transforms to a quadric with an equation in which the coefficient at x_0^2 is equal to 0. The polar line of p_1 with respect to Q_1 is $V(\sum_{i=1}^n \alpha_i t_i)$. It intersects Q_2 along a quadric of dimension $n-2$ in the hyperplane $t_0 = 0$. Using a transformation leaving $V(t_0)$ and Q_1 invariant, we transform Q_2' to another quadric such that the point $p_2 = [0, 1, 0, \ldots, 0]$ belongs to $V(t_0) \cap Q_2'$. This implies that the coefficients of the equation of Q_2' at t_0^2 and t_1^2 are equal to zero. Continuing in this way, we may assume that the equation of Q_2 is of the form $a_{nn} t_n^2 + \sum_{0 \le i < j \le n}^n a_{ij} t_i t_j = 0$. The trace condition is $a_{nn} \alpha_n^{-1} = 0$. It implies that $a_{nn} = 0$ and hence the point $p_{n+1} = [0, \ldots, 0, 1]$ is on Q_2.

The triangle with vertices $[1,0,\dots,0],\dots,[0,\dots,0,1]$ is a polar simplex of Q_1 which is inscribed in Q_2. □

Observe that, if $Q_1 = V(\sum t_i^2)$, the trace condition means that the conic Q_2 is defined by a harmonic polynomial with respect to the Laplace operator.

Definition 2.3.9 *A quadric Q_1 is called* apolar *to a quadric Q_2 if one of the equivalent conditions in Proposition 2.3.8 holds. If Q_1 is apolar to Q_2 and vice versa, the quadrics are called* mutually apolar.

The geometric interpretation of other invariants Θ_i is less clear. First note that a quadratic form q on a vector space E defines a quadratic form $\Lambda^k q$ on the space $\bigwedge^k E$. Its polar bilinear form is the map $\bigwedge^k b_q : \bigwedge^k E \to \bigwedge^k E^\vee = (\bigwedge^k E)^\vee$, where $b_q : E \to E^\vee$ is the polar bilinear form of q. Explicitly, the polar bilinear form $\bigwedge^k b_q$ is defined by the formula

$$(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k) = \det(b_q(v_i, w_j))$$

which we already used in Lemma 2.3.3.

If A is the symmetric matrix defining q, then the matrix defining $\bigwedge^k q$ is denoted by $A^{(k)}$ and is called the *k-th compound matrix* of A. If we index the rows and the columns of $A^{(k)}$ by an increasing sequence $J = (j_1, \dots, j_k) \subset \{1, \dots, n+1\}$, then the entry $A^{(k)}_{J,J'}$ of $A^{(k)}$ is equal to the (J, J')-minor $A_{J,J'}$ of A. Replacing each $A^{(k)}_{J,J'}$ with the minor $A_{\overline{J'},\overline{J}}$ taken with the sign $(-1)^{\epsilon(J,J')}$, we obtain the definition of the *adjugate k-th compound matrix* $\mathrm{adj}^{(k)}(A)$ (not to be confused with $\mathrm{adj}(A^{(k)})$). The Laplace formula for the determinant gives

$$A^{(k)}\mathrm{adj}^{(k)}(A) = \det(A) I.$$

If A is invertible, then $A^{(k)}$ is invertible and $(A^{(k)})^{-1} = \frac{1}{\det A}\mathrm{adj}(A^{(k)})$.

We leave it to the reader to check the following fact.

Proposition 2.3.10 *Let $Q_1 = V(q), Q_2 = V(q')$ be defined by symmetric matrices A, B and let $A^{(k)}$ and $B^{(k)}$ be their k-th compound matrices. Then*

$$\Theta_k(A, B) = \mathrm{Tr}(A^{(n+1-k)}\mathrm{adj}(B^{(k)})).$$

Example 2.3.11 Let $n = 3$. Then there is only one new invariant to interpret. This is $\Theta_2 = \mathrm{Tr}(A^{(2)}\mathrm{adj}(B^{(2)}))$. The compound matrices $A^{(2)}$ and $B^{(2)}$ are 6×6 symmetric matrices whose entries are 2×2-minors of A and B taken with an appropriate sign. Let $A = (a_{ij})$. The equation of the quadric defined

by $A^{(2)}$ is given by the *bordered determinant*

$$\det \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \xi_0 & \eta_0 \\ a_{10} & a_{11} & a_{12} & a_{13} & \xi_1 & \eta_1 \\ a_{20} & a_{21} & a_{22} & a_{23} & \xi_2 & \eta_2 \\ a_{30} & a_{31} & a_{32} & a_{33} & \xi_3 & \eta_3 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 & 0 & 0 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 & 0 & 0 \end{pmatrix} = 0. \tag{2.18}$$

The equation is called the *line-equation* or *complex equation* of the quadric Q defined by the matrix A. If we take the minors $\xi_i\eta_j - \xi_j\eta_i$ as Plücker coordinates in $|\bigwedge^2 \mathbb{C}^4|$, the line-equation parameterizes lines in $\mathbb{P}^3$ which are tangent to the quadric Q. This can be immediately checked by considering a parametric equation of a line $\lambda(\xi_0, \xi_1, \xi_2, \xi_3) + \mu(\eta_0, \eta_1, \eta_2, \eta_3)$, inserting it in the equation of the quadric and finding the condition when the corresponding quadratic form in λ, μ has a double root. In matrix notation, the condition is $(\xi A\xi)(\eta A\eta) - (\xi A\eta)^2 = 0$, which can be easily seen rewritten in the form of the vanishing of the bordered determinant. The intersection of the quadric defined by the matrix $A^{(2)}$ with the Klein quadric defining the Grassmannian of lines in $\mathbb{P}^3$ is an example of a *quadratic line complex*. We will discuss this and other quadratic line complexes in Chapter 10. Take $Q = V(\sum t_i^2)$. Then the bordered determinant becomes equal to

$$(\sum_{i=0}^{3} \xi_i^2)(\sum_{i=0}^{3} \eta_i) - (\sum_{i=0}^{3} \xi_i\eta_i)^2 = \sum_{0\le i<j\le 3} (\xi_i\eta_j - \xi_j\eta_i)^2 = \sum_{0\le i<j\le 3} p_{ij}^2,$$

where p_{ij} are the Plücker coordinates. We have

$$\Theta_2(A, B) = \mathrm{Tr}(B_2) = \sum_{0\le i<j\le 3} (b_{ij}b_{ji} - b_{ii}b_{jj}).$$

The coordinate line $t_i = t_j = 0$ touches the quadric Q_2 when $b_{ij}b_{ji} - b_{ii}b_{jj} = 0$. Thus Θ_2 vanishes when a polar tetrahedron of Q_1 has its edges touching Q_2.

It is clear that the invariants Θ_k are bihomogeneous of degree $(k, n+1-i)$ in coefficients of A and B. We can consider them as invariants of the group $\mathrm{SL}(n+1)$ acting on the product of two copies of the space of square symmetric matrices of size $n+1$. One can prove that the $n+1$ invariants Θ_i form a complete system of polynomial invariants of two symmetric matrices. This means that the polynomials Θ_i generate the algebra of invariant polynomials (see [580], p. 304).

One can use the invariants Θ_i to express different mutual geometric properties of two quadrics. We refer to [537] for many examples. We give only one example.

Theorem 2.3.12 *Two quadrics touch each other if and only if*

$$J = \mathsf{D}(\Theta_0, \dots, \Theta_{n+1}) = 0,$$

where D *is the discriminant of a binary form of degree* $n+1$.

Proof This follows from the description of the tangent space of the discriminant hypersurface of quadratic forms. The line defining the pencil of quadrics generated by the two quadrics does not intersect the discriminant hypersurface transversally if and only if one of quadrics in the pencil is of corank ≥ 2, or one of the quadrics has a singular point at the base locus of the pencil (see (1.45)). In the case of pencils the first condition implies the second one. Thus the condition for tangency is that one of the roots of the equation $\det(t_0A+t_1B) = 0$ is a multiple root. □

The invariant J is called the *tact-invariant* of two quadrics.[1] Note that two quadrics touch each other if and only if their intersection has a singular point.

Corollary 2.3.13 *The degree of the hypersurface of quadrics in* $\mathbb{P}^n$ *touching a given nonsingular quadric is equal to* $n(n+1)$.

Proof This follows from the known property of the discriminant of a binary form $\sum_{i=0}^{d} a_i t_0^{d-i} t_1^i$. If we assign the degree $(d-i, i)$ to each coefficient a_i, then the total degree of the discriminant is equal to $d(d-1)$. This can be checked, for example, by computing the discriminant of the form $a_0 t_0^d + a_d t_1^d$, which is equal to $d^d a_0^{d-1} a_d^{d-1}$ (see [239], p. 406). In our case, each Θ_k has bidegree $(n+1-k, k)$, and we get that the total bidegree is equal to $(n(n+1), n(n+1))$. Fixing one of the quadrics, we obtain the asserted degree of the hypersurface. □

2.3.3 Invariants of a pair of conics

In this case we have four invariants $\Theta_0, \Theta_1, \Theta_2, \Theta_3$, which are traditionally denoted by $\Delta, \Theta, \Theta', \Delta'$, respectively.

The polynomials

$$(R_0, R_1, R_2, R_3) = (\Theta\Theta', \Delta\Delta', \Theta'^3\Delta, \Theta^3\Delta')$$

are bihomogeneous of degrees $(3,3), (3,3), (6,6), (6,6)$. They define a rational map $\mathbb{P}^5 \times \mathbb{P}^5 \dashrightarrow \mathbb{P}(1,1,2,2)$. We have the obvious relation $R_0^3 R_1 - R_2 R_3 = 0$. After dehomogenization, we obtain rational functions

$$X = R_1/R_0^2,\ Y = R_2/R_0,\ Z = R_3/R_0^2$$

[1] The terminology is due to A. Cayley, taction = tangency.

such that $X = YZ$. The rational functions

$$Y = \Theta'\Delta/\Theta^2, \quad Z = \Theta\Delta'/\Theta'^2$$

generate the field of rational invariants of pairs of conics (see [536], p. 280). The polynomials R_0, R_1, R_2, R_3 generate the algebra of bihomogeneous invariants on $\mathbb{P}^5 \times \mathbb{P}^5$ with respect to the diagonal action of $\mathrm{SL}(4)$ and the GIT-quotient is isomorphic to the rational surface $V(t_0^3 t_1 - t_2 t_3)$ in the weighted projective space $\mathbb{P}(1,1,2,2)$. The surface is a normal surface with one singular point $[0,1,0,0]$ of type A_2. The singular point corresponds to a unique orbit of a pair of nonsingular conics (C, S) such that $C^\vee$ is apolar to S and $S^\vee$ is apolar to C. It is represented by the pair

$$t_0^2 + t_1^2 + t_2^2 = 0, \quad t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 = 0,$$

where $\epsilon = e^{2\pi i/3}$. The stabilizer subgroup of this orbit is a cyclic group of order 3 generated by a cyclic permutation of the coordinates.

Recall that the GIT-quotient parameterizes minimal orbits of semi-stable points. In our case, all semi-stable points are stable, and unstable points correspond to a pairs of conics, one of which has a singular point on the other conic.

Using the invariants $\Delta, \Theta, \Theta', \Delta'$, one can express the condition that the two conics are Poncelet related.

Theorem 2.3.14 *Let C and S be two nonsingular conics. A triangle inscribed in C and circumscribing S exists if and only if*

$$\Theta'^2 - 4\Theta\Delta' = 0.$$

Proof Suppose there is a triangle inscribed in C and circumscribing S. Applying a linear transformation, we may assume that the vertices of the triangle are the points $[1,0,0], [0,1,0]$ and $[0,0,1]$ and $C = V(t_0t_1 + t_0t_2 + t_1t_2)$. Let $S = V(g)$, where

$$g = at_0^2 + bt_1^2 + ct_2^2 + 2dt_0t_1 + 2et_0t_2 + 2ft_1t_2. \tag{2.19}$$

The triangle circumscribes S when the points $[1,0,0], [0,1,0], [0,0,1]$ lie on the dual conic $\check{S}$. This implies that the diagonal entries $bc - f^2, ac - e^2, ab - d^2$ of the matrix $\mathrm{adj}(B)$ are equal to zero. Therefore, we may assume that

$$g = \alpha^2 t_0^2 + \beta^2 t_1^2 + \gamma^2 t_2^2 - 2\alpha\beta t_0t_1 - 2\alpha\gamma t_0t_2 - 2\beta\gamma t_1t_2. \tag{2.20}$$

We get

$$\Theta' = \mathrm{Tr}\Big(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2\alpha\beta\gamma^2 & 2\alpha\gamma\beta^2 \\ 2\alpha\beta\gamma^2 & 0 & 2\beta\gamma\alpha^2 \\ 2\alpha\gamma\beta^2 & 2\beta\gamma\alpha^2 & 0 \end{pmatrix}\Big) = 4\alpha\beta\gamma(\alpha+\beta+\gamma),$$

$$\Theta = \mathrm{Tr}\Big(\begin{pmatrix} \alpha^2 & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & \beta^2 & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & \gamma^2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\Big) = -(\alpha+\beta+\gamma)^2,$$

$$\Delta' = -4(\alpha\beta\gamma)^2.$$

This checks that $\Theta'^2 - 4\Theta\Delta' = 0$.

Let us prove the sufficiency of the condition. Take a tangent line ℓ_1 to S intersecting C at two points x, y and consider tangent lines ℓ_2, ℓ_3 to S passing through x and y, respectively. The triangle with sides ℓ_1, ℓ_2, ℓ_3 circumscribes S and has two vertices on C. Choose the coordinates such that this triangle is the coordinate triangle. Then, we may assume that $C = V(at_0^2 + 2t_0t_1 + 2t_1t_2 + 2t_0t_2)$ and $S = V(g)$, where g is as in (2.20). Computing $\Theta'^2 - 4\Theta\Delta'$ we find that it is equal to zero if and only if $a = 0$. Thus the coordinate triangle is inscribed in C. □

Darboux's Theorem is another example of a poristic statement, with respect to the property of the existence of a polygon inscribed in one conic and circumscribing the other conic. Another example of a poristic statement is one of the equivalent properties of a pair of conics from Proposition 2.3.8: Given two nonsingular conics C and S, there exists a polar triangle of C inscribed in S, or, in other words, C is apolar to S.

Recall from Theorem 1.1.9 that any projective automorphism of $\mathbb{P}^n = |E|$ is a composition of two polarities $\phi, \psi : |E| \to |E^\vee|$.

Proposition 2.3.15 *Let C and S be two different nonsingular conics and $g \in \mathrm{Aut}(\mathbb{P}^2)$ be the composition of the two polarities defined by the conics. Then g is of order 3 if and only if C and S are mutually apolar.*

Proof Let A, B be symmetric 3×3 matrices corresponding to C and S. The conics C and S are mutually apolar if and only if $\mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. The projective transformation g is given by the matrix $X = AB^{-1}$. This transformation is of order 3 if and only if the characteristic polynomial $|X - \lambda I_3|$ of the matrix X has zero coefficients at λ, λ^2. Since $\mathrm{Tr}(X) = 0$, the coefficient at λ^2 is equal to zero. The coefficient at λ is equal to zero if and only if $\mathrm{Tr}(X^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. Thus g is of order 3 if and only if $\mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. □

Remark 2.3.16 It is immediate that any set of mutually apolar conics is linearly independent. Thus the largest number of mutually apolar conics is equal to six. The first example of a set of six mutually apolar conics was given by F. Gerbardi [241]. The following is a set of mutually apolar conics given by P. Gordan [256]:

$$
\begin{aligned}
&t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 = 0,\\
&t_0^2 + \epsilon^2 t_1^2 + \epsilon t_2^2 = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(t_0t_1 + t_0t_2 + t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(-t_0t_1 - t_0t_2 + t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(-t_0t_1 + t_0t_2 - t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(t_0t_1 - t_0t_2 - t_1t_2) = 0,
\end{aligned}
$$

where $\eta = e^{2\pi i/3}, r = \frac{-\sqrt{3}+\sqrt{-5}}{4}$. These six quadrics play an important role in the theory of invariants of the *Valentiner group* G, the subgroup of $\mathrm{PGL}(3)$ isomorphic to the alternating group $\mathfrak{A}_6$. in $\mathbb{C}^3$ with the algebra of invariants generated by three polynomials of degrees 6, 12 and 30. The invariant of degree 6 is the sum of cubes of the six mutually apolar quadratic forms. The invariant of degree 12 is their product. The invariant of degree 30 is also expressed in terms of the six quadratic forms but in a more complicated way (see [242], [256]). We refer to [248] for further discussion of mutually apolar conics.

Consider the set of polar triangles of C inscribed in S. We know that this set is either empty or of dimension ≥ 1. We consider each triangle as a set of its three vertices, i.e. as an effective divisor of degree 3 on S.

Proposition 2.3.17 *The closure X of the set of self-polar triangles with respect to C which are inscribed in S, if not empty, is a g^1_3, i.e. a linear pencil of divisors of degree 3.*

Proof First we use that two self-polar triangles with respect to C and inscribed in S which share a common vertex must coincide. In fact, the polar line of the vertex must intersect S at the vertices of the triangle. Then the assertion is proved using the argument from the proof of Proposition 2.2.4. □

Note that a general g^1_3 contains four singular divisors corresponding to ramification points of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^1$. In our case these divisors correspond to four intersection points of C and S.

Another example of a poristic statement is the following.

Theorem 2.3.18 *Let T and T' be two different triangles. The following assertions are equivalent:*

(i) there exists a conic S containing the vertices of the two triangles;
(ii) there exists a conic Σ touching the sides of the two triangles;
(iii) there exists a conic C with polar triangles T and T'.

Moreover, when one of the conditions is satisfied, there is an infinite number of triangles inscribed in S, circumscribed around Σ, and all of these triangles are polar triangles of C.

Proof (iii)$\Leftrightarrow$ (ii) According to Proposition 1.3.9, a conic C admits T as a polar triangle if the conics in the dual plane containing the sides of the triangle are all apolar to C. If T and T' are polar triangles of C, then the two nets of conics passing through the sides of the first and the second triangle intersect in the 4-dimensional space of apolar conics. The common conic is the conic Σ from (ii). Conversely, if Σ exists, the two nets contain a common conic and hence are contained in a 4-dimensional space of conics in the dual plane. The apolar conic is the conic C from (iii).

(iii)$\Leftrightarrow$ (i) This follows from the previous argument applying Corollary 2.1.4.

Let us prove the last assertion. Suppose one of the conditions of the Theorem is satisfied. Then we have the conics C, S, Σ with the asserted properties with respect to the two triangles T, T'. By Proposition 2.3.17, the set of self-polar triangles with respect to C inscribed in S is a g^1_3. By Proposition 2.2.4, the set of triangles inscribed in S and circumscribing Σ is also a g^1_3. Two g^1_3's with 2 common divisors coincide. □

Recall from Theorem 2.3.12 that the condition that two conics C and S touch each other is

$$27\Delta^2\Delta'^2 - 18\Theta\Theta'\Delta\Delta' + 4\Delta\Theta'^3 + 4\Delta'\Theta^3 - \Theta'^2\Theta^2 = 0. \tag{2.21}$$

The variety of pairs of touching conics is a hypersurface of bidegree $(6, 6)$ in $\mathbb{P}^5 \times \mathbb{P}^5$. In particular, conics touching a given conic is a hypersurface of degree 6 in the space of conics. This fact is used for the solution of the famous *Apollonius problem* in enumerative geometry: find the number of nonsingular conics touching five fixed general conics (see [231], Example 9.1.9).

Remark 2.3.19 Choose a coordinate system such that $C = V(t_0^2 + t_1^2 + t_2^2)$. Then the condition that S is Poncelet related to C with respect to triangles is easily seen to be equal to

$$c_2^2 - c_1 c_3 = 0,$$

where

$$\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3$$

is the characteristic polynomial of a symmetric matrix A defining S. This is

a quartic hypersurface in the space of conics. The polynomials c_1, c_2, c_3 generate the algebra of invariants of the group $\mathrm{SO}(3)$ acting on the space $V = S^2((\mathbb{C}^3)^\vee)$. If we use the decomposition $V = \mathcal{H}_q \oplus \mathbb{C}q$, where $q = t_0^2 + t_1^2 + t_2^2$ and $\mathcal{H}_q$ is the space of harmonic quadratic polynomials with respect to q, then the first invariant corresponds to the projection $\mathcal{H}_q \oplus \mathbb{C}q \to \mathbb{C}q$. Let $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ be the Veronese map with image equal to C. Then the pull-back map

$$\nu^* : V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$$

defines an isomorphism of the representation $\mathcal{H}_q$ of $\mathrm{SO}(3)$ with the representation $S^4((\mathbb{C}^2)^\vee)$ of $\mathrm{SL}(2)$. Under this isomorphism, the invariants c_2 and c_3 correspond to the invariants S and T on the space of binary quartics from Example 1.5.2. In particular, the fact that a harmonic conic is Poncelet related to C is equivalent to the corresponding binary quartic admiting an apolar binary quadric. Also, the discriminant invariant of degree 6 of binary quartics corresponds to the condition that a harmonic conic touches C.

2.3.4 The Salmon conic

One can also look for *covariants* or *contravariants* of a pair of conics, that is, rational maps $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(d)|$ or $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(d)|^\vee$ which are defined geometrically, i.e. not depending on a choice of projective coordinates.

Recall the definition of the *cross ratio* of four distinct ordered points $p_i = [a_i, b_i]$ on $\mathbb{P}^1$

$$R(p_1p_2; p_3p_4) = \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}, \tag{2.22}$$

where

$$p_i - p_j = \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = a_i b_j - a_j b_i.$$

It is immediately checked that the cross ratio does not take the values $0, 1, \infty$. It does not depend on the choice of projective coordinates. It is also invariant under a permutation of the four points equal to the product of two commuting transpositions. The permutation (12) changes R to $-R/(1-R)$ and the permutation (23) changes R to $1/R$. Thus there are at most six possible cross ratios for an ordered set of four points

$$R,\ \frac{1}{R},\ 1 - R,\ \frac{1}{1-R},\ \frac{R}{R-1},\ \frac{R-1}{R}.$$

The number of distinct cross ratios may be reduced to three or two. The first

case happens if and only if one of them is equal to -1 (the other ones will be 2 and $1/2$). The unordered set of four points in this case is called a *harmonic quadruple*. The second case happens when R satisfies $R^2+R+1=0$, i.e. R is one of two cubic roots of 1 not equal to 1. In this case we have an *equianharmonic quadruple*.

If we identify the projective space of binary forms of degree 2 with the projective plane, the relation (2.3) can be viewed as a symmetric hypersurface H of bidegree $(1,1)$ in $\mathbb{P}^2\times\mathbb{P}^2$. In particular, it makes sense to speak about harmonically conjugate pairs of maybe coinciding points. We immediately check that a double point is harmonically conjugate to a pair of points if and only if it coincides with one of the roots of this form.

We can extend the definition of the cross ratio to any set of points no three of which coincide by considering the cross ratios as the point

$$\mathbf{R}=[(p_1-p_2)(p_3-p_4),(p_1-p_3)(p_2-p_4)]\in\mathbb{P}^1. \tag{2.23}$$

It is easy to see that two points coincide if and only if $\mathbf{R}=[0,1],[1,1],[1,0]$. This corresponds to $R=0,1,\infty$.

Two pairs of points $\{p_1,p_2\}$ and $\{q_1,q_2\}$ are harmonically conjugate in the sense of definition (2.3) if and only if $R(p_1q_1;q_2p_2)=-1$. To check this, we may assume that p_1,p_2 are roots of $f=\alpha t_0^2+2\beta t_0t_1+\gamma t_1^2$ and q_1,q_2 are roots of $g=\alpha' t_0^2+2\beta' t_0t_1+\gamma' t_1^2$, where, for simplicity, we may assume that $\alpha,\alpha'\neq 0$ so that, in affine coordinates, the roots x,y of the first equations satisfy $x+y=-2\beta/\alpha, xy=\gamma/\alpha$ and similarly the roots of the second equation x',y' satisfy $x'+y'=-2\beta'/\alpha', x'y'=\gamma'/\alpha'$. Then

$$R(xx';y'y)=\frac{(x-x')(y'-y)}{(x-y')(x'-y)}=-1$$

if and only if

$$(x-x')(y'-y)+(x-y')(x'-y)=(x+y)(x'+y')-2xy-2x'y$$

$$=\frac{4\beta\beta'}{\alpha\alpha'}-\frac{2\gamma}{\alpha}-\frac{2\gamma'}{\alpha'}=-2\frac{\alpha\gamma'+\alpha'\gamma-2\beta\beta'}{\alpha\alpha'}=0.$$

So we see that the two pairs of roots form a harmonic quadruple if and only if (2.3) holds.

The expression $\alpha\gamma'+\alpha'\gamma-2\beta\beta''$ is an invariant of a pair (f,g) of binary quadratic forms. It is equal to the coefficient at t for the discriminant of $f+tg$. It is analogous to the invariants Θ and Θ' for a pair of conics.

The *Salmon conic* associated to a pair of conics C and C' is defined to be the locus $\mathsf{S}(C,C')$ of points x in $\mathbb{P}^2$ such that the pairs of the tangents through x to C and to C' are harmonically conjugate. Note that it makes sense even

when x lies on one of the conics. In this case one considers the corresponding tangent as the double tangent.

Let A be a square symmetric 3×3-matrix. The entries of the adjugate matrix $\mathrm{adj}(A)$ are quadratic forms in the entries of A. By polarization, we obtain

$$\mathrm{adj}(\lambda_0 A+\lambda_1 B)=\lambda_0^2\mathrm{adj}(A)+\lambda_0\lambda_1\mathrm{adj}(A,B)+\lambda_1^2\mathrm{adj}(B),$$

where $(A,B)\to \mathrm{adj}(A,B)$ is a bilinear function of A and B.

Theorem 2.3.20 *Let $C=V(q), C'=V(q')$, where q and q' are quadratic forms defined by symmetric matrices $A=(a_{ij})$ and $B=(b_{ij})$. Then the Salmon conic $\mathsf{S}(C,C')$ is defined by the matrix* $\mathrm{adj}(\mathrm{adj}(A),\mathrm{adj}(B))$.

Proof By duality, the pencil of lines through a point $x=[x_0,x_1,x_2]$ corresponds to the line $\ell_x=V(x_0u_0+x_1u_1+x_2u_2)$ in the dual plane with dual coordinates u_0,u_1,u_2. Without loss of generality, we may assume that $x_2=-1$. Let $C^\vee, C'^\vee$ be the dual conics defined by the matrices $\mathrm{adj}(A)=(A_{ij}), \mathrm{adj}(B)=(B_{ij})$. The intersection of the line ℓ_x with $C^\vee$ is equal to two points $[u_0,u_1,x_0u_0+x_1u_1]$ such that

$$(A_{00}+A_{02}x_0+A_{22}x_0^2)u_0^2+(A_{11}+A_{12}x_1+A_{22}x_1^2)u_1^2$$

$$+2(A_{22}x_0t_1+A_{02}x_1+A_{12}x_0+A_{01})u_0u_1=0.$$

Replacing A with B, we get the similar formula for the intersection of ℓ with $C'^\vee$. The intersection points $[u_0,u_1,x_0u_0+x_1u_1]$ correspond to the tangent lines to C and C' passing through the point x. By (2.3), they are harmonically conjugate if and only if

$$(A_{00}+A_{02}x_0+A_{22}x_0^2)(B_{11}+B_{12}x_1+B_{22}x_1^2)$$

$$+(B_{00}+B_{02}x_0+B_{22}x_0^2)(A_{11}+A_{12}x_1+A_{22}x_1^2)$$

$$-2(A_{22}x_0t_1+A_{02}x_1+A_{12}x_0+A_{01})(B_{22}x_0x_1+B_{02}x_1+B_{12}x_0+B_{01})=0.$$

This gives the equation of the Salmon conic $S(C,C')$:

$$(A_{22}B_{11}+A_{11}B_{22}-2A_{12}B_{12})x_0^2+(A_{00}B_{22}+A_{22}B_{00}-2A_{02}B_{02})x_1^2$$

$$+(A_{00}B_{11}+A_{11}B_{00}-2A_{01}B_{01})x^2+2(A_{02}B_{12}+A_{12}B_{02}-A_{22}B_{02}-A_{02}B_{22})x_0x_1$$

$$+2(A_{02}B_{11}+A_{11}B_{02}-A_{12}B_{01}-A_{01}B_{12})x_0x_2$$

$$+2(A_{00}B_{12}+A_{12}B_{00}-A_{02}B_{01}-A_{01}B_{02})x_1x_2=0.$$

It is easy to see that the symmetric matrix defining this quadratic form is equal to $\mathrm{adj}(\mathrm{adj}(A),\mathrm{adj}(B))$. □

Let $\mathsf{S}(C, C') = V(s)$. Consider the pencil generated by $C^\vee$ and $C'^\vee$. In matrix notation, it is equal to the pencil of matrices $\mathrm{adj}(A) + t\mathrm{adj}(B)$. The dual conics of this pencil form a quadratic family of conics defined by the matrices $\mathrm{adj}(\mathrm{adj}(A) + t\mathrm{adj}(B)) = |A|A + tS + t^2|B|B$, where S is the matrix defining the Salmon conic. Its members are tangent to the quartic curve $V(s^2 - 4|A||B|qq')$. Since the members of the linear pencil pass through the four points $C^\vee \cap C'^\vee$, all members of the quadratic family are tangent to the four common tangents of C and C'. Thus

$$V(s^2 - 4|A||B|qq') = V(l_1 l_2 l_3 l_4), \tag{2.24}$$

where $V(l_i)$ are the common tangents. This implies the following remarkable property of the Salmon conic.

Theorem 2.3.21 *Let C and C' be two conics such that the dual conics intersect at four distinct points representing the four common tangents of C and S. Then the eight tangency points lie on the Salmon conic associated with C and C'.*

Here is another proof of the theorem that does not use (2.24). Let x be a point where the Salmon conic meets C. Then the tangent line ℓ through x to C represents a double line in the harmonic pencil formed by the four tangents through x to C and S. As we remarked before, the conjugate pair of lines must contain ℓ. Thus ℓ is a common tangent to C and S and hence x is one of the eight tangency points. Conversely, the argument is reversible and shows that every tangency point lies on the Salmon conic.

The Salmon conic represents a covariant of pairs of conics. A similar construction gives a contravariant conic in the dual plane, called the *Salmon envelope conic* $\mathsf{S}'(C, C')$. It parameterizes lines which intersect the dual conics C and C' at two pairs of harmonically conjugate points. We leave it to the reader to show that its equation is equal to

$$(a_{22}b_{11} + a_{11}b_{22} - 2a_{12}b_{12})u_0^2 + (a_{00}b_{22} + a_{22}b_{00} - 2a_{02}b_{02})u_1^2$$

$$+(a_{00}b_{11} + a_{11}b_{00} - 2a_{01}b_{01})u_2^2 + 2(a_{02}b_{12} + a_{12}b_{02} - a_{22}b_{02} - a_{02}b_{22})u_0u_1$$

$$+2(a_{02}b_{11} + a_{11}b_{02} - a_{12}b_{01} - a_{01}b_{12})u_0u_2$$

$$+2(a_{00}b_{12} + a_{12}b_{00} - a_{02}b_{01} - a_{01}b_{02})u_1u_2 = 0.$$

If we write $\mathsf{S}'(C, C') = V(s')$, we find, as above, that $V(s'^2 - q^\vee q'^\vee)$ is equal to the union of four lines corresponding to intersection points of $C \cap C'$. This implies that the Salmon envelope conic passes through the eight points corresponding to the eight tangents of C and C' at the intersection points.

The equation of the Salmon conic is greatly simplified if we simultaneously diagonalize the quadrics q and q' defining C and C'. Assume $q = t_0^2 + t_1^2 + t_2^2, q' = at_0^2 + bt_1^2 + ct_2^2$. Then the equation of $\mathsf{S}(C, C')$ becomes

$$a(b+c)t_0^2 + b(c+a)t_1^2 + c(a+b)t_2^2 = 0,$$

and the equation of $\mathsf{S}'(C, C')$ becomes

$$(b+c)u_0^2 + (c+a)u_1^2 + (a+b)u_2^2 = 0.$$

By passing to the dual conic, we see that the dual conic $\mathsf{S}'(C, C')^\vee$ is different from $\mathsf{S}(C, C')$. Its equation is

$$(a+c)(a+b)t_0^2 + (a+b)(b+c)t_1^2 + (a+b)(b+c)t_2^2 = 0.$$

It can be expressed as a linear combination of the equations of C, C' and $\mathsf{S}(C, C')$

$$(a+c)(a+b)t_0^2+(a+b)(b+c)t_1^2+(a+b)(b+c)t_2^2 = (ab+bc+ac)(t_0^2+t_1^2+t_2^2)$$

$$+(a+b+c)(at_0^2 + bt_1^2 + ct_2^2) - (a(b+c)t_0^2 + b(c+a)t_1^2 + c(a+b)t_2^2).$$

Remark 2.3.22 The full system of covariants, and contravariants of a pair of conics is known (see [258], p. 286.) The curves $C, C', \mathsf{S}'(C, C')$, the Jacobian of C, C', and $\mathsf{S}(C, C')$ generate the algebra of covariants over the ring of invariants. The envelopes $C^\vee, C'^\vee, \mathsf{S}'(C, C')$, the Jacobian $C^\vee, C'^\vee$, and $\mathsf{S}'(C, C')$ generate the algebra of contravariants.

Exercises

2.1 Let E be a vector space of even dimension $n = 2k$ over a field K of characteristic 0 and $(e_1, \dots, e_n)$ be a basis in E. Let $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j \in \bigwedge^2 E^\vee$ and $A = (a_{ij})_{1 \le i \le j \le n}$ be the skew-symmetric matrix defined by the coefficients a_{ij}. Let $\bigwedge^k(\omega) = \omega \wedge \cdots \wedge \omega = ak! e_1 \wedge \cdots \wedge e_n$ for some $a \in F$. The element a is called the *pfaffian* of A and is denoted by $\mathrm{Pf}(A)$.

(i) Show that

$$\mathrm{Pf}(A) = \sum_{S \in \mathcal{S}} \epsilon(S) \prod_{(i,j) \in S} a_{ij},$$

where S is a set of pairs $(i_1, j_1), \dots, (i_k, j_k)$ such that $1 \le i_s < j_s \le 2k, s = 1, \dots, k$, $\{i_1, \dots, i_k, j_1, \dots, j_k\} = \{1, \dots, n\}$, $\mathcal{S}$ is the set of such sets S, $\epsilon(S) = 1$ if the permutation $(i_1, j_1, \dots, i_k, j_k)$ is even and -1 otherwise.

(ii) Compute $\mathrm{Pf}(A)$ when $n = 2, 4, 6$.

(iii) Show that, for any invertible matrix C,

$$\mathrm{Pf}({}^tC \cdot A \cdot C) = \det(C)\mathrm{Pf}(A).$$

(iv) Using (iii) prove that

$$\det(A) = \mathrm{Pf}(A)^2.$$

(iv) Show that

$$\mathrm{Pf}(A) = \sum_{i=1}^{n} (-1)^{i+j-1} \mathrm{Pf}(A_{ij}) a_{ij},$$

where A_{ij} is the matrix of order $n-2$ obtained by deleting the i-th and j-th rows and columns of A.

(v) Let B be a skew-symmetric matrix of odd order $2k-1$ and B_i be the matrix of order $2k-2$ obtained from B by deleting the i-th row and i-th column. Show that the vector $(\mathrm{Pf}(B_1), \ldots, (-1)^{i+1}\mathrm{Pf}(B_i), \ldots, \mathrm{Pf}(B_{2k-1}))$ is a solution of the equation $B \cdot x = 0$.

(vi) Show that the rank of a skew-symmetric matrix A of any order n is equal to the largest m such that there exist $i_1 < \ldots < i_m$ such that the matrix $A_{i_1 \ldots i_m}$ obtained from A by deleting i_j-th rows and columns, $j = 1, \ldots, m$, has nonzero pfaffian .

2.2 Let $\mathsf{V} = \nu_2(\mathbb{P}^2)$ be a Veronese surface in $\mathbb{P}^5$, considered as the space of conics in $\mathbb{P}^2$.

(i) Let Λ be a plane in $\mathbb{P}^5$ and $\mathcal{N}_\Lambda$ be the net of conics in $\mathbb{P}^2$ cut out by hyperplanes containing Λ. Show that Λ is a trisecant plane if and only if the set of base points of $\mathcal{N}_\Lambda$ consists of ≥ 3 points (counting with multiplicities). Conversely, a net of conics through three points defines a unique trisecant plane.

(ii) Show that the nets of conics with two base points, one of them is infinitely near, forms an irreducible divisor in the variety of trisecant planes.

(iii) Using (ii), show that the anticanonical divisor of degenerate triangles is irreducible.

(iv) Show that the trisecant planes intersecting the Veronese plane at one point (corresponding to net of conics with one base point of multiplicity 3) define a smooth rational curve in the boundary of the variety of self-polar triangles. Show that this curve is equal to the set of singular points of the boundary.

2.3 Let $U \subset (\mathbb{P}^2)^{(3)}$ be the subset of the symmetric product of $\mathbb{P}^2$ parameterizing the sets of three distinct points. For each set $Z \in U$ let L_Z be the linear system of conics containing Z. Consider the map $f : U \to G_2(\mathbb{P}^5), Z \mapsto L_Z \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$.

(i) Consider the divisor D in U parameterizing sets of 3 distinct collinear points. Show that $f(D)$ is a closed subvariety of $G_2(\mathbb{P}^5)$ isomorphic to $\mathbb{P}^2$.

(ii) Show that the map f extends to the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ of 0-cycles Z with $h^0(\mathcal{O}_Z) = 3$.

(iii) Show that the closure $\bar{D}$ of $\pi^{-1}(D)$ in the Hilbert scheme is isomorphic to a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$ and the restriction of f to $\bar{D}$ is the projection map to its base.

(iv) Define the map $\tilde{f} : \mathcal{P} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$, which assigns to a point in the fiber $p^{-1}(Z)$ the corresponding conic in the net of conics though Z. Show that the fiber of $\tilde{f}$ over a nonsingular conic C is isomorphic to the Fano variety of self-polar triangles of the dual conic $C^\vee$.

(v) Let $\mathcal{P}^s = \tilde{f}^{-1}(\mathsf{D}_2(2))$ be the preimage of the hypersurface of singular conics. Describe the fibres of the projections $p : \mathcal{P}^s \to (\mathbb{P}^2)^{[3]}$ and $\tilde{f} : \mathcal{P}^s \to \mathsf{D}_2(2)$.

2.4 Identify $\mathbb{P}^1$ with its image under a Veronese map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$.

(i) Show that any involution of $\mathbb{P}^1$ (i.e. an automorphism of order 2) coincides with the involution of the Veronese conic obtained by projection from a point not lying on the conic (called the center of the involution).
(ii) Show that two involutions of $\mathbb{P}^1$ without common fixed points commute if and only if the two pairs of fixed points are harmonically conjugate.
(iii) Show that the product of three involutions is an involution if their centers are collinear (J. Valles). The converse is known for any number of involutions.

2.5 Prove that two unordered pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are harmonically conjugate if and only if there is an involution of $\mathbb{P}^1$ with fixed points a, b that switches c and d.

2.6 Prove the following *Hesse's Theorem*. If two pairs of opposite vertices of a quadrilateral are each conjugate for a conic, then the third pair is also conjugate. Such a quadrilateral is called a *Hesse quadrilateral*. Show that four lines form a polar quadrilateral for a conic if and only if it is a Hesse quadrilateral.

2.7 A tetrad of points p_1, p_2, p_3, p_4 in the plane is called *self-conjugate* with respect to a nonsingular conic if no three points are collinear and the pole of each side $\overline{p_i p_j}$ lies on the opposite side $\overline{p_k p_l}$.

(i) Given two conjugate triangles, show that the vertices of one of the triangles together with the center of perspectivity form a self-conjugate tetrad.
(ii) Show that the four lines with poles equal to p_1, p_2, p_3, p_4 form a polar quadrilateral of the conic and any nondegenerate polar quadrilateral is obtained in this way from a self-conjugate tetrad.
(iii) Show that any polar triangle of a conic can be extended to a polar quadrilateral.

2.8 Extend Darboux's Theorem to the case of two tangent conics.

2.9 Show that the secant lines of a Veronese curve R_m in $\mathbb{P}^m$ are parameterized by the surface in the Grassmannian $G_1(\mathbb{P}^m)$ isomorphic to $\mathbb{P}^2$. Show that the embedding of $\mathbb{P}^2$ into the Grassmannian is given by the Schwarzenberger bundle.

2.10 Let U be a 2-dimensional vector space. Use the construction of curves of degree $n-1$ Poncelet related to a conic to exhibit an isomorphism of linear representations $\bigwedge^2(S^n U)$ and $S^{n-1}(S^2 U)$ of $\mathrm{SL}(U)$.

2.11 Assume that the pencil of sections of the Schwarzenberger bundle $\mathcal{S}_{n,C}$ has no base points. Show that the Poncelet curve associated to the pencil is nonsingular at a point x defined by a section s from the pencil if and only if the scheme of zeros $Z(s)$ is reduced.

2.12 Find a geometric interpretation of vanishing of the invariants Θ, Θ' from (2.16) in the case when C or S is a singular conic.

2.13 Let p_1, p_2, p_3, p_4 be four distinct points on a nonsingular conic C. Show that the triangle with the vertices $A = \overline{p_1 p_3} \cap \overline{p_2 p_4}$, $B = \overline{p_1 p_2} \cap \overline{p_3 p_4}$ and $C = \overline{p_1 p_4} \cap \overline{p_2 p_3}$ is a self-conjugate triangle with respect to C.

2.14 Show that two pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ with a common point are nver harmonically conjugate.

2.15 Let (a, b, c, d) be a quadrangle in $\mathbb{P}^2$, and p, q be the intersection points of two pairs of opposite sides $\overline{ab}, \overline{cd}$ and $\overline{bc}, \overline{ad}$. Let p', q' be the intersection points of the line $\overline{pq}$ with the diagonals $\overline{ac}$ and $\overline{bd}$. Show that the pairs (p, q) and (p', q') are harmonically conjugate .

2.16 Show that the pair of points on a diagonal of a complete quadrilateral defined by its sides is harmonically conjugate to the pair of points defined by intersection with other two diagonals.

2.17 Show that a general net of conics admits a common polar quadrangle.

2.18 Show that four general conics admit a unique common polar quadrangle.

2.19 Find the condition on a pair of conics expressing that the associate Salmon conic is degenerate.

2.20 Show that the triangle formed by any three tangents to two general conics is in perspective with any three of common points.

2.21 Show that the set of $2n+2$ vertices of two self-polar $(n+1)$-hedra of a quadric in $\mathbb{P}^n$ impose one less condition on quadrics. In particular, two self-polar triangles lie on a conic, two self-polar tetrahedra are the base points of a net of quadrics.

2.22 A hexad of points in $\mathbb{P}^3$ is called self-conjugate with respect to a nonsingular quadric if no four are on the plane and the pole of each plane spanned by three points lies on the plane spanned by the remaining three points.. Show that the quadric admits a nondegenerate polar hexahedron whose planes are polar planes of points in the hexad. Conversely, any nondegenerate polar hexahedron of the quadric is obtained in this way from a self-conjugate tetrad.

2.23 Show that the variety of sums of five powers of a nonsingular quadric surface is isomorphic to the variety of self-conjugate pentads of points in $\mathbb{P}^3$.

2.24 Consider 60 Pascal lines associated with a hexad of points on a conic. Prove the following properties of the lines.

(i) There are 20 points at which three of Pascal lines intersect, called the *Steiner points*.
(ii) The 20 Steiner points lie on 15 lines, each containing four of the points (the *Plücker lines*).
(iii) There are 60 points each contained in three Plücker lines (the *Kirkman points*).

2.25 Prove the following generalization of Pascal's Theorem. Consider the 12 intersection points of a nonsingular quadric surface Q with six edges of a tetrahedron T with vertices p_1, p_2, p_3, p_4. For each vertex p_i choose one of the 12 points on each edge $\overline{p_i p_j}$ and consider the plane Λ_i spanned by these three points. Show that the four lines in which each of these four planes meats the opposite face of the tetrahedron are rulings of a quadric. This gives 32 quadrics associated to the pair (T, Q) [95], p. 400, [21], v. 3, Ex. 15, [494], p. 362.

2.26 Let $\Theta_0, \ldots, \Theta_4$ be the invariants of a pair of quadric surfaces.

(i) Show that the five products $\Theta_2, \Theta_0\Theta_4, \Theta_1\Theta_3, \Theta_1^2\Theta_4, \Theta_3^2\Theta_0$ generate the algebra of invariants of bidegrees (m, n) with $m = n$.
(ii) Show that the GIT-quotient of ordered pairs of quadrics by the group $\mathrm{SL}(4)$ is isomorphic to the hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$ given by the equation $t_1 t_2^2 - t_3 t_4 = 0$.
(iii Show that the GIT-quotient has a singular line and its general point corresponds to the orbit of the pair $V(\sum t_i^2), V((t_0^2 - t_1^2) + a(t_2^2 - t_3^2))$.

Historical Notes

There is a great number of books dealing with the analytic geometry of conics. The most comprehensive source for the history of the subject is Coolidge's book [127]. Many facts and results about real conics treated in a synthetic way can be found in text-books in projective geometry. Coxeter's small book [134] is one of the best.

The theory of polarity for conics goes back to Poncelet [452]. Polar triangles and tetrahedra of a conic and a quadric surface were already studied by P. Serret [529]. In particular, he introduced the notion of a self-conjugate triangles, quadrangles and pentagons. They were later intensively studied by T. Reye [461], [466] and R. Sturm [552], B. 3. The subject of their study was called the *Polarraum*, i.e. a pair consisting of a projective space together with a nonsingular quadric.

Pascal's Theorem was discovered by B. Pascal in 1639 when he was 16 years old [431] but not published until 1779 [431]. It was independently rediscovered by C. MacLaurin in 1720 [374]. A large number of results about the geometry and combinatorics of 60 Pascal lines assigned to six points on a conic have been discovered by J. Steiner, J. Kirkman, A. Cayley, G. Salmon, L. Cremona and others. A good survey of these results can be found in Note 1 in Baker's book [21], v.2, and Notes in Salmon's book [491]. We will return to this in Chapter 9.

Poncelet's Closure Theorem which is the second part of Darboux's Theorem 2.2.2 was first discoverd by J. Poncelet himself [452]. We refer the reader to the excellent account of the history of the Poncelet related conics in [49]. A good elementary discussion of Poncelet's Theorem and its applications can be found in Flatto's book [222]. Other elementary and non-elementary treatments of the Poncelet properties and their generalizations can be found in [26], [27], [118], [120], [265], [266].

The relationship between Poncelet curves and vector bundles is discussed in [577], [409], [578], [582]. The Schwarzenberger bundles were introduced in [508]. We followed the definition given in [178]. The papers [397] and [298], [299] discuss the compactification of the variety of conjugate triangles. The latter two papers of N. Hitchin also discuss an interesting connection with Painleve equations.

The notion of the apolarity of conics is due to T. Reye [464]. However, J. Rosanes [479] used this notion before under the name conjugate conics. In the same paper he also studied the representation of a conic as a sum of four squares of linear forms. The condition (2.8) for conjugate conics was first discovered by O. Hesse in [290]. He also proved that this property is poristic.

The condition for Poncelet relation given in terms of invariants of a pair of conics (Theorem 2.3.14) was first discovered by A. Cayley [76], [81].

The theory of invariants of two conics and two quadric surfaces was first developed by G. Salmon (see [491], [493], vol. 1). The complete system of invariants, covariants and contravariants of a pair of conics was given by J. Grace and A. Young [258]. P. Gordan has given a complete system of 580 invariants, covariants and contravariants of a pair of quadric surfaces [255]. Later H.W. Turnbull was able to reduce it to 123 elements [579]. In series of papers by J. Todd one can find further simplifications and more geometric interpretations of the system of combinants of two quadric surfaces [573], [574]. A good expositions of the theory of invariants can be found in Sommerville's and Todd's books [537], [575]. The latter book contains many examples and exercises, some of which were borrowed here.

Chasles' Theorem 2.3.4 about the covariant quadric was proven by him in [92] and reproved later by N. Ferrers [218] . A special case was known earlier to E. Bobillier [45]. Chasles' generalization of Pascal's Theorem to quadric surfaces can be found in [95]. Baker's book [21], v. 3, gives a good exposition of polar properties of quadric surfaces.

The proof of Theorem 2.3.21 is due to J. Coolidge [127], Chapter VI, §3. The result was known to G. von Staudt [540] (see [127], p. 66) and can be also found in Salmon's book on conics [491], p. 345. Although Salmon writes in the footnote on p. 345 that "I believe that I was the first to direct the attention to the importance of this conic in the theory of two conics", this conic was already known to Ph. La Hire [353] (see [127], p. 44). In Sommerville's book [536], the Salmon conic goes under the name *harmonic conic-locus* of two conics.

3

Plane cubics

3.1 Equations

3.1.1 Elliptic curves

There are many excellent expositions of the theory of elliptic curves from their many aspects: analytical, algebraic and arithmetical (a short survey can be found in Hartshorne's book [282], Chapter IV). We will be brief here.

Let X be a nonsingular projective curve of genus 1. By Riemann-Roch, for any divisor D of degree $d \geq 1$, we have $\dim H^0(X, \mathcal{O}_X(D)) = d$. If $d > 2$, the complete linear system $|D|$ defines an isomorphism $X \to C$, where C is a curve of degree d in $\mathbb{P}^{d-1}$ (called an *elliptic normal curve* of degree d). If $d = 2$, the map is of degree 2 onto $\mathbb{P}^1$. The divisor classes of degree 0 are parameterized by the Jacobian variety $\mathrm{Jac}(X)$ isomorphic to X. Fixing a point x_0 on X, the group law on $\mathrm{Jac}(X)$ transfers to a group law on X by assigning to a divisor class $\mathfrak{d}$ of degree 0 the divisor class $\mathfrak{d} + x_0$ of degree 1 represented by a unique point on X. The group law becomes

$$x \oplus y = z \in |x + y - x_0|. \tag{3.1}$$

The translation automorphisms of X act transitively on the set $\mathrm{Pic}^d(X)$ of divisor classes of degree d. This implies that two elliptic normal curves are isomorphic if and only if they are projectively equivalent. In the case $d = 2$, this implies that two curves are isomorphic if and only if the two sets of four branch points of the double cover are projectively equivalent.

In this Chapter we will be mainly interested in the case $d = 3$. The image of X is a nonsingular plane cubic curve. There are two known normal forms for its equation. The first one is the *Weierstrass form* and the second one is the *Hesse form*. We will deal with the Hesse form in the next subsection. Let us start with the Weierstrass form.

By Proposition 1.1.17, $C = V(f)$ has an inflection point p_0. Without loss of

generality, we may assume that $p_0 = [0, 0, 1]$ and the inflection tangent line at this point has the equation $t_0 = 0$. The projection from p_0 is the double cover $C \to \mathbb{P}^1$. It has ramification branch points, the intersection points of C with the first polar. There are four tangent lines to C containing p_0. One of them is $V(t_0)$. The first polar $V(\frac{\partial f}{\partial t_2})$ of the point p_0 is a singular conic that intersects C at the tangency points of the four tangents, we immediately obtain that it consists of the line $V(t_0)$ and a line $V(t_2 + at_1 + bt_0)$ not passing through the point p_0. Changing the coordinates, we may assume that the line is equal to $V(t_2)$. Now the equation of C takes the form

$$t_0t_2^2 + \alpha t_1^3 + \beta t_1^2t_0 + \gamma t_1t_0^2 + \delta t_0^3 = 0,$$

where $\alpha \neq 0$. Replacing t_1 with $t_1 + \frac{\beta}{3\alpha}t_0$, and scaling the coordinates, we may assume that $\alpha = 1$ and $\beta = 0$. This gives us the *Weierstrass equation* of a nonsingular cubic:

$$t_0t_2^2 + t_1^3 + at_1t_0^2 + bt_0^3 = 0. \tag{3.2}$$

It is easy to see that C is nonsingular if and only if the polynomial $x^3 + ax + b$ has no multiple roots, or, equivalently, its discriminant $\Delta = 4a^3 + 27b^2$ is not equal to zero.

Two Weierstrass equations define isomorphic elliptic curves if and only if there exists a projective transformation transforming one equation to another. It is easy to see that it happens if and only if $(\alpha', \beta') = (\lambda^3\alpha, \lambda^2\beta)$ for some nonzero constant λ. This can be expressed in terms of the *absolute invariant*

$$j = 2^6 3^3 \frac{a^3}{4a^3 + 27b^2}. \tag{3.3}$$

Two elliptic curves are isomorphic if and only if their absolute invariants are equal.[1]

The projection $[t_0, t_1, t_2] \mapsto [t_0, t_1]$ exhibits C as a double cover of $\mathbb{P}^1$. Its ramification points are the intersection points of C and its polar conic $V(t_0t_2)$. The cover has four branch points $[1, \lambda], [0, 1]$, where $\lambda^3 + a\lambda + b = 0$. The corresponding points $[1, \lambda, 0]$, and $[0, 0, 1]$ on C are the ramification points. If we choose $p_0 = [0, 0, 1]$ to be the zero point in the group law on C, then $2p \sim 2p_0$ for any ramification point p implies that p is a 2-torsion point. Any 2-torsion point is obtained in this way.

It follows from the above computation that any nonsingular plane cubic $V(f)$ is projectively isomorphic to the plane cubic $V(t_2^2t_0 + t_1^3 + at_1t_0^2 + bt_0^3)$. The functions $S : f \mapsto a/27, T : f \mapsto 4b$ can be extended to the *Aronhold*

[1] The coefficient $1728 = 2^6 3^3$ is needed to make this work in characteristic 2 and 3, otherwise j would not be defined for example when $a = 1, b = 0$ in characteristic 2.

invariants S and T of degrees 4 and 6 of a ternary cubic form. The explicit expressions of S and T in terms of the coefficients of f are rather long and can be found in many places (e.g. [182], [492]).

Fixing an order on the set of branch points, and replacing them by a projectively equivalent set, we may assume that the cubic polynomial $x^3 + ax + b$ is equal to $-x(x-1)(x-\lambda)$. This gives an affine equation of C

$$y^2 = x(x-1)(x-\lambda),$$

called the *Legendre equation.*

The number λ is equal to the cross ratio $R(q_1q_2; q_3q_4)$ of the four ordered branch points $(q_1, q_2, q_3, q_4) = (0, \lambda, 1, \infty)$. The absolute invariant (3.3) is expressed in terms of λ to give the following formula:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}. \tag{3.4}$$

Remark 3.1.1 For any binary form $g(t_0, t_1)$ of degree 4 without multiple zeros, the equation

$$t_2^2 + g(t_0, t_1) = 0 \tag{3.5}$$

defines an elliptic curve X in the weighted projective plane $\mathbb{P}(1, 1, 2)$. The four zeros of g are the branch points of the projection $X \to \mathbb{P}^1$ to the first two coordinates. So, every elliptic curve can be given by such an equation. The coefficients a, b in the Weierstrass equation are expressed in terms of the invariants S and T of binary quartics from Example 1.5.2. We have $a = -4S, b = -4T$. In particular.

$$j = \frac{27S(g)^3}{S(g)^3 - 27T(g)^2}.$$

Definition 3.1.2 *A nonsingular plane cubic $V(f)$ with Weierstrass Equation (3.2) is called* harmonic *(resp.* equianharmonic*) if $b = 0$ (resp. $a = 0$).*

We leave it to the reader to prove the following.

Theorem 3.1.3 *Let $C = V(f)$ be a nonsingular plane cubic and c be any point on C. The following conditions are equivalent.*

(i) C is a harmonic (resp. equianharmonic cubic).
(ii) The absolute invariant $j = 1728$ (resp. $j = 0$).
(iii) The set of cross ratios of four roots of the polynomial $t_0(t_1^3 + at_1t_0^2 + bt_0^3)$ is equal to $\{-1, 2, \frac{1}{2}\}$ (resp. consists of two primitive cube roots of -1).

(iv) The group of automorphisms of C leaving the point c invariant is a cyclic group of order 4 (resp. 6).

Note that C is a harmonic cubic if and only if the invariant T of degree on the space of binary quartic forms (1.82) vanishes on the binary form g in Equation (3.5). A quartic binary form on which T vanishes is called a *harmonic binary quartic*. We know that a binary form g is harmonic if and only if it admits an apolar binary quadratic form. One can check that this form is nondegenerate if and only if g has no multiple zeros. In this case it can be written as a sum of two powers of linear forms $l_1^4 + l_2^4$. This exhibits an obvious symmetry of order 4. Changing coordinates we can reduce the form to $t_0^4 - t_1^4 = (t_0^2 + t_1^2)(t_0^2 - t_1^2)$. The pairs of zeros of the factors are harmonically conjugate pairs of points. This explains the name harmonic cubic.

Theorem 3.1.3 gives a geometric interpretation for the vanishing of the quadratic invariant S (1.82) on the space of binary quartics. It vanishes if and only if there exists a projective transformation of order 3 leaving the zeros of a binary forms invariant.

Another useful model of an elliptic curve is an elliptic normal quartic curve C in $\mathbb{P}^3$. There are two types of nondegenerate quartic curves in $\mathbb{P}^3$ that differ by the dimension of the linear system of quadrics containing the curve. In terminology of classical algebraic geometry, a space quartic curve is of the *first species* if the dimension is equal to 1, quartics of the *second species* are those which lie on a unique quadric. Elliptic curves are nonsingular quartics of the first species. The proof is rather standard (see, for example, [278]). By Lemma 8.6.1 from Chapter 8, we can write C as the intersection of two simultaneously diagonalized quadrics

$$Q_1 = V(\sum_{i=0}^{3} t_i^2), \quad Q_2 = V(\sum_{i=0}^{3} a_i t_i^2).$$

The pencil $\lambda Q_1 + \mu Q_2$ contains exactly four singular members corresponding to the parameters $[-a_i, 1], i = 0, 1, 2, 3$. The curve C is isomorphic to the double cover of $\mathbb{P}^1$ branched over these four points. This can be seen in many ways. Later we will present one of them, a special case of Weil's Theorem on the intersection of two quadrics (the same proof can be found in Harris's book [278], Proposition 22.38). Changing a basis in the pencil of quadrics containing C, we can reduce the equations of C to the form

$$t_0^2 + t_1^2 + t_2^2 = t_1^2 + \lambda t_2^2 + t_3^2 = 0. \tag{3.6}$$

The absolute invariant of E is expressed via formula (3.4).

3.1.2 The Hesse equation

Classical geometers rarely used Weierstrass equations. They preferred *Hesse's canonical equations* of cubic curves:

$$t_0^3 + t_1^3 + t_2^3 + 6\alpha t_0 t_1 t_2 = 0. \tag{3.7}$$

Let us see that any nonsingular cubic can be reduced to this form by a linear change of variables.

Since any tangent line at an inflection point intersects the curve with multiplicity 3, applying (1.23), we obtain that the curve has exactly 9 inflection points. Using the group law on an elliptic cubic curve with an inflection point $\mathfrak{o}$ as the zero, we can interpret any inflection point as a 3-torsion point. This of course agrees with the fact the group $X[3]$ of 3-torsion points on an elliptic curve X is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

Let H be a subgroup of order 3 of X. Since the sum of elements of this group add up to 0, we see that the corresponding three inflection points p, q, r satisfy $p + q + r \sim 3\mathfrak{o}$. It is easy to see that the rational function on C with the divisor $p + q + r - 3\mathfrak{o}$ can be obtained as the restriction of the rational function $m(t_0, t_1, t_2)/l_0(t_0, t_1, t_2)$, where $V(m)$ defines the line containing the points p, q, r and $V(l_0)$ is the tangent to C at the point $\mathfrak{o}$. There are three cosets with respect to each subgroup H. Since the sum of elements in each coset is again equal to zero, we get 12 lines, each containing three inflection points. Conversely, if a line contains three inflection points, the sum of these points is zero, and it is easy to see that the three points forms a coset with respect to some subgroup H. Each element of $(\mathbb{Z}/3\mathbb{Z})^3$ is contained in four cosets (it is enough to check this for the zero element).

A triangle containing the inflection points is called an *inflection triangle*. There are four inflection triangles and the union of their sides is the set of 12 lines from above. The configuration of 12 lines and 9 points, each line contains 3 points, and each point lies on four lines is the famous *Hesse arrangement of lines* $(12_3, 9_4)$.

Consider the polar conic of an inflection point. It splits into the union of the tangent line at the point and another line, called the *harmonic polar line* of the inflection point.

Lemma 3.1.4 *Let x be a point on a nonsingular cubic C. Any line ℓ passing through x intersects C at points y, z which are harmonically conjugate to the pair x, w, where w is the intersection point of the line and the conic polar $P_x(C)$.*

We will prove this property later in a more general case when C is a curve of degree d and x is its point of multiplicity $(d-2)$ (see Proposition 7.3.18).

Proposition 3.1.5 *Let a, b, c be three collinear inflection points. The harmonic polar lines of three inflection points on a line ℓ intersect at the opposite vertex of the inflection triangle containing ℓ.*

Proof Let Δ be the inflection triangle with side ℓ containing the points a, b, c. Consider the three lines ℓ_i through a which join a with one of the inflection point x_i on the side of Δ. Let z_i be the other inflection point on ℓ_i (lying on the other side). By the previous Lemma, the harmonic polar line intersects each ℓ_i at a point y_i such that the cross ratio $R(ay_i; t_i z_i)$ is constant. This implies that the harmonic polar line is the line in the pencil of lines through the vertex that, together with the two sides and the line passing through a, make the same cross ratio in the pencil. Since the same is true for harmonic polar lines of the points b and c, we get the assertion. □

It follows from the previous Proposition that the nine harmonic polar lines intersect by three at 12 edges of the inflection triangles, and each vertex belongs to four lines. This defines the *dual Hesse arrangement of lines* $(9_4, 12_3)$. It is combinatorially isomorphic to the arrangement of lines in the dual plane which is defined from the Hesse line arrangement via duality.

Now it is easy to reduce a nonsingular cubic curve $C = V(f)$ to the Hesse canonical form. Choose coordinates such that one of the inflection triangles is the coordinate triangle. Let q be one of its vertices, say $q = [1, 0, 0]$, and x be an inflection point on the opposite line $V(t_0)$. Then $P_x(C)$ is the union of the tangent to C at x and the harmonic polar of x. Since the latter passes through q, we have $P_{q^2x}(C) = P_{xq^2}(C) = 0$. Thus the polar line $P_{q^2}(C)$ intersects the line $V(t_0)$ at three points. This can happen only if $P_{q^2}(C) = V(t_0)$. Hence $V(\frac{\partial^2 f}{\partial t_0^2}) = V(t_0)$ and f has no terms $t_0^2 t_1, t_0^2 t_2$. We can write

$$f = at_0^3 + bt_1^3 + ct_2^2 + dt_0 t_1 t_2.$$

Since C is nonsingular, it is immediately checked that the coefficients a, b, c are not equal to zero. After scaling the coordinates, we arrive at the Hesse canonical form.

It is easy to check, by taking partials, that the condition that the curve given by the Hesse canonical form is nonsingular is

$$1 + 8\alpha^3 \neq 0. \tag{3.8}$$

By reducing the Hesse equation to a Weierstrass forms one can express the

Aronhold invariants S, T and the absolute invariant j in terms of the parameter α in (3.7):

$$\mathsf{S} = \alpha - \alpha^4, \tag{3.9}$$

$$\mathsf{T} = 1 - 20\alpha^3 - 8\alpha^6, \tag{3.10}$$

$$j = \frac{64(\alpha - \alpha^4)^3}{(1 + 8\alpha^3)^3}. \tag{3.11}$$

3.1.3 The Hesse pencil

Since the cubic C and its four inflection triangles pass through the same set of nine points, the inflection points of C, they belong to a pencil of cubic curves. This pencil is called the *Hesse pencil*. It is spanned by C and one of the inflection triangles, say the coordinate triangle. Thus the Hesse pencil is defined by the equation

$$\lambda(t_0^3 + t_1^3 + t_2^3) + \mu t_0 t_1 t_2 = 0. \tag{3.12}$$

Its base points are

$$\begin{array}{lll} [0,1,-1], & [0,1,-\epsilon], & [0,1,-\epsilon^2], \\ [1,0,-1], & [1,0,-\epsilon^2], & [1,0,-\epsilon], \\ [1,-1,0], & [1,-\epsilon,0], & [1,-\epsilon^2,0], \end{array} \tag{3.13}$$

where $\epsilon = e^{2\pi i/3}$. They are the nine inflection points of any nonsingular member of the pencil. The singular members of the pencil correspond to the values of the parameters

$$(\lambda, \mu) = (0,1),\ (1,-3),\ (1,-3\epsilon),\ (1,-3\epsilon^2).$$

The last three values correspond to the three values of α for which the Hesse equation defines a singular curve.

Any triple of lines containing the nine base points belongs to the pencil and forms its singular member. Here they are:

$$\begin{array}{c} V(t_0), \quad V(t_1), \quad V(t_2), \\ V(t_0 + t_1 + t_2),\ V(t_0 + \epsilon t_1 + \epsilon^2 t_2),\ V(t_0 + \epsilon^2 t_1 + \epsilon t_2), \\ V(t_0 + \epsilon t_1 + t_2),\ V(t_0 + \epsilon^2 t_1 + \epsilon^2 t_2),\ V(t_0 + t_1 + \epsilon t_2), \\ V(t_0 + \epsilon^2 t_1 + t_2),\ V(t_0 + \epsilon t_1 + \epsilon t_2),\ V(t_0 + t_1 + \epsilon^2 t_2). \end{array} \tag{3.14}$$

We leave to a suspicious reader to check that

$$(t_0+t_1+t_2)(t_0+\epsilon t_1+\epsilon^2 t_2)(t_0+\epsilon^2 t_1+\epsilon t_2)=t_0^3+t_1^3+t_2^3-3t_0t_1t_2,$$

$$(t_0+\epsilon t_1+t_2)(t_0+\epsilon^2 t_1+\epsilon^2 t_2)(t_0+t_1+\epsilon t_2)=t_0^3+t_1^3+t_2^3-3\epsilon t_0t_1t_2,$$

$$(t_0+\epsilon^2 t_1+t_2)(t_0+\epsilon t_1+\epsilon t_2)(t_0+t_1+\epsilon^2 t_2)=t_0^3+t_1^3+t_2^3-3\epsilon^2 t_0t_1t_2.$$

The 12 lines (3.14) and nine inflection points (3.13) form the Hesse configuration corresponding to any nonsingular member of the pencil.

Choose $[0,1,-1]$ to be the zero point in the group law on C. Then we can define an isomorphism of groups $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ by sending $[1,0]$ to $[0,1,-\epsilon]$, $[0,1]$ to $[1,0,-1]$. The points of the first row in (3.13) are the subgroup H generated by $\phi([1,0])$. The points of the second row are the coset of H containing $\phi([0,1])$.

Remark 3.1.6 Note that, varying α in $\mathbb{P}^1 \setminus \{-\frac{1}{2}, -\frac{\epsilon}{2}, -\frac{\epsilon^2}{2}, \infty\}$, we obtain a family of elliptic curves X_α defined by Equation (3.7) with a fixed isomorphism $\phi_\alpha : (\mathbb{Z}/3\mathbb{Z})^2 \to X_\alpha[3]$. After blowing up the 9 base points, we obtain a rational surface $S(3)$

$$f : S(3) \to \mathbb{P}^1 \tag{3.15}$$

defined by the rational map $\mathbb{P}^2 - \to \mathbb{P}^1, [t_0,t_1,t_2] \mapsto [t_0t_1t_2, t_0^3+t_1^3+t_2^3]$. The fiber of f over a point $(a,b) \in \mathbb{P}^2$ is isomorphic to the member of the Hesse pencil corresponding to $(\lambda,\mu) = (-b,a)$. It is known that (3.15) is a *modular family* of elliptic curves with level 3, i.e. the universal object for the fine moduli space of pairs (X,ϕ), where X is an elliptic curve and $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ is an isomorphism of groups. There is a canonical isomorphism $\mathbb{P}^1 \cong Y$, where Y is the modular curve of level 3, i.e. a nonsingular compactification of the quotient of the upper half-plane $\mathcal{H} = \{a+bi \in \mathbb{C} : b>0\}$ by the group

$$\Gamma(3) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : A \equiv I_3 \mod 3\},$$

which acts on $\mathcal{H}$ by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. The boundary of $H/\Gamma(3)$ in Y consists of four points (the cusps). They correspond to the singular members of the Hesse pencil.

3.1.4 The Hesse group

The *Hesse group* G_{216} is the group of projective transformations which preserve the Hesse pencil of cubic curves. First, we see the obvious symmetries

generated by the transformations

$$\tau : [t_0, t_1, t_2] \mapsto [t_0, \epsilon_3 t_1, \epsilon_3^2 t_2],$$
$$\sigma : [t_0, t_1, t_2] \mapsto [t_2, t_0, t_1].$$

They define a projective representation of the group $(\mathbb{Z}/3\mathbb{Z})^2$.

If we fix the group law by taking the origin to be $[0, 1, -1]$, then τ induces on each nonsingular fiber the translation automorphism by the point $[0, 1, -\epsilon]$ and σ is the translation by the point $[1, 0, -1]$.

Theorem 3.1.7 *The Hesse group G_{216} is a group of order 216 isomorphic to the semi-direct product*

$$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{F}_3),$$

where the action of $\mathrm{SL}(2, \mathbb{F}_3)$ *on* $(\mathbb{Z}/3\mathbb{Z})^2$ *is the natural linear representation.*

Proof Let $g \in G_{216}$. It transforms a member of the Hesse pencil to another member. This defines a homomorphism $G_{216} \to \mathrm{Aut}(\mathbb{P}^1)$. An element of the kernel K leaves each member of the pencil invariant. In particular, it leaves invariant the curve $V(t_0 t_1 t_2)$. The group of automorphisms of this curve is generated by homotheties $[t_0, t_1, t_2] \mapsto [t_0, at_1, bt_2]$ and permutation of coordinates. Suppose σ induces a homothety. Since it also leaves invariant the curve $V(t_0^3 + t_1^3 + t_2^3)$, we must have $1 = a^3 = b^3$. To leave invariant a general member we also need that $a^3 = b^3 = bc$. This implies that g belongs to the subgroup generated by the transformation σ. An even permutation of coordinates belongs to a subgroup generated by the transformation τ. The odd permutation $\sigma_0 : [t_0, t_1, t_2] \mapsto [t_0, t_2, t_1]$ acts on the group of 3-torsion points of each nonsingular fiber as the negation automorphism $x \mapsto -x$. Thus we see that

$$K \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \langle \sigma_0 \rangle.$$

Now let I be the image of the group G_{216} in $\mathrm{Aut}(\mathbb{P}^1)$. It acts by permuting the four singular members of the pencil and thus leaves the set of zeros of the binary form

$$\Delta = (8t_1^3 + t_0^3)t_0$$

invariant. It follows from the invariant theory that this implies that H is a subgroup of $\mathfrak{A}_4$. We claim that $H = \mathfrak{A}_4$. Consider the projective transformations given by the matrices

$$\sigma_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \end{pmatrix}.$$

The transformations $\sigma_0, \sigma_1, \sigma_2$ generate a subgroup isomorphic to the quaternion group Q_8 with center generated by σ_0. The transformation

$$\sigma_3 : [t_0, t_1, t_2] \mapsto [\varepsilon t_0, t_2, t_1]$$

satisfies $\sigma_3^3 = \sigma_0$. It acts by sending a curve C_α from (3.7) to $C_{\varepsilon\alpha}$. It is easy to see that the transformations $\sigma_1, \sigma_2, \sigma_3, \tau$ generate the group isomorphic to $\mathrm{SL}(2, \mathbb{F}_3)$. Its center is (σ_0) and the quotient by the center is isomorphic to $\mathfrak{A}_4$. In other words, this group is the binary tetrahedral group. Note that the whole group can be generated by transformations $\sigma, \tau, \sigma_0, \sigma_1$. □

Recall that a linear operator $\sigma \in \mathrm{GL}(E)$ of a complex vector space E of dimension $n+1$ is called a *complex reflection* if it is of finite order and the rank of $\sigma - \mathrm{id}_E$ is equal to 1. The kernel of $\sigma - \mathrm{id}_E$ is a hyperplane in E, called the *reflection hyperplane* of σ. It is invariant with respect to σ and its stabilizer subgroup is a cyclic group. A *complex reflection group* is a finite subgroup G of $\mathrm{GL}(E)$ generated by complex reflections. One can choose a unitary inner product on E such that any complex reflection σ from E can be written in the form

$$s_{v,\eta} : x \mapsto x + (\eta - 1)(x, v)v,$$

where v is a vector of norm 1 perpendicular to the reflection hyperplane H_v of σ, and η is a nontrivial root of unity of order equal to the order of σ.

Recall the basic facts about complex reflection groups (see, for example, [539]):

- The algebra of invariants $S(E)^G \cong \mathbb{C}[t_0, \dots, t_n]^G$ is freely generated by $n+1$ invariant polynomials $f_0, \dots, f_n$ (geometrically, $E/G \cong \mathbb{C}^{n+1}$).
- The product of degrees d_i of the polynomials $f_0, \dots, f_n$ is equal to the order of G.
- The number of complex reflections in G is equal to $\sum(d_i - 1)$.

All complex reflections group were classified by G. Shephard and J. Todd [531]. There are five conjugacy classes of complex reflection subgroups of $\mathrm{GL}(3, \mathbb{C})$. Among them is the group G isomorphic to a central extension of degree 3 of the Hesse group. It is generated by complex reflections $s_{v,\eta}$ of order 3, where the reflection line H_v is one of the 12 lines (3.14) in $\mathbb{P}^2$ and v is the unit normal vector (a, b, c) of the line $V(at_0 + bt_1 + ct_2)$. Note that each reflection $s_{v,\eta}$ leaves invariant the hyperplanes with a normal vector orthogonal to v. For example, $s_{(1,0,0),\varepsilon}$ leaves invariant the line $V(t_0)$. This implies that each of the 12 complex reflections leaves the Hesse pencil invariant. Thus the image of G in $\mathrm{PGL}(3, \mathbb{C})$ is contained in the Hesse group. It follows from the

classification of complex reflection groups (or could be checked directly, see [539]) that it is equal to the Hesse group and the subgroup of scalar matrices from G is a cyclic group of order 3.

Each of the 12 reflection lines defines two complex reflections. This gives 24 complex reflections in G. This number coincides with the number of elements of order 3 in the Hesse group and so there are no more complex reflections in G. Let $d_1 \leq d_2 \leq d_3$ be the degrees of the invariants generating the algebra of invariants of G. We have $d_1 + d_2 + d_3 = 27, d_1 d_2 d_3 = 648$. This easily gives $d_1 = 6, d_2 = 9, d_3 = 12$. There are obvious reducible curves of degree 9 and 12 in $\mathbb{P}^2$ invariant with respect to G. The curve of degree 9 is the union of the polar harmonic lines. Each line intersects a nonsingular member of the pencil at nontrivial 2-torsion points with respect to the group law defined by the corresponding inflection point. The equation of the union of nine harmonic polar lines is

$$f_9 = (t_0^3 - t_1^3)(t_0^3 - t_2^3)(t_1^3 - t_2^3) = 0. \tag{3.16}$$

The curve of degree 12 is the union of the 12 lines (3.14). Its equation is

$$f_{12} = t_0 t_1 t_2 [27 t_0^3 t_1^3 t_2^3 - (t_0^3 + t_1^3 + t_2^3)^3] = 0. \tag{3.17}$$

A polynomial defining an invariant curve is a *relative invariant* of G (it is an invariant with respect to the group $G' = G \cap \mathrm{SL}(3, \mathbb{C})$). One checks that the polynomial f_9 is indeed an invariant, but the polynomial f_{12} is only a relative invariant. So, there exists another curve of degree 12 whose equation defines an invariant of degree 12. What is this curve? Recall that the Hesse group acts on the base of the Hesse pencil via the action of the tetrahedron group $\mathfrak{A}_4$. It has three orbits with stabilizers of order 2,3 and 3. The first orbit consists of six points such that the fibres over these points are harmonic cubics. The second orbit consists of four points such that the fibres over these points are equianharmonic cubics. The third orbit consists of four points corresponding to singular members of the pencil. It is not difficult to check that the product of the equations of the equianharmonic cubics defines an invariant of degree 12. Its equation is

$$f'_{12} = (t_0^3 + t_1^3 + t_2^3)[(t_0^3 + t_1^3 + t_2^3)^3 + 216 t_0^3 t_1^3 t_2^3] = 0. \tag{3.18}$$

An invariant of degree 6 is

$$f_6 = 7(t_0^6 + t_1^6 + t_2^6) - 6(t_0^3 + t_2^3 + t_3^3)^2. \tag{3.19}$$

The product of the equations defining 6 harmonic cubics is an invariant of degree 18

$$f_{18} = (t_0^3 + t_1^3 + t_2^3)^6 - 540 t_0^3 t_1^3 t_2^3 (t_0^3 + t_1^3 + t_2^3)^3 - 5832 t_0^6 t_1^6 t_2^6 = 0. \tag{3.20}$$

3.2 Polars of a plane cubic

3.2.1 The Hessian of a cubic hypersurface

Let $X = V(f)$ be a cubic hypersurface in $\mathbb{P}^n$. We know that the Hessian $\mathrm{He}(X)$ is the locus of points $a \in \mathbb{P}^n$ such that the polar quadric $P_a(X)$ is singular. Also we know that, for any $a \in \mathrm{He}(X)$,

$$\mathrm{Sing}(P_a(X)) = \{b \in \mathbb{P}^2 : D_b(D_a(f)) = 0\}.$$

Since $P_b(P_a(X)) = P_a(P_b(X))$ we obtain that $b \in \mathrm{He}(X)$.

Theorem 3.2.1 *The Hessian* $\mathrm{He}(X)$ *of a cubic hypersurface* X *contains the Steinerian* $\mathrm{St}(X)$. *If* $\mathrm{He}(X) \neq \mathbb{P}^n$, *then*

$$\mathrm{He}(X) = \mathrm{St}(X).$$

For the last assertion one needs only to compare the degrees of the hypersurfaces. They are equal to $n + 3$.

In particular, the rational map, if defined,

$$\mathrm{st}_X^{-1} : \mathrm{St}(X) \to \mathrm{He}(X), a \mapsto \mathrm{Sing}(P_a(X)) \tag{3.21}$$

is a birational automorphism of the Hessian hypersurface. We have noticed this already in Chapter 1.

Proposition 3.2.2 *Assume* X *has only isolated singularities. Then* $\mathrm{He}(X) = \mathbb{P}^n$ *if and only if* X *is a cone over a cubic hypersurface in* $\mathbb{P}^{n-1}$.

Proof Let $W = \{(a, b) \in \mathbb{P}^n \times \mathbb{P}^n : P_{a,b^2}(X) = 0\}$. For each $a \in \mathbb{P}^n$, the fiber of the first projection over the point a is equal to the first polar $P_a(X)$. For any $b \in \mathbb{P}^n$, the fiber of the second projection over the point b is equal to the second polar $P_{b^2}(X) = V(\sum \partial_i f(b)t_i)$. Let $U = \mathbb{P}^n \setminus \mathrm{Sing}(X)$. For any $b \in U$, the fiber of the second projection is a hyperplane in $\mathbb{P}^n$. This shows that $p_2^{-1}(U)$ is nonsingular. The restriction of the first projection to U is a morphism of nonsingular varieties. The general fiber of this morphism is a regular scheme over the general point of $\mathbb{P}^n$. Since we are in characteristic 0, it is a smooth scheme. Thus there exists an open subset $W \subset \mathbb{P}^n$ such that $p_1^{-1}(W) \cap U$ is nonsingular. If $\mathrm{He}(X) = 0$, all polar quadrics $P_a(X)$ are singular, and a general polar must have singularities inside of $p_2^{-1}(\mathrm{Sing}(X))$. This means that $p_1(p_2^{-1}(\mathrm{Sing}(X))) = \mathbb{P}^n$. For any $x \in \mathrm{Sing}(X)$, all polar quadrics contain x and either all of them are singular at x or there exists an open subset $U_x \subset \mathbb{P}^n$ such that all quadrics $P_a(X)$ are nonsingular at x for $a \in U_x$. Suppose that, for any $x \in \mathrm{Sing}(X)$, there exists a polar quadric which is nonsingular at x. Since the number of isolated singular points is finite, there

will be an open set of points $a \in \mathbb{P}^n$ such that the fiber $p_1^{-1}(a)$ is nonsingular in $p_2^{-1}(\mathrm{Sing}(X))$. This is a contradiction. Thus, there exists a point $c \in \mathrm{Sing}(X)$ such that all polar quadrics are singular at x. This implies that c is a common solution of the systems of linear equations $\mathrm{He}(f_3)(a) \cdot X = 0, a \in \mathbb{P}^n$. Thus the first partials of f_3 are linearly dependent. Now we apply Proposition 1.1.6 to obtain that X is a cone. □

Remark 3.2.3 The example of a cubic hypersurface in $\mathbb{P}^4$ from Remark 1.1.16 shows that the assumption on singular points of X cannot be weakened. The singular locus of the cubic hypersurface is the plane $t_0 = t_1 = 0$.

3.2.2 The Hessian of a plane cubic

Consider a plane cubic $C = V(f)$ with equation in the Hesse canonical form (3.7). The partials of $\frac{1}{3}f$ are

$$t_0^2 + 2\alpha t_1 t_2, \quad t_1^2 + 2\alpha t_0 t_2, \quad t_2^2 + 2\alpha t_0 t_1. \tag{3.22}$$

Thus the Hessian of C has the following equation:

$$\mathrm{He}(C) = \begin{vmatrix} t_0 & \alpha t_2 & \alpha t_1 \\ \alpha t_2 & t_1 & \alpha t_0 \\ \alpha t_1 & \alpha t_0 & t_2 \end{vmatrix} = (1 + 2\alpha^3) t_0 t_1 t_2 - \alpha^2 (t_0^3 + t_1^3 + t_2^3). \tag{3.23}$$

In particular, the Hessian of the member of the Hesse pencil corresponding to the parameter $(\lambda, \mu) = (1, 6\alpha),\ \alpha \neq 0$, is equal to

$$t_0^3 + t_1^3 + t_2^3 - \frac{1 + 2\alpha^3}{\alpha^2} t_0 t_1 t_2 = 0, \tag{3.24}$$

or, if $(\lambda, \mu) = (1, 0)$ or $(0, 1)$, then the Hessian is equal to $V(t_0 t_1 t_2)$.

Lemma 3.2.4 *Let C be a nonsingular cubic in a Hesse's canonical form. The following assertions are equivalent:*

(i) $\dim \mathrm{Sing}(P_a(C)) > 0$;
(ii) $a \in \mathrm{Sing}(\mathrm{He}(C))$;
(iii) $\mathrm{He}(C)$ *is the union of three nonconcurrent lines;*
(iv) C *is isomorphic to the Fermat cubic* $V(t_0^3 + t_1^3 + t_1^3)$;
(v) $\mathrm{He}(C)$ *is a singular cubic;*
(vi) C *is an equianharmonic cubic;*
(vii) $\alpha(\alpha^3 - 1) = 0$.

Proof Use the Hesse equation for a cubic and for its Hessian. We see that $\mathrm{He}(C)$ is singular if and only if either $\alpha = 0$ or $1 + 8(-\frac{1+2\alpha^3}{6\alpha^2})^3 = 0$. Obviously, $\alpha = 1$ is a solution of the second equation. Other solutions are ϵ, ϵ^2. This corresponds to $\mathrm{He}(C)$, where C is of the form $V(t_0^3+t_1^3+t_1^3)$, or is given by the equation

$$t_0^3 + t_1^3 + t_2^3 + 6\epsilon^i t_0 t_1 t_2 = (\epsilon^i t_0 + \epsilon t_1 + t_2)^3 + (t_0 + \epsilon^i t_1 + t_2)^3$$

$$+(t_0 + t_1 + \epsilon^i t_2)^3 = 0,$$

where $i = 1, 2$, or

$$t_0^3 + t_1^3 + t_2^3 + 6t_0 t_1 t_2 = (t_0 + t_1 + t_2)^3 + (t_0 + \epsilon t_1 + \epsilon^2 t_2)^3$$

$$+(t_0 + \epsilon^2 t_1 + \epsilon t_2)^3 = 0.$$

This computation proves the equivalence of (iii), (iv), (v), and (vii).

Assume (i) holds. Then the rank of the Hessian matrix is equal to 1. It is easy to see that the first two rows are proportional if and only if $\alpha(\alpha^3 - 1) = 0$. Thus (i) is equivalent to (vii), and hence to (iii), (iv), (v) and (vii). The point a is one of the three intersection points of the lines such that the cubic is equal to the sum of the cubes of linear forms defining these lines. Direct computation shows that (ii) holds. Thus (i) implies (ii).

Assume (ii) holds. Again the previous computations show that $\alpha(\alpha^3 - 1) = 0$ and the Hessian curve is the union of three lines. Now (i) is directly verified.

The equivalence of (iv) and (vi) follows from Theorem 3.1.3 since the transformation $[t_0, t_1, t_2] \to [t_1, t_0, e^{2\pi i/3} t_2]$ generates a cyclic group of order 6 of automorphisms of C leaving the point $[1, -1, 0]$ fixed. □

Corollary 3.2.5 *Assume that $C = V(f)$ is not projectively isomorphic to the Fermat cubic. Then the Hessian cubic is nonsingular, and the map $a \mapsto \mathrm{Sing}(P_a(C))$ is an involution on $\mathrm{He}(C)$ without fixed points.*

Proof The only unproved assertion is that the involution does not have fixed points. A fixed point a has the property that $D_a(D_a(f)) = D_{a^2}(f) = 0$. It follows from Theorem 1.1.5 that this implies that $a \in \mathrm{Sing}(C)$. □

Remark 3.2.6 Consider the Hesse pencil of cubics with parameters $(\lambda, \mu) = (\alpha_0, 6\alpha_1)$

$$C_{(\alpha_0,\alpha)} = V(\alpha_0(t_0^3 + t_1^3 + t_2^3) + 6\alpha_1 t_0 t_1 t_2).$$

Taking the Hessian of each curve from the pencil we get the pencil

$$H_{(\alpha_0,\alpha)} = V(\alpha_0 t_0^3 + t_1^3 + t_2^3 + 6\alpha_1 t_0 t_1 t_2).$$

The map $C_{(\alpha_0,\alpha)} \to H_{(\alpha_0,\alpha)}$ defines a regular map

$$\mathfrak{h} : \mathbb{P}^1 \to \mathbb{P}^1, \quad [\alpha_0, \alpha_1] \mapsto [t_0, t_1] = [-\alpha_0\alpha_1^2, \alpha_0^3 + 2\alpha_1^3]. \tag{3.25}$$

This map is of degree 3. For a general value of the inhomogeneous parameter $\lambda = t_1/t_0$, the preimage consists of three points with inhomogeneous coordinate $\alpha = \alpha_1/\alpha_0$ satisfyfing the cubic equation

$$6\lambda\alpha^3 - 2\alpha^2 + 1 = 0. \tag{3.26}$$

We know that the points $[\alpha_0, \alpha_1] = [0, 1], [1, -\frac{1}{2}], [1, -\frac{\epsilon}{2}], [1, -\frac{\epsilon^2}{2}]$ correspond to singular members of the λ-pencil. These are the branch points of the map $\mathfrak{h}$. Over each branch point we have two points in the preimage. The points

$$(\alpha_0, \alpha_1) = [1, 0], [1, 1], [1, \epsilon], [1, \epsilon^2]$$

are the ramification points corresponding to equianharmonic cubics. A nonramication point in the preimage corresponds to a singular member.

Let $C_\alpha = C_{(1,\alpha)}$. If we fix a group law on a $H_\alpha = \mathrm{He}(C_\alpha)$, we will be able to identify the involution described in Corollary 3.2.5 with the translation automorphism by a nontrivial 2-torsion point η. Given a nonsingular cubic curve H together with a fixed-point-free involution τ, there exists a unique nonsingular cubic C_α such that $H = H_\alpha$ and the involution τ is the involution described in the corollary. Thus the three roots of Equation (3.26) can be identified with 3 nontrivial torsion points on H_α. We refer the reader to Exercise 3.2 for a reconstruction of C_α from the pair (H_α, η).

Recall that the Cayleyan curve of a plane cubic C is the locus of lines $\overline{pq}$ in the dual plane such that $a \in \mathrm{He}(C)$ and b is the singular point of $P_a(C)$. Each such line intersects $\mathrm{He}(C)$ at three points a, b, c. The following gives the geometric meaning of the third intersection point.

Proposition 3.2.7 *Let c be the third intersection point of a line $\ell \in \mathrm{Cay}(C)$ and $\mathrm{He}(C)$. Then ℓ is a component of the polar $P_d(C)$ whose singular point is c. The point d is the intersection point of the tangents of $\mathrm{He}(C)$ at the points a and b.*

Proof From the general theory of linear system of quadrics, applied to the net of polar conics of C, we know that ℓ is a Reye line, i.e. it is contained in some polar conic $P_d(C)$ (see subsection 1.1.7). The point d must belong to $\mathrm{He}(C)$ and its singular point c belongs to ℓ. Thus c is the third intersection point of ℓ with C.

It remains for us to prove the last assertion. Chose a group law on the curve $\mathrm{He}(C)$ by fixing an inflection point as the zero point. We know that the Steinerian involution is defined by the translation $x \mapsto x \oplus \eta$, where η is a fixed

2-torsion point. Thus $b = a \oplus \eta$. It follows from the definition of the group law on a nonsingular cubic that the tangents $\mathbb{T}_a(\mathrm{He}(C))$ and $\mathbb{T}_b(\mathrm{He}(C))$ intersect at a point d on $\mathrm{He}(C)$. We have $d \oplus 2a = 0$, hence $d = -2a$. Since a, b, c lie on a line, we get $c = -a - b$ in the group law. After subtracting, we get $d - c = b - a = \eta$. Thus the points x and c are an orbit of the Steinerian involution. This shows that c is the singular point of $P_d(C)$. By Proposition 1.2.5, $P_d(C)$ contains the points a, b. Thus $\overline{ab}$ is a component of $P_d(C)$. □

It follows from the above Proposition that the Cayleyan curve of a nonsingular cubic C parameterizes the line components of singular polar conics of C. It is also isomorphic to the quotient of $\mathrm{He}(C)$ by the Steinerian involution from Corollary 3.2.5. Since this involution does not have fixed points, the quotient map $\mathrm{He}(C) \to \mathrm{Cay}(C)$ is an unramified cover of degree 2. In particular, $\mathrm{Cay}(C)$ is a nonsingular curve of genus 1.

Let us find the equation of the Cayleyan curve. A line ℓ belongs to $\mathrm{Cay}(X)$ if and only if the restriction of the linear system of polar conics of X to ℓ is of dimension 1. This translates into the condition that the restriction of the partials of X to ℓ is a linearly dependent set of three binary forms. So, write ℓ in the parametric form as the image of the map $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[u, v] \mapsto [a_0u + b_0v, a_1u + b_1v, a_2u + b_2v]$. The condition of the linear dependence is given by

$$\det \begin{pmatrix} a_0^2 + 2\alpha a_1a_2 & 2a_0b_0 + 2\alpha(a_1b_2 + a_2b_1) & b_0^2 + 2\alpha b_1b_2 \\ a_1^2 + 2\alpha a_0a_2 & 2a_1b_1 + 2\alpha(a_0b_2 + a_2b_0) & b_1^2 + 2\alpha b_0b_2 \\ a_2^2 + 2\alpha a_0a_1 & 2a_2b_2 + 2\alpha(a_0b_1 + a_1b_0) & b_2^2 + 2\alpha b_0b_1 \end{pmatrix} = 0.$$

The coordinates of ℓ in the dual plane are

$$[u_0, u_1, u_2] = [a_1b_2 - a_2b_1, a_2b_0 - a_0b_2, a_0b_1 - a_1b_0].$$

Computing the determinant, we find that the equation of $\mathrm{Cay}(X)$ in the coordinates u_0, u_1, u_2 is

$$u_0^3 + u_1^3 + u_2^3 + 6\alpha' u_0u_1u_2 = 0, \tag{3.27}$$

where $\alpha' = (1 - 4\alpha^3)/6\alpha$. Note that this agrees with the degree of the Cayleyan curve found in Proposition 1.1.26. Using formula (3.9) for the absolute invariant of the curve, this can be translated into an explicit relationship between the absolute invariant of an elliptic curve C and the isogenous elliptic curve $C/(\tau_e)$, where τ_e is the translation automorphism by a nontrivial 2-torsion point e.

3.2.3 The dual curve

Write the equation of a general line in the form $t_2 = u_0t_0 + u_1t_1$ and plug in Equation (3.7). The corresponding cubic equation has a multiple root if and only if the line is a tangent. We have

$$(u_0t_0 + u_1t_1)^3 + t_0^3 + t_1^3 + 6\alpha t_0t_1(u_0t_0 + u_1t_1)$$

$$= (u_0^3+1)t_0^3 + (u_1^3+1)t_1^3 + (3u_0^2u_1 + 6\alpha u_0)t_0^2t_1 + (3u_0u_1^2 + 6\alpha u_1)t_0t_1^2 = 0.$$

The condition that there is a multiple root is that the discriminant of the homogeneous cubic form in t_0, t_1 is equal to zero. Using the formula (1.81) for the discriminant of a cubic form, after plugging in, we obtain

$$(3u_0^2u_1+6\alpha u_0)^2(3u_0u_1^2+6\alpha u_1)^2+18(3u_0^2u_1+6\alpha u_0)(3u_0u_1^2+6\alpha u_1)(u_0^3+1)(u_1^3+1)$$

$$-4(u_0^3+1)(3u_0u_1^2+6\alpha u_1)^3 - 4(u_1^3+1)(3u_1u_0^2+6\alpha u_0)^3 - 27(u_0^3+1)^2(u_1^3+1)^2$$

$$= -27 + 864u_0^3u_1^3\alpha^3 + 648u_0^2u_1^2\alpha - 648\alpha^2u_0u_1^4 - 648\alpha^2u_0^4u_1 + 648\alpha^2u_0u_1$$

$$+1296\alpha^4u_0^2u_1^2 - 27u_1^6 - 27u_0^6 + 54u_0^3u_1^3 - 864u_1^3\alpha^3 - 864u_0^3\alpha^3 - 54u_1^3 - 54u_0^3 = 0.$$

It remains for us to homogenize the equation and divide by (-27) to obtain the equation of the dual curve

$$u_0^6 + u_1^6 + u_2^6 - (2 + 32\alpha^3)(u_0^3u_1^3 + u_0^3u_2^3 + u_2^3u_1^3)$$
$$-24\alpha^2u_0u_1u_2(u_0^3 + u_1^3 + u_2^3) - (24\alpha + 48\alpha^4)u_0^2u_1^2u_2^2 = 0. \qquad (3.28)$$

According to the Plücker formula (1.50), the dual curve of a nonsingular plane cubic has nine cusps. They correspond to the inflection tangents of the original curve. The inflection points are given in (3.12). Computing the equations of the tangents, we find the following singular points of the dual curve:

$$[-2m, 1, 1],\ [1, -2\alpha, 1],\ [1, 1, -2\alpha],\ [-2\alpha\varepsilon, \varepsilon^2, 1],\ [-2\alpha\varepsilon, 1, \varepsilon^2],$$

$$[\varepsilon^2, -2\alpha\varepsilon, 1],\ [1, -2\alpha\varepsilon, \varepsilon^2],\ [1, \varepsilon^2, -2\alpha\varepsilon],\ [\varepsilon^2, 1, -2\alpha].$$

The tangent of C at an inflection point a is a component of the polar conic $P_a(C)$, hence connects a to the singular point of the polar conic. This implies that the tangent line belongs to the Cayleyan curve $\mathrm{Cay}(C)$, hence the Cayleyan curve contains the singular points of the dual cubic. The pencil of plane curves of degree 6 spanned by the dual cubic $C^\vee$ and the Cayleyan cubic taken with multiplicity 2 is an example of an *Halphen pencil* of index 2 of curves of degree 6 with nine double base points (see Exercises to Chapter 7).

3.2.4 Polar s-gons

Since, for any three general points in $\mathbb{P}^2$, there exists a plane cubic singular at these points (the union of three lines), a general ternary cubic form does not admit polar triangles. Of course this is easy to see by counting constants.

By Lemma 3.2.4, a nonsingular cubic admits a polar triangle if and only if it is an equianharmonic cubic. Its polar triangle is unique. Its sides are the three first polars of C which are double lines.

Proposition 3.2.8 *A plane cubic admits a polar triangle if and only if either it is a Fermat cubic or it is equal to the union of three distinct concurrent lines.*

Proof Suppose $C = V(l_1^3 + l_2^3 + l_3^3)$. Without loss of generality, we may assume that l_1 is not proportional to l_2. Thus, after a linear change of coordinates, $C = V(t_0^3 + t_1^3 + l^3)$. If $l(t_0, t_1, t_2)$ does not depend on t_2, the curve C is the union of three distinct concurrent lines. Otherwise, we can change coordinates to assume that $l = t_2$ and get a Fermat cubic. □

By counting constants, a general cubic admits a *polar quadrangle*. It is clear that a polar quadrangle $\{[l_1], \dots, [l_4]\}$ is nondegenerate if and only if the linear system of conics in the dual plane through the points $[l_i]$ is an irreducible pencil (i.e. a linear system of dimension 1 whose general member is irreducible). This allows us to define a *nondegenerate generalized polar quadrangle* of C as a generalized quadrangle Z of C such that $|\mathcal{I}_Z(2)|$ is an irreducible pencil.

Let $g(t_0, t_1)$ be a binary form of degree 3. Its polar 3-hedron is the divisor of zeros of its apolar form of degree 3. Thus

$$\mathrm{VSP}(g, 3) \cong |\mathrm{AP}_3(g)|^\vee \cong \mathbb{P}^2. \tag{3.29}$$

This implies that any ternary cubic form $f = t_2^3 + g(t_0, t_1)$ admits degenerate polar quadrangles.

Also, if $C = V(g(t_0, t_1))$ is the union of three concurrent lines then any four distinct nonzero linear forms l_1, l_2, l_3, l_4 form a degenerate quadrangle of C. In fact, using the Van der Monde determinant, we obtain that the cubes $l_1^3, l_2^3, l_3^3, l_4^3$ form a basis in the space of binary cubic forms. So the variety of sums of four powers of C is isomorphic to the variety of four distinct points in $\mathbb{P}^1$. Its closure $\mathrm{VSP}(C, 4)$ in the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{P}^2)$ is isomorphic to $(\mathbb{P}^1)^{(4)} \cong \mathbb{P}^4$.

Lemma 3.2.9 *C admits a degenerate polar quadrangle if and only if it is one of the following curves:*

(i) an equianharmonic cubic;
(ii) a cuspidal cubic;

(iii) the union of three concurrent lines (not necessarily distinct).

Proof We only have to prove the converse. Suppose

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3,$$

where l_1, l_2, l_3 vanish at a common point a which we identify with a vector in E. We have

$$\frac{1}{3}D_a(f) = l_1(a)l_1^2 + l_2(a)l_2^2 + l_3(a)l_3^2 + l_4(a)l_4^2 = l_4(a)l_4^2.$$

This shows that the first polar $P_a(V(f))$ is either the whole $\mathbb{P}^2$ or the double line $2\ell = V(l_4^2)$. In the first case C is the union of three concurrent lines. Assume that the second case occurs. We can choose coordinates such that $a = [0, 0, 1]$ and $\ell = V(t_2)$. Write

$$f = g_0 t_2^3 + g_1 t_2^2 + g_2 t_2 + g_3,$$

where g_k are homogeneous forms of degree k in variables t_0, t_1. Then $D_a(f) = \partial_2 f = 3t_2^2 g_0 + 2t_2 g_1 + g_2$. This can be proportional to t_2^2 only if $g_1 = g_2 = 0, g_0 \neq 0$. Thus $V(f) = V(g_0 t_2^3 + g_3(t_0, t_1))$. If g_3 has no multiple linear factors, we get an equianharmonic cubic. If g_3 has a linear factor with multiplicity 2, we get a cuspidal cubic. Finally, if g_3 is a cube of a linear form, we reduce the latter to the form t_1^3 and get three concurrent lines.

□

Remark 3.2.10 We know that all equianharmonic cubics are projectively equivalent to the Fermat cubic. The orbit of the Fermat cubic $V(t_0^3+t_1^3+t_2^3)$ is somorphic to the homogeneous space $\mathrm{PSL}(3)/G$, where $G = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathfrak{S}_3$. Its closure in $|S^3(E^\vee)|$ is a hypersurface F and consists of curves listed in the assertion of the previous Lemma and also reducible cubics equal to the unions of irreducible conics with its tangent lines. The explicit equation of the hypersurface F is given by the *Aronhold invariant* S of degree 4 in the coefficients of the cubic equation. A nice expression for the invariant S in terms of a pfaffian of a skew-symmetric matrix was given by G. Ottaviani [421].

Lemma 3.2.11 *The following properties are equivalent:*

(i) $\mathrm{AP}_1(f) \neq \{0\}$*;*
(ii) $\dim \mathrm{AP}_2(f) > 2$*;*
(iii) $V(f)$ *is equal to the union of three concurrent lines.*

Proof By the apolarity duality,

$$(A_f)_1 \times (A_f)_2 \to (A_f)_3 \cong \mathbb{C},$$

we have

$$\dim(A_f)_1 = 3 - \dim \mathrm{AP}_1(f) = \dim(A_f)_2 = 6 - \dim \mathrm{AP}_2(f).$$

Thus $\dim \mathrm{AP}_2(f) = 3 + \dim \mathrm{AP}_1(f)$. This proves the equivalence of (i) and (ii). By definition, $\mathrm{AP}_1(f) \neq \{0\}$ if and only if $D_\psi(f) = 0$ for some nonzero linear operator $\psi = \sum a_i \partial_i$. After a linear change of variables, we may assume that $\psi = \partial_0$, and then $\partial_0(f) = 0$ if and only if C does not depend on t_0, i.e. C is the union of three concurrent lines. □

Lemma 3.2.12 *Let Z be a generalized polar quadrangle of f. Then $|\mathcal{I}_Z(2)|$ is a pencil of conics in $|E^\vee|$ contained in the linear system $|\mathrm{AP}_2(f)|$. If Z is nondegenerate, then the pencil has no fixed component. Conversely, let Z be a 0-dimensional cycle of length 4 in $|E|$. Assume that $|\mathcal{I}_Z(2)|$ is an irreducible pencil contained in $|\mathrm{AP}_2(f)|$. Then Z is a nondegenerate generalized polar quadrangle of f.*

Proof The first assertion follows from the definition of nondegeneracy and Proposition 1.3.12. Let us prove the converse. Let $V(\lambda q_1 + \mu q_2)$ be the pencil of conics $|\mathcal{I}_Z(2)|$. Since $\mathrm{AP}(f)$ is an ideal, the linear system L of cubics of the form $V(q_1 l_1 + q_2 l_2)$, where l_1, l_2 are linear forms, is contained in $|\mathrm{AP}_3(f)|$. Obviously, it is contained in $|\mathcal{I}_Z(3)|$. Since $|\mathcal{I}_Z(2)|$ has no fixed part we may choose q_1 and q_2 with no common factors. Then the map $E^\vee \oplus E^\vee \to I_Z(3)$ defined by $(l_1, l_2) \to q_1 l_1 + q_2 l_2$ is injective, hence $\dim L = 5$. Assume $\dim |\mathcal{I}_Z(3)| \geq 6$. Choose three points in general position on an irreducible member C of $|\mathcal{I}_Z(2)|$ and three non-collinear points outside C. Then find a cubic K from $|\mathcal{I}_Z(3)|$ which passes through these points. Then K intersects C with total multiplicity $4 + 3 = 7$, hence contains C. The other component of K must be a line passing through three non-collinear points. This contradiction shows that $\dim |\mathcal{I}_Z(3)| = 5$ and we have $L = |\mathcal{I}_Z(3)|$. Thus $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$ and, by Proposition 1.3.12, Z is a generalized polar quadrangle of C.

□

Note that not every point in $\mathrm{Hilb}^4(\mathbb{P}^2)$ can be realized as a generalized quadrangle of a ternary cubic. Each point in the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{P}^2)$ is the union of subschemes supported at one point. Let us recall analytic classification of closed subschemes $V(I)$ of length $h \leq 4$ supported at one point (see [54]).

- $h = 1$: $I = (x, y)$;
- $h = 2$: $I = (x, y^2)$;
- $h = 3$: $I = (x, y^3), (x^2, xy, y^2)$;

- $h = 4$: $I = (x, y^4), (x^2, y^2), (x^2, xy, y^3)$.

The subschemes Z of length 4 that cannot be realized as the base scheme of a pencil of conics, are those which contain a subscheme analytically isomorphic to one of the following schemes $V(x, y^3), V(x, y^4), V(x^2, xy, y^2)$, or $V(x^2, xy, y^3)$.

Theorem 3.2.13 *Assume that C is neither an equianharmonic cubic, nor a cuspidal cubic, nor the union of three concurrent lines. Then*

$$\mathrm{VSP}(f, 4) \cong |\mathrm{AP}_2(f)|^\vee \cong \mathbb{P}^2.$$

If C is nonsingular, the complement of $\Delta = \mathrm{VSP}(f, 4) \setminus \mathrm{VSP}(f, 4)^o$ is a curve of degree 6 isomorphic to the dual of a nonsingular cubic curve. If C is a nodal cubic, then Δ is the union of a quartic curve isomorphic to the dual quartic of C and two lines. If C is the union of a nonsingular conic and a line intersecting it transversally, Δ is the union of a conic and two lines. If C is the union of a conic and its tangent line, then $\Delta = \mathrm{VSP}(f, 4)$.

Proof We will start with the case when C is nonsingular. We know that its equation can be reduced to the Hesse canonical form (3.7). The space of apolar quadratic forms is spanned by $\alpha u_0 u_1 - u_2^2$, $\alpha u_1 u_2 - u_0^2$, $\alpha u_0 u_2 - u_1^2$. It is equal to the net of polar conics of the curve C' in the dual plane given by the equation

$$u_0^3 + u_1^3 + u_2^3 - 6\alpha u_0 u_1 u_2 = 0, \quad \alpha(\alpha^3 - 1) \neq 0. \tag{3.30}$$

The net $|\mathrm{AP}_2(f)|$ is base-point-free. Its discriminant curve is a nonsingular cubic, the Hessian curve of the curve C'. The generalized quadrangles are parameterized by the dual curve $\mathrm{He}(C')^\vee$. All pencils are irreducible, so there are no degenerate generalized quadrangles. Generalized quadrangles correspond to tangent lines of the discriminant cubic. So,

$$\mathrm{VSP}(f, 4) = |\mathrm{AP}_2(f)|^\vee, \tag{3.31}$$

and $\mathrm{VSP}(f, 4) \setminus \mathrm{VSP}(f, 4)^o = \mathrm{He}(C')^\vee$.

Next, assume that $C = V(t_2^2 t_0 + t_1^3 + t_1^2 t_0)$ is an irreducible nodal cubic.

The space of apolar quadratic forms is spanned by $u_0^2, u_1 u_2, u_2^2 - u_1^2 + 3u_0 u_1$. The net $|\mathrm{AP}_2(f)|$ is base-point-free. Its discriminant curve is an irreducible nodal cubic D. So, all pencils are irreducible, and (3.31) holds. Generalized quadrangles are parameterized by the union of the dual quartic curve $D^\vee$ and the pencil of lines through the double point.

Next, assume that $C = V(t_0^3 + t_0 t_1 t_2)$ is the union of an irreducible conic and a line which intersects the conic transversally.

The space of apolar quadratic forms is spanned by $u_1^2, u_2^2, 6u_1u_2 - u_0^2$. The net $|\mathrm{AP}_2(f)|$ is base-point-free. It is easy to see that its discriminant curve is the union of a conic and a line intersecting the conic transversally. The line component defines the pencil generated by $V(u_1^2)$ and $V(u_2^2)$. It has no fixed part but its members are singular. So, all generalized quadrangles are nondegenerate and (3.31) holds. The locus of generalized quadrangles consists of a conic and two lines.

Next, assume that $V(f) = V(t_0t_1t_2)$ is the union of three nonconcurrent lines.

The net $|\mathrm{AP}_2(f)|$ of apolar conics is generated by $V(u_0^2), V(u_1^2), V(u_2^2)$. It is base-point-free. The discriminant curve is the union of three nonconcurrent lines representing pencils of singular conics which have no fixed component. Thus any pencil not containing a singular point of the discriminant curve defines a nondegenerate polar quadrangle. A pencil containing a singular point defines a nondegenerate generalized polar quadrangle. Again (3.31) holds and $\mathrm{VSP}(f,4) \setminus \mathrm{VSP}(f,4)^o$ consists of three nonconcurrent lines.

Finally, let $C = V(t_0(t_0t_1 + t_2^2))$ be the union of an irreducible conic and its tangent line. We check that $\mathrm{AP}_2(f)$ is spanned by $u_1^2, u_1u_2, u_2^2 - u_0u_1$. The discriminant curve is a triple line. It corresponds to the pencil $V(\lambda u_1^2 + \mu u_1u_2)$ of singular conics with the fixed component $V(u_1)$. There are no polar quadrangles. Consider the subscheme Z of degree 4 in the affine open set $u_0 \neq 0$ defined by the ideal supported at the point $[1,0,0]$ with ideal at this point generated by $(u_1/u_0)^2, u_1u_2/u_0^2$, and $(u_2/u_0)^2$. The linear system $|\mathcal{I}_Z(3)|$ is of dimension 5 and consists of cubics of the form $V(u_0u_1(au_1 + bu_2) + g_3(u_1,u_2))$. One easily computes $\mathrm{AP}_3(f)$. It is generated by the polynomial $u_0u_2^2 - u_0^2u_1$ and all monomials except $u_0^2u_1$ and $u_0u_2^2$. We see that $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$. Thus Z is a degenerate generalized polar quadrangle of C and (3.31) holds.

□

Remark 3.2.14 We know already the variety $\mathrm{VSP}(f,4)$ in the case when C is the union of concurrent lines. In the remaining cases which have been excluded, the variety $\mathrm{VSP}(f,4)$ is a reducible surface. Its description is too involved to discuss it here. For example, if C is an equianharmonic cubic, it consists of four irreducible components. Three components are isomorphic to $\mathbb{P}^2$. They are disjoint and each contains an open dense subset parametrizing degenerate polar quadrangles. The fourth component contains an open subset of base schemes of irreducible pencils of apolar conics. It is isomorphic to the blow-up of $|\mathrm{AP}_2|^\vee$ at three points corresponding to reducible pencils. Each of

the first three components intersects the fourth component along one of the three exceptional curves.

3.3 Projective generation of cubic curves

3.3.1 Projective generation

Suppose we have m different r-dimensional linear systems $|L_i|$ of hypersurfaces of degrees d_i in $\mathbb{P}^n$. Choose projective isomorphisms $\phi_i : \mathbb{P}^r \to |L_i|$ and consider the variety

$$Z = \{(\lambda, x) \in \mathbb{P}^r \times \mathbb{P}^n : x \in \phi_1(\lambda) \cap \ldots \cap \phi_m(\lambda)\}. \tag{3.32}$$

The expected dimension of a general fiber of the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^r$ is equal to $n - m$. Assume

- Z is irreducible of dimension $r + n - m$;
- the second projection $\mathrm{pr}_2 : Z \to \mathbb{P}^n$ is of finite degree k on its image X.

Under these assumptions, X is an irreducible subvariety of dimension $r + n - m$.

Proposition 3.3.1

$$\deg X = s_r(d_1, \ldots, d_m)/k,$$

where s_r is the r-th elementary symmetric function in m variables.

Proof It is immediate that Z is a complete intersection in $\mathbb{P}^r \times \mathbb{P}^n$ of m divisors of type $(1, d_i)$. Let Π be a general linear subspace in $\mathbb{P}^n$ of codimension $n - m + r$. We use the intersection theory from [231]. Let $\bar{h}_1$ and $\bar{h}_2$ be the natural generators of $H^2(\mathbb{P}^r \times \mathbb{P}^n, \mathbb{Z})$ equal to the preimages of the cohomology classes h_1, h_2 of a hyperplane in $\mathbb{P}^r$ and $\mathbb{P}^n$, respectively. We have $(\mathrm{pr}_2)_*([Z]) = k[X]$. By the projection formula,

$$(\mathrm{pr}_2)_*([Z]) = (\mathrm{pr}_2)_*(\prod_{j=1}^{m}(\bar{h}_1 + d_j\bar{h}_2)) = (\mathrm{pr}_2)_*(\sum_{j=1}^{m} s_j(d_1, \ldots, d_m)\bar{h}_1^j\bar{h}_2^{m-j})$$

$$= \sum_{j=1}^{m} s_j(d_1, \ldots, d_m)h_2^{m-j}(\mathrm{pr}_2)_*(\bar{h}_1^j) = s_r(d_1, \ldots, d_m)h_2^{m-r}.$$

Intersecting with h_2^{n-m+r}, we obtain that $k \deg X = s_r(d_1, \ldots, d_m)$.

□

Since through a general point in $\mathbb{P}^n$ passes a unique member of a pencil, $k = 1$ if $r = 1$.

The following example is *Steiner's construction* of rational normal curves of degree n in $\mathbb{P}^n$. We have already used it in the case of conics, referring the reader for the details to [267].

Example 3.3.2 Let $r = 1, m = n$ and $d_1 = \ldots = d_n = 1$. Let $p_1, \ldots, p_n$ be linearly independent points in $\mathbb{P}^n$ and let $\mathcal{P}_i$ be the pencil of hyperplanes passing through the codimension 2 subspace spanned by all points except p_i. Choose a linear isomorphism $\phi_i : \mathbb{P}^1 \to \mathcal{P}_i$ such that the common hyperplane H spanned by all the points corresponds to different parameters $\lambda \in \mathbb{P}^1$.

Let $H_i(\lambda) = \phi_i(\lambda)$. A line contained in the intersection $H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ meets H, and hence H meets each $H_i(\lambda)$. If H is different from each $H_i(\lambda)$, this implies that the base loci of the pencils $\mathcal{P}_i$ meet. However this contradicts the assumption that the points p_i are linearly independent. If $H = H_i(\lambda)$ for some i, then $H \cap H_j(\lambda)$ is equal to the base locus of $\mathcal{P}_j$. Thus the intersection$H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ consists of the point p_i. This shows that, under the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^1$, the incidence variety (3.32) is isomorphic to $\mathbb{P}^1$. In particular, all the assumptions on the pencils $\mathcal{P}_i$ are satisfied with $k = 1$. Thus the image of Z in $\mathbb{P}^n$ is a rational curve R_n of degree n. If $\phi_i(\lambda) = H$, then the previous argument shows that $p_i \in R_n$. Thus all points $p_1, \ldots, p_n$ lie on R_n. Since all rational curves of degree n in $\mathbb{P}^n$ are projectively equivalent, we obtain that any such curve can be projectively generated by n pencils of hyperplanes.

More generally, let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be n pencils of hyperplanes. Since a projective isomorphism $\phi_i : \mathbb{P}^1 \to \mathcal{P}_i$ is uniquely determined by the images of three different points, we may assume that $\phi_i(\lambda) = V(\lambda_0 l_i + \lambda_1 m_i)$ for some linear forms l_i, m_i. Then the intersection of the hyperplanes $\phi_1(\lambda) \cap \ldots \cap \phi_n(\lambda)$ consists of one point if and only if the system of n linear equations with $n + 1$ unknowns

$$\lambda_0 l_1 + \lambda_1 m_1 = \ldots = \lambda_0 l_n + \lambda_1 m_n = 0$$

has a 1-dimensional space of solutions. Under some genericity assumption on the choice of the pencils, we may always assume it. This shows that the rational curve R_n is projectively generated by the pencils, and its equations are expressed by the condition that

$$\mathrm{rank} \begin{pmatrix} l_0 & l_1 & \ldots & l_n \\ m_0 & m_1 & \ldots & m_n \end{pmatrix} \leq 1.$$

Observe that the maximal minors of the matrix define quadrics in $\mathbb{P}^n$ of rank ≤ 4.

Example 3.3.3 Take two pencils $\mathcal{P}_i$ of planes in $\mathbb{P}^3$ through skew lines ℓ_i. Choose a linear isomorphism $\phi : \mathbb{P}^1 \to \mathcal{P}_i$. Then the union of the lines $\phi_1(\lambda) \cap \phi_2(\lambda)$ is equal to a quadric surface in $\mathbb{P}^3$ containing the lines ℓ_1, ℓ_2.

3.3.2 Projective generation of a plane cubic

We consider a special case of the previous construction where $n = 2, r = 1$ and $m = 2$. By Proposition 3.3.1, X is a curve of degree $d_1 + d_2$. Assume that the base locus of the pencil $\mathcal{P}_i$ consists of d_i^2 distinct points and the two base loci have no points in common. It is clear that the union of the base loci is the set of $d_1^2 + d_2^2$ points on X.

Take a pencil of lines $\mathcal{P}_1$ and a pencil of conics $\mathcal{P}_2$. We obtain a cubic curve C containing the base point of the pencil of lines and four base points of the pencil of conics. The pencil $\mathcal{P}_2$ cuts out on C a g_2^1. We will use the following.

Lemma 3.3.4 *For any g_2^1 on an irreducible reduced plane cubic curve, the lines spanned by the divisor from g_2^1 intersect at one point on the curve.*

Proof The standard exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_C(1) \to 0$$

gives an isomorphism $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong H^0(C, \mathcal{O}_C(1))$. It shows that the pencil g_2^1 is cut out by a pencil of lines. Its base point is the point whose existence is asserted in the Lemma. □

The point of intersection of lines spanned by the divisors from a g_2^1 was called by Sylvester the *coresidual point* of C (see [492], p. 134).

Let C be a nonsingular plane cubic. Pick up four points on C, no three of them lying on a line. Consider the pencil of conics through these points. Let q be the coresidual point of the g_2^1 on C defined by the pencil. Then the pencil of lines through q and the pencil of conics projectively generate C.

Note that the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^1$ is a degree 2 cover defined by the g_2^1 cut out by the pencil of conics. It has four branch points corresponding to lines $\phi_1(\lambda)$ which touch the conic $\phi_2(\lambda)$.

There is another way to projectively generate a cubic curve. This time we take three nets of lines with fixed isomorphisms ϕ_i to $\mathbb{P}^2$. Explicitly, if $\lambda = [\lambda_0, \lambda_1, \lambda_2] \in \mathbb{P}^2$ and $\phi_i(\lambda) = V(a_0^{(i)} t_0 + a_1^{(i)} t_1 + a_2^{(i)} t_2)$, where $a_j^{(i)}$ are linear forms in $\lambda_0, \lambda_1, \lambda_2$, then C is given by the equation

$$\det \begin{pmatrix} a_0^{(1)} & a_1^{(1)} & a_2^{(1)} \\ a_0^{(2)} & a_1^{(2)} & a_2^{(2)} \\ a_0^{(3)} & a_1^{(3)} & a_2^{(3)} \end{pmatrix} = 0.$$

This is an example of a determinantal equation of a plane curve which we will study in detail in the next Chapter.

3.4 Invariant theory of plane cubics

3.4.1 Mixed concomitants

The classical invariant theory dealt with objects more general that invariants of homogeneous forms. Let E, as usual, denote a complex vector space of dimension $n+1$. Recall that the main object of study in the invariant theory is a *mixed combinant*, an element Φ of the tensor product

$$\bigotimes_{i=1}^{r} S^{m_i}(S^{d_i}(E^\vee))^\vee \otimes \bigotimes_{i=1}^{k} S^{p_i}(E^\vee) \otimes \bigotimes_{i=1}^{s} S^{q_i}(E)$$

which is invariant with respect to the natural linear representation of $\mathrm{SL}(E)$ on the tensor product. We will be dealing here only with the cases when $r = 1, k, s \leq 1$. If $k = s = 0$, Φ is an *invariant* of degree m_1 on the space $S^d(E^\vee)$. If $k = 1, s = 0$, then Φ is a *covariant* of degree m and order p. If $k = 0, s = 1$, then Φ is a *contravariant* of degree m and class q. If $k = s = 1$, then Φ is a *mixed concomitant* of degree m, order p and class q.

Choosing a basis $u_0, \dots, u_n$ in E, and the dual basis $t_0, \dots, t_n$ in $E^\vee$, one can write an invariant $\Phi \in S^m(S^d(E^\vee))^\vee = S^m(S^d(E))$ as a homogeneous polynomial of degree m in coefficients of a general polynomial of degree d in $u_0, \dots, u_n$ which are expressed as monomials of degree d in $u_0, \dots, u_n$. Via polarization, we can consider it as a multihomogeneous function of degree $(d, \dots, d)$ on $(E^*)^m$. *Symbolically*, it is written as a product of w sequences $(i_1 \dots i_n)$ of numbers from $\{1, \dots, m\}$ such that each number appears d times. The relation

$$(n+1)w = md$$

must hold. In particular, there are no invariants if $n+1$ does not divide md. The number w is called the *weight* of the invariant. When we apply a linear transformation, it is multiplied by the w-th power of the determinant.

A covariant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^p(E^\vee)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d and of degree p in coordinates $t_0, \dots, t_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d, \dots, d, p)$ on $(E^\vee)^m \times E$. Symbolically, it can be written as a product of w expressions $(j_0 \dots j_n)$ and p expressions $(i)_x$,

where each number from $\{1,\dots,m\}$ appears d times. We must have

$$(n+1)w + pn = md.$$

A contravariant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^q(E)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d and of degree q in $u_0,\dots,u_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d,\dots,d,q)$ on $(E^\vee)^m \times E^\vee$. Symbolically, it can be written as a product of w expressions $(j_0 \dots j_n)$ and q expressions $(i_1 \dots i_n)_u$. We have

$$(n+1)w + qn = md.$$

A mixed concomitant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^p(E^\vee) \otimes S^q(E)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d, of degree p in $t_0,\dots,t_n$, and of degree q in $u_0,\dots,u_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d,\dots,d,p,q)$ on $(E^\vee)^m \times E \times E^\vee$. Symbolically, it can be written as a product of w expressions $(j_0,\dots,j_n)$, p expressions $(i)_x$ and q expressions $(i_1,\dots,i_n)_\xi$, where each number from $\{1,\dots,m\}$ appears d times. We have

$$(n+1)w + (a+b)n = md.$$

Note that instead of numbers $1,\dots,m$ classics often employed m letters $a,b,c,\dots$.

For example, we have met already the Aronhold invariants S and T of degrees 4 and 6 of a ternary cubic form. Their symbolic expressions are

$$\mathsf{S} = (123)(124)(134)(234) = (abc)(abd)(acd)(bcd),$$
$$\mathsf{T} = (123)(124)(135)(256)(456)^2 = (abc)(abd)(ace)(bef)(def)^2.$$

3.4.2 Clebsch's transfer principle

This principle allows one to relate invariants of polynomials in n variables to contravariants and covariants of polynomials in $n+1$ variables.

Start from an invariant Φ of degree m on the space $S^d((\mathbb{C}^n)^\vee)$ of homogeneous polynomials of degree d. We will "transfer it" to a contravariant $\tilde{\Phi}$ on the space of polynomials of degree d in $n+1$ variables. First we fix a volume form ω on E. A basis in a hyperplane $U \subset E$ defines a linear isomorphism $\mathbb{C}^n \to U$. We call a basis admissible if the pull-back of the volume form under this linear map is equal to the standard volume form $e_1 \wedge \dots \wedge e_n$. For

any $\alpha \in E^\vee$, choose an admissible basis $(v_1^\alpha, \dots, v_n^\alpha)$ in $\mathrm{Ker}(\alpha)$. For any $(l_1, \dots, l_m) \in (E^\vee)^m$, we obtain n vectors in $\mathbb{C}^n$, the columns of the matrix

$$A = \begin{pmatrix} l_1(v_1^\alpha) & \dots & l_m(v_1^\alpha) \\ \vdots & \vdots & \vdots \\ l_1(v_n^\alpha) & \dots & l_m(v_n^\alpha) \end{pmatrix}.$$

The value of Φ on this set of vectors can be expressed as a linear combination of the product of maximal minors $|A_I|$, where each column occurs d times. It is easy to see that each minor $A_{i_1 \dots i_n}$ is equal to the value of $l_{i_1} \wedge \dots \wedge l_{i_n} \in \bigwedge^n E^\vee$ on $v_{i_1}^u \wedge \dots \wedge v_{i_n}^u$ under the canonical pairing

$$\bigwedge^n E^\vee \times \bigwedge^n E \to \mathbb{C}.$$

Our choice of a volume from on E allows us to identify $\bigwedge^n E$ with $E^\vee$. Thus any minor can be considered as multilinear function on $(E^\vee)^m \times E^\vee$ and its value does not depend on the choice of an admissible basis in $\mathrm{Ker}(u)$. Symbolically, $(i_1 \dots i_n)$ becomes the bracket expression $(i_1 \dots i_n)_u$. This shows that the invariant Φ, by restricting to the subspaces $\mathrm{Ker}(\alpha)$, defines a contravariant $\tilde{\Phi}$ on $S^d(E^\vee)$ of degree m and class $q = md/n$.

Example 3.4.1 Let Φ be the discriminant of a quadratic form in n variables. It is an invariant of degree $m = n$ on the space of quadratic forms. Its symbolic notation is $(12 \dots n)^2$. Its transfer to $\mathbb{P}^n$ is a contravariant $\tilde{\Phi}$ of degree n and class $q = 2n/n = 2$. Its symbolic notation is $(12 \dots n)_u^2$. Considered as map $\tilde{\Phi} : S^2 E^\vee \to S^2 E$, the value of $\tilde{\Phi}(q)$ on $u \in E^\vee$ is the discriminant of the quadratic form obtained from restriction of q to $\mathrm{Ker}(u)$. It is equal to zero if and only if the hyperplane $V(u)$ is tangent to the quadric $V(q)$. Thus $V(\tilde{\Phi}(q))$ is the dual quadric $V(q)^\vee$.

Example 3.4.2 Consider the quadratic invariant S on the space of binary forms of even degree $d = 2k$ with symbolic expression $(12)^{2k}$. We write a general binary form $f \in S^d(U)$ of degree d symbolically,

$$f = (\xi_0 t_0 + \xi_1 t_1)^{2k} = (\eta_0 t_0 + \eta_1 t_1)^{2k},$$

where (ξ_0, ξ_1) and (η_0, η_1) are two copies of a basis in U and (t_0, t_1) is its dual basis. Then the coefficients of f are equal to $\binom{d}{j} a_j$, where $a_j = \xi_0^j \xi_1^{2k-j} = \eta_0^j \eta_1^{2k-j}$. Thus S is equal to

$$(\xi_0 \eta_1 - \xi_1 \eta_0)^{2k} = \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} (\xi_0 \eta_1)^j (\xi_1 \eta_0)^{2k-j}$$

$$= \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} (\xi_0^j \xi_1^{2k-j})(\eta_1 \eta_0^{2k-j})$$

$$= \sum_{j=0}^{2k} (-1)^j \tbinom{d}{j} a_j a_{2k-j} = 2\Big(\sum_{j=0}^{k} (-1)^j \tbinom{2k}{j} a_j a_{2k-j} + \tfrac{1}{2}\tbinom{2k}{k} a_k^2\Big).$$

We have already encountered this invariant in the case $d = 3$ (see Subsection 1.5.1).

The transfer of S is the contravariant of degree 2 and class d with symbolic expression $(abu)^d$. For example, when $d = 4$, its value on a quartic ternary form f is a quartic form in the dual space which vanishes on lines which cut out on $V(f)$ a harmonic set of 4 points. The transfer T of the invariant of degree 3 on the space of quartic binary forms defines a contravariant of class 6. Its value on a quartic ternary form is a ternary form of degree 6 in the dual space which vanishes on the set of lines which cut out in $V(f)$ an equianharmonic set of 4 points.

One can also define Clebsch's transfer of covariants of degree m and order p, keeping the factors i_x in the symbolic expression. The result of the transfer is a mixed concomitant of degree m, order p and class md/n.

3.4.3 Invariants of plane cubics

Since this material is somewhat outside of the topic of the book, we state some of the facts without proof, referring to classical sources for the invariant theory (e.g. [113], t. 2, [492]).

We know that the ring of invariants of ternary cubic forms is generated by the Aronhold invariants S and T. Let us look for covariants and contravariants. As we know from Subsection 1.5.1, any invariant of binary form of degree 3 is a power of the discriminant invariant of order 4, and the algebra of covariants is generated over the ring of invariants by the identical covariant $\mathsf{U} : f \mapsto f$, the Hessian covariant H of order 2 with symbolic expression $(ab)a_x b_x$, and the covariant $\mathsf{J} = \mathrm{Jac}(f, \mathsf{H})$ of degree 3 and order 6 with symbolic expression $(ab)^2(ca)b_x c_x^2$. Clebsch's transfer of the discriminant is a contravariant F of degree 4 and class 6. Its symbolic expression is $(abu)^2(cdu)^2(acu)(bdu)$. Its value on a general ternary cubic form is the form defining the dual cubic curve. Clebsch's transfer of H is a mixed concomitant Θ of degree 2, order 2 and class

2. Its symbolic expression is $(abu)^2 a_x b_x$. Explicitly, up to a constant factor,

$$\Theta = \det \begin{pmatrix} f_{00} & f_{01} & f_{02} & u_0 \\ f_{10} & f_{11} & f_{12} & u_1 \\ f_{20} & f_{21} & f_{22} & u_2 \\ u_0 & u_1 & u_2 & 0 \end{pmatrix}, \tag{3.33}$$

where $f_{ij} = \frac{\partial^2 f}{\partial t_i \partial t_j}$.

The equation $\Theta(f, x, u) = 0$, for fixed x, is the equation of the dual of the polar conic $P_x(V(f))$. The equation $\Theta(f, x, u) = 0$, for fixed u, is the equation of the locus of points x such that the first polar $P_x(V(f))$ is tangent to the line $V(u)$. It is called the *poloconic* of the line $V(u)$. Other description of the poloconic can be found in Exercise 3.3.

The Clebsch transfer of J is a mixed concomitant Q of degree 3, order 3 and class 3. Its symbolic expression is $(abu)^2 (cau) c_x^2 b_x$. The equation $\mathsf{Q}(f, x, u) = 0$, for fixed u, is the equation of the cubic curve such that second polars $P_{x^2}(V(f))$ of its points intersect $V(u)$ at a point conjugate to x with respect to the poloconic of $V(u)$. A similar contravariant is defined by the condition that it vanishes on the set of pairs (x, u) such that the line $V(u)$ belongs to the Salmon envelope conic of the polars of x with respect to the curve and its Hessian curve.

An obvious covariant of degree 3 and order 3 is the Hessian determinant $\mathsf{H} = \det \mathrm{He}(f)$. Its symbolic expression is $(abc)^2 a_x b_x c_x$. Another covariant G is defined by the condition that it vanishes on the locus of points x such that the Salmon conic of the polar of x with respect to the curve and its Hessian curve passes through x. It is of degree 8 and order 6. Its equation is the following bordered determinant

$$\begin{pmatrix} f_{00} & f_{01} & f_{02} & h_0 \\ f_{10} & f_{11} & f_{12} & h_1 \\ f_{20} & f_{21} & f_{22} & h_2 \\ h_0 & h_1 & h_2 & 0 \end{pmatrix},$$

where $f_{ij} = \frac{\partial^2 f}{\partial t_i \partial t_j}, h_i = \frac{\partial \mathsf{H}(f)}{\partial t_i}$ (see [77],[113], t. 2, p. 313). The algebra of covariants is generated by $\mathsf{U}, \mathsf{H}, \mathsf{G}$ and the *Brioschi covariant* [56] $J(f, \mathsf{H}, \mathsf{G})$ whose value on the cubic (3.7) is equal to

$$(1 + 8\alpha^3)(t_1^3 - t_2^3)(t_2^3 - t_0^3)(t_0^3 - t_1^3).$$

Comparing this formula with (3.16), we find that it vanishes on the union of 9 harmonic polars of the curve. The square of the Hermite covariant is a polynomial in $\mathsf{U}, \mathsf{H}, \mathbf{G}$.

The Cayleyan of a plane cubic defines a contravariant P of degree 3 and class

3. Its symbolic expression is $(abc)(abu)(acu)(bcu)$. Its value on the curve in the Hesse form is given in (3.27). There is also a contravariant Q of degree 5 and class 3. In analogy with the form of the word Hessian, A. Cayley gave them the names the *Pippian* and the *Quippian* [78]. If $C = V(f)$ is given in the Hesse form (3.7), then

$$\mathsf{Q}(f) = V((1 - 10\alpha^3)(u_0^2 + u_1^3 + u_2^3) - 6\alpha^2(5 + 4\alpha^3)u_0u_1u_2).$$

The full formula can be found in Cayley's paper [77]). He also gives the formula

$$\mathsf{H}(6a\mathsf{P} + b\mathsf{Q}) = (-2Ta^3 + 48S^2a2b + 18TSab^2 + (T^3 + 16S^2)b^3\mathsf{P}$$

$$+(8\mathsf{S}a^3 + 3\mathsf{T}a^2b - 24\mathsf{S}^2ab^2 - \mathsf{T}\mathsf{S}^2b^3)\mathsf{Q},$$

where the product of a covariant and a contravariant is considered as the composition of the corresponding equivariant maps.

According to A. Clebsch, $\mathsf{Q}(f)$ vanishes on the locus of lines whose poloconics with respect to the Cayleyan of C are apolar to their poloconics with respect to C. Also, according to W. Milne and D. Taylor, $\mathsf{Q}(f)$ is the locus of lines which intersect C at three points such that the polar line of the Hessian curve $\mathsf{H}(f)$ with respect two of the points is tangent to $\mathsf{H}(f)$ at the third point (see [384]). This is similar to the property of the Pippian which is the set of lines which intersect C at three points such that the polar line with respect to two of the points is tangent to C at the third point. The algebra of contravariants is generated by $\mathsf{F}, \mathsf{P}, \mathsf{Q}$ and the *Hermite contravariant* [287]. Its value on the cubic in the Hesse form is equal to

$$(1 + 8\alpha^3)(u_1^3 - u_2^3)(u_2^3 - u_0^3)(u_0^3 - u_1^3).$$

It vanishes on the union of nine lines corresponding to the inflection points of the curve. The square of the Hermite contravariant is a polynomial in $\mathsf{F}, \mathsf{P}, \mathsf{Q}$.

Exercises

3.1 Find the Hessian form of a nonsingular cubic given by the Weierstrass equation.

3.2 Let $H = \mathrm{He}(C)$ be the Hessian cubic of a nonsingular plane cubic curve C that is not an equianharmonic cubic. Let $\tau : H \to H$ be the Steinerian automorphism of H that assigns to $a \in H$ the unique singular point of $P_a(C)$.

(i) Let $\tilde{H} = \{(a, \ell) \in H \times (\mathbb{P}^2)^\vee : \ell \subset P_a(C)\}$. Show that the projection $p_1 : \tilde{H} \to H$ is an unramified double cover.
(ii) Show that $\tilde{H} \cong H/\langle\tau\rangle \cong C$.

3.3 Let $C = V(f) \subset \mathbb{P}^2$ be a nonsingular cubic.

(i) Show that the set $K(\ell)$ of second polars of C with respect to points on a fixed line ℓ is a dual conic of the poloconic of C with respect to ℓ.
(ii) Show that $K(\ell)$ is equal to the set of poles of ℓ with respect to polar conics $P_x(C)$, where $x \in \ell$.
(iii) What happens to the conic $K(\ell)$ when the line ℓ is tangent to C?
(iv) Show that the set of lines ℓ such that $K(\ell)$ is tangent to ℓ is the dual curve of C.
(v) Let $\ell = V(a_0t_0 + a_1t_1 + a_2t_2)$. Show that $K(\ell)$ can be given by the equation

$$g(a,t) = \det \begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ a_0 & \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 \partial t_1} & \frac{\partial^2 f}{\partial t_0 \partial t_2} \\ a_1 & \frac{\partial^2 f}{\partial t_1 \partial t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ a_2 & \frac{\partial^2 f}{\partial t_2 \partial t_0} & \frac{\partial^2 f}{\partial t_2 \partial t_1} & \frac{\partial^2 f}{\partial t_2^2} \end{pmatrix} = 0.$$

(vi) Show that the dual curve $C^\vee$ of C can be given by the equation (the *Schläfli equation*)

$$\det \begin{pmatrix} 0 & \xi_0 & \xi_1 & \xi_2 \\ \xi_0 & \frac{\partial^2 g(\xi,t)}{\partial t_0^2}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_0 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_0 \partial t_2}(\xi) \\ \xi_1 & \frac{\partial^2 g(\xi,t)}{\partial t_1 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_1^2}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_1 \partial t_2}(\xi) \\ \xi_2 & \frac{\partial^2 g(\xi,t)}{\partial t_2 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_2 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_2^2}(\xi) \end{pmatrix}.$$

3.4 Let $C \subset \mathbb{P}^{d-1}$ be an elliptic curve embedded by the linear system $\left|\mathcal{O}_C(dp_0)\right|$, where p_0 is a point in C. Assume $d = p$ is prime.

(i) Show that the image of any p-torsion point is an osculating point of C, i.e., a point such that there exists a hyperplane (an *osculating hyperplane*) which intersects the curve only at this point.
(ii) Show that there is a bijective correspondence between the sets of cosets of $(\mathbb{Z}/p\mathbb{Z})^2$ with respect to subgroups of order p and hyperplanes in $\mathbb{P}^{p-1}$ which cut out in C the set of p osculating points.
(iii) Show that the set of p-torsion points and the set of osculating hyperplanes define a $(p_{p+1}^2, p(p+1)_p)$-configuration of p^2 points and $p(p+1)$ hyperplanes (i.e. each point is contained in $p+1$ hyperplanes and each hyperplane contains p points).
(iv) Find a projective representation of the group $(\mathbb{Z}/p\mathbb{Z})^2$ in $\mathbb{P}^{p-1}$ such that each osculating hyperplane is invariant with respect to some cyclic subgroup of order p of $(\mathbb{Z}/p\mathbb{Z})^2$.

3.5 A point on a nonsingular cubic is called a *sextactic point* if there exists an irreducible conic intersecting the cubic at this point with multiplicity 6. Show that there are 27 sextactic points.

3.6 The pencil of lines through a point on a nonsingular cubic curve C contains four tangent lines. Show that the twelve contact points of three pencils with collinear base points on C lie on 16 lines forming a configuration $(12_4, 16_3)$ (the *Hesse-Salmon configuration*).

3.7 Show that the cross ratio of the four tangent lines of a nonsingular plane cubic curve which pass through a point on the curve does not depend on the point.

3.8 Prove that the second polar of a nonsingular cubic C with respect to the point a on

the Hessian $\mathrm{He}(C)$ is equal to the tangent line $\mathbb{T}_b(\mathrm{He}(C))$, where b is the singular point of the polar conic $P_a(C)$.

3.9 Let a, b be two points on the Hessian curve $\mathrm{He}(C)$ forming an orbit with the respect to the Steinerian involution. Show that the line $\overline{ab}$ is tangent to $\mathrm{Cay}(C)$ at some point d. Let c be the third intersection point of $\mathrm{He}(C)$ with the line $\overline{ab}$. Show that the pairs (a, b) and (c, d) are harmonically conjugate.

3.10 Show that from each point a on the $\mathrm{He}(C)$ one can pass three tangent lines to the dual curve of $\mathrm{Cay}(C)$. Let b be the singular point of $P_a(C)$. Show that the set of the three tangent lines consists of the line $\overline{ab}$ and the components of the reducible polar conic $P_b(C)$.

3.11 Let $C = V(\sum_{0\le i\le j\le k\le 2} a_{ijk}t_it_jt_k)$. Show that the Cayleyan curve $\mathrm{Cay}(C)$ can be given by the equation

$$\det\begin{pmatrix} a_{000} & a_{001} & a_{002} & \xi_0 & 0 & 0 \\ a_{110} & a_{111} & a_{112} & 0 & \xi_1 & 0 \\ a_{220} & a_{221} & a_{222} & 0 & 0 & \xi_2 \\ 2a_{120} & 2a_{121} & 2a_{122} & 0 & \xi_2 & \xi_1 \\ 2a_{200} & 2a_{201} & 2a_{202} & \xi_2 & 0 & \xi_0 \\ 2a_{010} & 2a_{011} & 2a_{012} & \xi_1 & \xi_0 & 0 \end{pmatrix} = 0$$

[113], p. 245.

3.12 Show that any general net of conics is equal to the net of polars of some cubic curve. Show that the curve parameterizing the irreducible components of singular members of the net coincides with the Cayleyan curve of the cubic (it is called the *Hermite curve* of the net).

3.13 Show that the group of projective transformations leaving a nonsingular plane cubic invariant is a finite group of order 18, 36 or 54. Determine these groups.

3.14 Find all ternary cubics C such that $\mathrm{VSP}(C, 4)^o = \emptyset$.

3.15 Show that a plane cubic curve belongs to the closure of the Fermat locus if and only if it admits a first polar equal to a double line or the whole space.

3.16 Show that any plane cubic curve can be projectively generated by three pencils of lines.

3.17 Given a nonsingular conic K and a nonsingular cubic C, show that the set of points x such that $P_x(C)$ is inscribed in a self-polar triangle of K is a conic.

3.18 A complete quadrilateral is inscribed in a nonsingular plane cubic. Show that the tangent lines at the two opposite vertices intersect at a point on the curve. Also, show that the three points obtained in this way from the three pairs of opposite vertices are collinear.

3.19 Let $\mathfrak{o}$ be a point in the plane outside of a nonsingular plane cubic C. Consider the six tangents to C from the point $\mathfrak{o}$. Show that there exists a conic passing through the six points on C which lie on the tangents but not equal to the tangency points. It is called the *satellite conic* of C [141]. Show that this conic is tangent to the polar conic $P_{\mathfrak{o}}(C)$ at the points where it intersects the polar line $P_{\mathfrak{o}^2}(C)$.

3.20 Show that two general plane cubic curves C_1 and C_2 admit a common polar pentagon if and only if the planes of apolar conics $|\mathrm{AP}_2(C_1)|$ and $|\mathrm{AP}_2(C_2)|$ intersect.

3.21 Let C be a nonsingular cubic and K be its apolar cubic in the dual plane. Prove that, for any point on C, there exists a conic passing through this point such that the remaining five intersection points with C form a polar pentagon of K [499].

3.22 Let p, q be two distinct points on a nonsingular plane cubic curve. Starting from an arbitrary point p_1 find the third intersection point q_1 of the line $\overline{pp_1}$ with C, then define p_2 as the third intersection point of the line $\overline{qq_1}$ with C, and continue in this way to define a sequence of points $p_1, q_1, p_2, q_2, \ldots, q_k, p_{k+1}$ on C. Show that $p_{k+1} = p_1$ if and only if $p - q$ is a k-torsion point in the group law on C defined by a choice of some inflection point as the zero point. The obtained polygon $(p_1, q_1, \ldots, q_k, p_1)$ is called the *Steiner polygon* inscribed in C.

3.23 Show that the polar conic $P_x(C)$ of a point x on a nonsingular plane cubic curve C cuts out on C the divisor $2x + a + b + c + d$ such that the intersection points $\overline{ab} \cap \overline{cd}$, $\overline{ac} \cap \overline{bd}$ and $\overline{ad} \cap \overline{bc}$ lie on C.

3.24 Show that any intersection point of a nonsingular cubic C and its Hessian curve is a sextactic point on the latter.

3.25 Fix three pairs (p_i, q_i) of points in the plane in general position. Show that the closure of the locus of points x such that the three pairs of lines $\overline{xp_i}, \overline{xq_i}$ are members of a g^1_2 in the pencil of lines through x is a plane cubic.

3.26 Fix three points p_1, p_2, p_3 in the plane and three lines ℓ_1, ℓ_2, ℓ_3 in general position. Show that the set of points x such that the intersection points of $\overline{xp_i}$ with ℓ_i are collinear is a plane cubic curve [261].

Historical Notes

The theory of plane cubic curves originates from the works of I. Newton [413] and his student C. MacLaurin [375]. Newton was the first to classify real cubic curves, and he also introduced the Weierstrass equation. Much later, K. Weierstrass showed that the equation can be parameterized by elliptic functions, the Weierstrass functions $\wp(z)$ and $\wp(z)'$. The parameterization of a cubic curve by elliptic functions was widely used for defining a group law on the cubic. We refer to [495] for the history of the group law on a cubic curve. Many geometric results on cubic curves follow simply from the group law and were first discovered without using it. For example, the fact that the line joining two inflection points contains the third inflection point was discovered by MacLaurin much earlier before the group law was discovered. The book of Clebsch and Lindemann [113] contains many applications of the group law to the geometry of cubic curves.

The Hesse pencil was introduced and studied by O. Hesse [288],[289]. The pencil was also known as the *syzygetic pencil* (see [113]). It was widely used as a canonical form for a nonsingular cubic curve. More facts about the Hesse pencil and its connection to other constructions in modern algebraic geometry can be found in [14].

The Cayleyan curve first appeared in Cayley's paper [72]. The Schläfli equation of the dual curve from the Exercises was given by L. Schläfli in [496]. Its modern proof can be found in [239].

The polar polygons of plane cubics were first studied by F. London [366]. London proves that the set of polar 4-gons of a general cubic curve are base points of apolar pencils of conics in the dual plane. A modern treatment of some of these results is given in [177] (see also [460] for related results). A beautiful paper by G. Halphen [275] discusses the geometry of torsion points on plane cubic curves.

Poloconics of a cubic curve are studied extensively in Durège's book [194]. The term belongs to L. Cremona [141] (conic polar in Salmon's terminology). O. Schlessinger proved in [499] that any polar pentagon of a nonsingular cubic curve can be inscribed in an apolar cubic curve.

The projective generation of a cubic curve by a pencil and a pencil of conics was first given by M. Chasles. Other geometric ways to generate a plane cubic are discussed in Durège's book [194]. Steiner polygons inscribed in a plane cubic were introduced by J. Steiner in [541]. His claim that their existence is an example of a porism was given without proof. The proof was later supplied by A. Clebsch [106].

The invariants S and T of a cubic ternary form were first introduced by Aronhold [11]. G. Salmon gave the explicit formulas for them in [492]. The basic covariants and contravariants of plane cubics were given by A. Cayley [77]. He also introduced 34 basic concomitants [90]. They were later studied in detail by A. Clebsch and P. Gordan [110]. The fact that they generate the algebra of concomitants was first proved by P. Gordan and M. Noether [253], and by S. Gundelfinger [272]. A simple proof for the completeness of the set of basic covariants was given by L. Dickson [167]. One can find an exposition on the theory of invariants of ternary cubics in classical books on the invariant theory [258], [212].

Cremona's paper [141] is a fundamental source of the rich geometry of plane curves, and in particular, cubic curves. Other good sources for the classical geometry of cubic curves are books by Clebsch and Lindemann [113], t. 2, by H. Durège [194], by G. Salmon [492], by H. White [601] and by H. Schroter [503].

4

Determinantal equations

4.1 Plane curves

4.1.1 The problem

Let us consider the following problem. Let $f(t_0, \dots, t_n)$ be a homogeneous polynomial of degree d, find a $d \times d$ matrix $A = (l_{ij}(t))$ with linear forms as its entries such that

$$f(t_0, \dots, t_n) = \det(l_{ij}(t)). \tag{4.1}$$

We say that two determinantal representations defined by matrices A and A' are *equivalent* if there exists two invertible matrices X,Y with constant entries such that $A' = XAY$. One may ask to describe the set of equivalence classes of determinantal representations.

First, let us reinterpret this problem geometrically and coordinate-free. Let E be a vector space of dimension $n+1$ and let U, V be vector spaces of dimension d. A square matrix of size $d \times d$ corresponds to a linear map $U^\vee \to V$, or an element of $U \otimes V$. A matrix with linear forms corresponds to an element of $E^\vee \otimes U \otimes V$, or a linear map $\phi' : E \to U \otimes V$.

We shall assume that the map ϕ' is injective (otherwise the hypersurface $V(f)$ is a cone, so we can solve our problem by induction on the number of variables). Let

$$\phi : |E| \to |U \otimes V| \tag{4.2}$$

be the regular map of the associated projective spaces. Let $\mathcal{D}_d \subset |U \otimes V|$ be the *determinantal hypersurface* parameterizing non-invertible linear maps $U^\vee \to V$. If we choose bases in U, V, then $\mathcal{D}_d$ is given by the determinant of a square matrix (whose entries will be coordinates in $U \otimes V$). The preimage of $\mathcal{D}_d$ in $|E|$ is a hypersurface of degree d. Our problem is to construct such a map ϕ in order that a given hypersurface is obtained in this way.

Note that the singular locus $\mathcal{D}_d^{\text{sing}}$ of $\mathcal{D}_d$ corresponds to matrices of corank ≥ 2. It is easy to see that its codimension in $|U \otimes V|$ is equal to 4. If the image of $|E|$ intersects $\mathcal{D}_d^{\text{sing}}$, then $\phi^{-1}(\mathcal{D}_d)$ will be a singular hypersurface. So, a nonsingular hypersurface $V(f)$ of dimension ≥ 3 cannot be given by a determinantal equation. However, it still could be true for the hypersurface $V(f^k)$.

4.1.2 Plane curves

Let us first consider the case of nonsingular plane curves $C = V(f) \subset \mathbb{P}^2$. Assume that C has a determinantal equation. As we explained earlier, the image of the map ϕ does not intersect $\mathcal{D}_d^{\text{sing}}$. Thus, for any $x \in C$, the corank of the matrix $\phi(x)$ is equal to 1 (here we consider a matrix up to proportionality since we are in the projective space). The null-space of this matrix is a 1-dimensional subspace of $U^\vee$, i.e., a point in $\mathbb{P}(U)$. This defines a regular map

$$\mathfrak{l} : C \to \mathbb{P}(U), \quad x \mapsto |\text{Ker}(\phi(x))|.$$

Now let ${}^t\phi(x) : V^\vee \to U$ be the transpose map. In coordinates, it corresponds to the transpose matrix. Its null-space is isomorphic to $\text{Im}(\phi(x))^\perp$ and is also one-dimensional. So we have another regular map

$$\mathfrak{r} : C \to \mathbb{P}(V), \quad x \mapsto |\text{Ker}({}^t\phi(x))|.$$

Let

$$\mathcal{L} = \mathfrak{l}^*\mathcal{O}_{\mathbb{P}(U)}(1), \quad \mathcal{M} = \mathfrak{r}^*\mathcal{O}_{\mathbb{P}(V)}(1).$$

These are invertible sheaves on the curve C. We can identify U with $H^0(C, \mathcal{L})$ and V with $H^0(C, \mathcal{M})$ (see Lemma 4.1.3 below). Consider the composition of regular maps

$$\psi : C \xrightarrow{(\mathfrak{l},\mathfrak{r})} \mathbb{P}(U) \times \mathbb{P}(V) \xrightarrow{s_2} \mathbb{P}(U \otimes V), \tag{4.3}$$

where s_2 is the Segre map. It follows from the definition of the Segre map, that the tensor $\psi(x)$ is equal to $\mathfrak{l}(x) \otimes \mathfrak{r}(x)$. It can be viewed as a linear map $U \to V^\vee$. In coordinates, the matrix of this map is the product of the column vector defined by $\mathfrak{r}(x)$ and the row vector defined by $\mathfrak{l}(x)$. It is a rank 1 matrix equal to the adjugate matrix of the matrix $A = \phi(x)$ (up to proportionality). Consider the rational map

$$\text{Adj} : |U \otimes V| - \to \mathbb{P}(U \otimes V) \tag{4.4}$$

defined by taking the adjugate matrix. Recall that the adjugate matrix should be considered as a linear map $\bigwedge^{d-1} U^\vee \to \bigwedge^{d-1} V$ and we can identify

$|U^\vee \otimes V^\vee|$ with $|\bigwedge^{d-1} U \otimes \bigwedge^{d-1} V|$. Although Adj is not well-defined on vector spaces, it is well-defined, as a rational map, on the projective spaces (see Example 1.1.4). Let $\Psi = \mathrm{Adj} \circ \phi$, then ψ is equal to the restriction of Ψ to C. Since Adj is defined by polynomials of degree $d-1$ (after we choose bases in U, V), we have

$$\Psi^* \mathcal{O}_{\mathbb{P}(U\otimes V)}(1) = \mathcal{O}_{|E|}(d-1).$$

This gives

$$\psi^* \mathcal{O}_{\mathbb{P}(U\otimes V)}(1) = \mathcal{O}_{|E|}(d-1) \otimes \mathcal{O}_C = \mathcal{O}_C(d-1).$$

On the other hand, we get

$$\psi^* \mathcal{O}_{\mathbb{P}(U\otimes V)}(1) = (s_2 \circ (\mathfrak{l}, \mathfrak{r}))^* \mathcal{O}_{\mathbb{P}(U\otimes V)}(1)$$

$$= (\mathfrak{l}, \mathfrak{r})^* \big(s_2^* \mathcal{O}_{\mathbb{P}(U\otimes V)}(1)\big) = (\mathfrak{l}, \mathfrak{r})^* \big(p_1^* \mathcal{O}_{\mathbb{P}(U)}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}(V)}(1)\big)$$

$$= \mathfrak{l}^* \mathcal{O}_{\mathbb{P}(U)}(1) \otimes \mathfrak{r}^* \mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{L} \otimes \mathcal{M}.$$

Here $p_1 : \mathbb{P}(U) \times \mathbb{P}(V) \to |U|$, $p_2 : \mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(V)$ are the projection maps. Comparing the two isomorphisms, we obtain

Lemma 4.1.1

$$\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C(d-1). \tag{4.5}$$

Remark 4.1.2 It follows from Example 1.1.4 that the rational map (4.4) is given by the polars of the determinantal hypersurface. In fact, if $A = (t_{ij})$ is a matrix with independent variables as entries, then $\frac{\partial \det(A)}{\partial t_{ij}} = M_{ij}$, where M_{ij} is the ij-th cofactor of the matrix A. The map Adj is a birational map since $\mathrm{Adj}(A) = A^{-1} \det(A)$ and the map $A \to A^{-1}$ is obviously invertible. So, the determinantal equation is an example of a homogeneous polynomial such that the corresponding polar map is a birational map. Such a polynomial is called a *homaloidal polynomial* (see [180]).

Lemma 4.1.3 *Let $g = \frac{1}{2}(d-1)(d-2)$ be the genus of the curve C. Then*

(i) $\deg(\mathcal{L}) = \deg(\mathcal{M}) = \frac{1}{2}d(d-1) = g - 1 + d$;
(ii) $H^0(C, \mathcal{L}) \cong U$, $H^0(C, \mathcal{M}) \cong V$;
(iii) $H^i(C, \mathcal{L}(-1)) \cong H^i(C, \mathcal{M}(-1)) = \{0\}$, $i = 0, 1$;
(iv) $H^1(C, \mathcal{L}) = H^1(C, \mathcal{M}) = 0$.

Proof Let us first prove (iii). A nonzero section of $H^0(C, \mathcal{L}(-1))$ is a section of $\mathcal{L}$ that defines a hyperplane in $\mathbb{P}(U)$ which intersects $\mathfrak{l}(C)$ along a divisor $\mathfrak{l}(D)$, where D is a divisor on C cut out by a line. Since all such divisors D are linearly equivalent, we see that for any line ℓ the divisor $\mathfrak{l}(\ell \cap C)$ is cut out by a hyperplane in $\mathbb{P}(U)$. Choose ℓ such that it intersects C at d distinct points $x_1, \ldots, x_d$. Choose bases in U and V. The image of $\phi(\ell)$ in $|U \otimes V| = \mathbb{P}(\mathrm{Mat}_d)$ is a pencil of matrices $\lambda A + \mu B$. We know that there are d distinct values of (λ, μ) such that the corresponding matrix is of corank 1. Without loss of generality, we may assume that A and B are invertible matrices. So we have d distinct λ_i such that the matrix $A + \lambda_i B$ is singular. Let u^i span $\mathrm{Ker}(A + \lambda_i B)$. The corresponding points in $\mathbb{P}(U)$ are equal to the points $\mathfrak{l}(t_i)$. We claim that the vectors $u^1, \ldots, u^d$ are linearly independent vectors in $\mathbb{P}(U)$. The proof is by induction on d. Assume $a_1 u^1 + \cdots + a_d u^d = 0$. Then $Au^i + \lambda_i B u^i = 0$ for each $i = 1, \ldots, d$, gives

$$0 = A\big(\sum_{i=1}^{d} a_i u^i\big) = \sum_{i=1}^{d} a_i A u^i = -\sum_{i=1}^{d} a_i \lambda_i B u^i.$$

We also have

$$0 = B\big(\sum_{i=1}^{d} a_i u^i\big) = \sum_{i=1}^{d} a_i B u^i.$$

Multiplying the second equality by λ_d and adding it to the first one, we obtain

$$\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i) B u^i = B\big(\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i) u^i\big) = 0.$$

Since B is invertible, this gives

$$\sum_{i=1}^{d-1} a_i(\lambda_i - \lambda_d) u^i = 0.$$

By induction, the vectors $u^1, \ldots, u^{d-1}$ are linearly independent. Since $\lambda_i \neq \lambda_d$, we obtain $a_1 = \ldots = a_{d-1} = 0$. Since $u^d \neq 0$, we also get $a_d = 0$.

Since $u^1, \ldots, u^d$ are linearly independent, the points $\mathfrak{l}(x_i)$ span $\mathbb{P}(U)$. Hence no hyperplane contains these points. This proves that $H^0(C, \mathcal{L}(-1)) = 0$. Similarly, we prove that $H^0(C, \mathcal{M}(-1)) = 0$. Applying Lemma 4.1.1, we get

$$\mathcal{L}(-1) \otimes \mathcal{M}(-1) \cong \mathcal{O}_C(d-3) = \omega_C, \tag{4.6}$$

where ω_C is the canonical sheaf on C. By duality,

$$H^i(C, \mathcal{M}(-1)) \cong H^{1-i}(C, \mathcal{L}(-1)),\ i = 0, 1.$$

This proves (iii). Let us prove (i) and (ii). Let h be a section of $\mathcal{O}_C(1)$ with subscheme of zeros equal to H. The multiplication by h defines an exact sequence

$$0 \to \mathcal{L}(-1) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_H \to 0.$$

After passing to cohomology and applying (iii), we obtain $H^1(C, \mathcal{L}) = 0$. Replacing $\mathcal{L}$ with $\mathcal{M}$ and repeating the argument, we obtain that $H^1(C, \mathcal{M}) = 0$. This checks (iv).

We know that $\dim H^0(C, \mathcal{L}) \geq \dim U = d$. Applying Riemann-Roch, we obtain

$$\deg(\mathcal{L}) = \dim H^0(C, \mathcal{L}) + g - 1 \geq d + g - 1.$$

Similarly, we get

$$\deg(\mathcal{M}) \geq d + g - 1.$$

Adding up, and applying Lemma 4.1.1, we obtain

$$d(d-1) = \deg \mathcal{O}_C(d-1) = \deg(\mathcal{L}) + \deg(\mathcal{M}) \geq 2d + 2g - 2 = d(d-1).$$

Thus all the inequalities above are the equalities, and we get assertions (i) and (ii).

□

Now we would like to prove the converse. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on C satisfying (4.5) and the properties from the previous Lemma hold. It follows from property (iv) and the Riemann-Roch Theorem that

$$\dim U = \dim V = d.$$

Let $\mathfrak{l} : C \to \mathbb{P}(U), \mathfrak{r} : C \to \mathbb{P}(V)$ be the maps given by the complete linear systems $|\mathcal{L}|$ and $|\mathcal{M}|$. We define $\psi : C \to \mathbb{P}(U \otimes V)$ to be the composition of $(\mathfrak{l}, \mathfrak{r})$ and the Segre map s_2. It follows from property (4.5) that the map ψ is the restriction of the map

$$\Psi : |E| \to \mathbb{P}(U \otimes V)$$

given by a linear system of plane curves of degree $d-1$. We can view this map as a tensor in $S^{d-1}(E^\vee) \otimes U^\vee \otimes V^\vee$. In coordinates, it is a $d \times d$ matrix $A(t)$ with entries from the space of homogeneous polynomials of degree $d-1$. Since $\Psi|_C = \psi$, for any point $x \in C$, we have rank $A(x) = 1$. Let M be a 2×2 submatrix of $A(t)$. Since $\det M(x) = 0$ for $x \in C$, we have $f \mid \det M$. Consider a 3×3 submatrix N of $A(t)$. We have $\det \operatorname{adj}(N) = \det(N)^2$. Since the entries of $\operatorname{adj}(N)$ are determinants of 2×2 submatrices, we see that $f^3 \mid$

$\det(N)^2$. Since C is irreducible, this immediately implies that $f^2 \mid \det(N)$. Continuing in this way, we obtain that f^{d-2} divides all cofactors of the matrix A. Thus $B = f^{2-d}\text{adj}(A)$ is a matrix with entries in $E^\vee$. It defines a linear map $E \to U \otimes V$ and corresponding regular map of projective spaces

$$\phi : |E| \to |U \otimes V|$$

whose composition with the map Adj : $|U\otimes V| \to \mathbb{P}(U\otimes V)$ coincides with Ψ. Since rank $B =$ rank adj(A), and rank $A(x) = 1$, we get that rank $B(x) = d-1$ for any $x \in C$. So, if $\det B$ is not identically zero, we obtain that $V(\det(B))$ is a hypersurface of degree d vanishing on C, hence $\det(B) = \lambda f$ for some $\lambda \in \mathbb{C}^*$. This shows that $C = V(\det(B))$. To see that $\det(B) \neq 0$, we have to use property (iii) of Lemma 4.1.3. Reversing the proof of this property, we see that, for a general line ℓ in $|E|$, the images of the points $x_i \in \ell \cap C$ in $\mathbb{P}(U) \times \mathbb{P}(V)$ are the points (u^i, v^i) such that the u^i's span $\mathbb{P}(U)$ and the v^i's span $\mathbb{P}(V)$. The images of the x_i's in $\mathbb{P}(U \otimes V)$ under the map Ψ span a subspace L of dimension $d-1$. If we choose coordinates so that the points u^i and v^i are defined by the unit vectors $(0,\dots,1,\dots,0)$, then L corresponds to the space of diagonal matrices. The image of the line ℓ under Ψ is a Veronese curve of degree $d-1$ in L. A general point $\Psi(x), x \in \ell$, on this curve does not belong to any hyperplane in L spanned by $d-1$ points x_i's, thus it can be written as a linear combination of the points $\Psi(t_i)$ with nonzero coefficients. This represents a matrix of rank d. This shows that $\det A(x) \neq 0$ and hence $\det(B(x)) \neq 0$.

To sum up, we have proved the following theorem.

Theorem 4.1.4 *Let $C \subset \mathbb{P}^2$ be a nonsingular plane curve of degree d. Let $\text{Pic}(C)^{g-1}$ be the Picard variety of isomorphism classes of invertible sheaves on C of degree $g-1$. Let $\Theta \subset \text{Pic}^{g-1}(C)$ be the subset parameterizing invertible sheaves $\mathcal{F}$ with $H^0(C,\mathcal{F}) \neq \{0\}$. Let $\mathcal{L}_0 \in \text{Pic}^{g-1}(C) \setminus \Theta$, and $\mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$. Then $U = H^0(C, \mathcal{L}_0(1))$ and $V = H^0(C, \mathcal{M}_0(1))$ have dimension d and there is a unique regular map $\phi : \mathbb{P}^2 \to |U\otimes V|$ such that C is equal to the preimage of the determinantal hypersurface D_d. The composition of the restriction of ϕ to C and the map* Adj $: |U \otimes V| \to \mathbb{P}(U \otimes V)$ *is equal to the composition of the map $(\mathfrak{l}, \mathfrak{r}) : C \to \mathbb{P}(U) \times \mathbb{P}(V)$ and the Segre map. The maps $\mathfrak{l} : C \to \mathbb{P}(U)$ and $\mathfrak{r} : C \to \mathbb{P}(V)$ are given by the complete linear systems $|\mathcal{L}_0(1)|$ and $|\mathcal{M}_0(1)|$ and coincide with the maps $x \mapsto |\text{Ker}(\phi(x))|$ and $x \mapsto |\text{Ker}({}^t\phi(x))|$, respectively. Conversely, given a map $\phi : \mathbb{P}^2 \to |U \otimes V|$ such that $C = \phi^{-1}(\mathsf{D}_d)$, there exists a unique $\mathcal{L}_0 \in \text{Pic}^{g-1}(C)$ such that $U \cong H^0(C, \mathcal{L}_0(1))$, $V \cong H^0(C, \omega_C(1) \otimes \mathcal{L}_0^{-1})$ and the map ϕ is defined by $\mathcal{L}$ as above.*

Remark 4.1.5 Let X be the set of $d \times d$ matrices A with entries in $E^\vee$ such that $f = \det A$. The group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ acts on the set by

$$(\sigma_1, \sigma_2) \cdot A = \sigma_1 \cdot A \cdot \sigma_2^{-1}.$$

It follows from the Theorem that the orbit space X/G is equal to $\mathrm{Pic}^{g-1}(C) \setminus \Theta$.

We map $\mathcal{L}_0 \mapsto \mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$ is an involution on $\mathrm{Pic}^{g-1} \setminus \Theta$. It corresponds to the involution on X defined by taking the transpose of the matrix.

4.1.3 The symmetric case

Let us assume that the determinant representation of a plane irreducible curve C of degree d is given by a pair of equal invertible sheaves $\mathcal{L} = \mathcal{M}$. It follows from Lemmas 4.1.1 and 4.1.3 that

- $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(d-1)$;
- $\deg(\mathcal{L}) = \frac{1}{2}d(d-1)$;
- $H^0(C, \mathcal{L}(-1)) = \{0\}$.

Recall that the canonical sheaf ω_C is isomorphic to $\mathcal{O}_C(d-3)$. Thus

$$\mathcal{L}(-1)^{\otimes 2} \cong \omega_C. \tag{4.7}$$

Definition 4.1.6 *Let X be a curve with a canonical invertible sheaf ω_X (e.g. a nonsingular curve, or a curve on a nonsingular surface). An invertible sheaf θ whose tensor square is isomorphic to ω_X is called a* theta characteristic. *A theta characteristic is called* even *(resp.* odd*) if* $\dim\ H^0(X, \mathcal{N})$ *is even (resp. odd).*

Using this definition, we can express (4.7) by saying that

$$\mathcal{L} \cong \theta(1),$$

where θ is an even theta characteristic (because $H^0(C, \theta) = \{0\}$). Of course, the latter condition is stronger. An even theta characteristic with no nonzero global sections (resp. with nonzero global sections) is called a *non-effective theta characteristic* (resp. *effective theta characteristic*).

Rewriting the previous Subsection under the assumption that $\mathcal{L} = \mathcal{M}$, we obtain that $U = V$. The maps $\mathfrak{l} = \mathfrak{r}$ are given by the linear systems $|\mathcal{L}|$ and define a map $(\mathfrak{l}, \mathfrak{l}) : C \to \mathbb{P}(U) \times \mathbb{P}(U)$. Its composition with the Segre map $\mathbb{P}(U) \times \mathbb{P}(U) \to \mathbb{P}(U \otimes U)$ and the projection to $\mathbb{P}(S^2(U^\vee))$ defines a map

$$\psi : C \to \mathbb{P}(S^2(U^\vee)) = |S^2 U|.$$

In coordinates, the map is given by

$$\psi(x) = \tilde{\mathfrak{l}}(x) \cdot {}^t\tilde{\mathfrak{l}}(x),$$

where $\tilde{\mathfrak{l}}(x)$ is the column of projective coordinates of the point $\mathfrak{l}(x)$. It is clear that the image of the map ψ is contained in the variety of rank 1 quadrics in $|U^\vee|$. It follows from the proof of Theorem 4.1.4 that there exists a linear map $\phi : \mathbb{P}^2 \to |S^2(U^\vee)|$ such that its composition with the rational map defined by taking the adjugate matrix is equal, after restriction to C, to the map ψ. The image of ϕ is a net N of quadrics in $|U|$. The image $\phi(C)$ is the locus of singular quadrics in N. For each point $x \in C$, we denote the corresponding quadric by Q_x. The regular map $\mathfrak{l}$ is defined by assigning to a point $x \in C$ the singular point of the quadric Q_x. The image X of C in $|U|$ is a curve of degree equal to $\deg \mathcal{L} = \frac{1}{2}d(d-1)$.

Proposition 4.1.7 *The restriction map*

$$r : H^0(|U|, \mathcal{O}_{|U|}(2)) \to H^0(X, \mathcal{O}_X(2))$$

is bijective. Under the isomorphism

$$H^0(X, \mathcal{O}_X(2)) \cong H^0(C, \mathcal{L}^{\otimes 2}) \cong H^0(C, \mathcal{O}_C(d-1)),$$

the space of quadrics in $|U|$ is identified with the space of plane curves of degree $d-1$. The net of quadrics N is identified with the linear system of first polars of the curve C.

Proof Reversing the proof of property (iii) from Lemma 4.1.3 shows that the image of C under the map $\psi : C \to \mathbb{P}(U \otimes V)$ spans the space. In our case, this implies that the image of C under the map $C \to |S^2(U^\vee)|$ spans the space of quadrics in the dual space. If the image of C in $\mathbb{P}(U)$ were contained in a quadric Q, then Q would be apolar to all quadrics in the dual space, a contradiction. Thus the restriction map r is injective. Since the spaces have the same dimension, it must be surjective.

The composition of the map $i : \mathbb{P}^2 \to |\mathcal{O}_{|U|}(2)|, x \mapsto Q_x$, and the isomorphism $|\mathcal{O}_{|U|}(2)| \cong |\mathcal{O}_{\mathbb{P}^2}(d-1)|$ is a map $s : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(d-1)|$. A similar map s' is given by the first polars $x \mapsto P_x(C)$. We have to show that the two maps coincide. Recall that $P_x(C) \cap C = \{c \in C : x \in \mathbb{T}_c(C)\}$. In the next Lemma we will show that the quadrics $Q_x, x \in \mathbb{T}_c(C)$, form the line in N of quadrics passing through the singular point of Q_c equal to $\mathfrak{r}(c)$. This shows that the quadric $Q_{\mathfrak{r}(x)}$ cuts out in $\mathfrak{l}(C)$ the divisor $\mathfrak{r}(P_x(C) \cap C)$. Thus the curves $s(x)$ and $s'(x)$ of degree $d-1$ cut out the same divisor on C, hence they coincide. □

Lemma 4.1.8 *Let $W \subset S^d(U^\vee)$ be a linear subspace, and $|W|^{\mathrm{s}}$ be the locus of singular hypersurfaces. Assume $x \in |W|^{\mathrm{s}}$ is a nonsingular point of $|W|^{\mathrm{s}}$. Then the corresponding hypersurface has a unique ordinary double point y and the embedded tangent space $\mathbb{T}_x(|W|^{\mathrm{s}})$ is equal to the hyperplane of hypersurfaces containing y.*

Proof Assume $W = S^d(V^\vee)$. Then $|W|^{\mathrm{s}}$ coincides with the discriminant hypersurface $\mathsf{D}_d(|U|)$ of singular degree d hypersurfaces in $|U|$. If $|W|$ is a proper subspace, then $|W|^{\mathrm{s}} = |W| \cap \mathsf{D}_d(|U|)$. Since $x \in |W|^{\mathrm{s}}$ is a nonsingular point and the intersection is transversal, $\mathbb{T}_x(|W|^{\mathrm{s}}) = \mathbb{T}_x(\mathsf{D}_d(|U|)) \cap |W|$. This proves the assertion.

□

We see that a pair (C, θ), where C is a plane irreducible curve and θ is a non-effective even theta characteristic on C defines a net N of quadrics in $|H^0(C, \theta(1))^\vee|$ such that $C = \mathsf{N}^{\mathrm{s}}$. Conversely, let N be a net of quadrics in $\mathbb{P}^{d-1} = |V|$. It is known that the singular locus of the discriminant hypersurface $\mathsf{D}_2(d-1)$ of quadrics in $\mathbb{P}^{d-1}$ is of codimension 2. Thus a general net N intersects $\mathsf{D}_2(d-1)$ transversally along a nonsingular curve C of degree d. This gives a representation of C as a symmetric determinant and hence defines an invertible sheaf $\mathcal{L}$ and a non-effective even theta characteristic θ. This gives a dominant rational map of varieties of dimension $(d^2 + 3d - 16)/2$

$$G(3, S^2(U^\vee))/\mathrm{PGL}(U) - \to |\mathcal{O}_{\mathbb{P}^2}(d)|/\mathrm{PGL}(3). \tag{4.8}$$

The degree of this map is equal to the number of non-effective even theta characteristics on a general curve of degree d. We will see in the next chapter that the number of even theta characteristics is equal to $2^{g-1}(2^g + 1)$, where $g = (d-1)(d-2)/2$ is the genus of the curve. A curve C of odd degree $d = 2k+3$ has a unique vanishing even theta characteristic equal to $\theta = \mathcal{O}_C(k)$ with $h^0(\theta) = (k+1)(k+2)/2$. A general curve of even degree does not have vanishing even theta characteristics.

4.1.4 Contact curves

Let

$$(\mathfrak{l}, \mathfrak{r}) : C \to \mathbb{P}(U) \times \mathbb{P}(V) \subset \mathbb{P}(U \otimes V)$$

be the embedding of C given by the determinant representation. By restriction, it defines a linear map

$$r : |\mathcal{L}| \times |\mathcal{M}| = |U| \times |V| \to |\mathcal{L} \otimes \mathcal{M}| \cong |\mathcal{O}_C(d-1)|, (D_1, D_2) \mapsto \langle D_1, D_2 \rangle, \tag{4.9}$$

where $\langle D_1, D_2\rangle$ is the unique curve of degree $d-1$ that cuts out the divisor $D_1 + D_2$ on C. Consider the variety

$$F = \{(x, D_1, D_2) \in \mathbb{P}^2 \times |U| \times |V| : x \in \langle D_1, D_2\rangle\}.$$

It is a hypersurface in $\mathbb{P}^2 \times |U| \times |V|$ of type $(d-1, 1, 1)$. Choose a basis $(u_0, \dots, u_{d-1})$ in U and a basis $(v_0, \dots, v_{d-1})$ in V. They will serve as projective coordinates in $\mathbb{P}(U)$ and $\mathbb{P}(V)$. Let $A = (l_{ij})$ define the determinantal representation of C.

Proposition 4.1.9 *The incidence variety F is given by the equation*

$$\det \begin{pmatrix} l_{11} & \dots & l_{1d} & u_0 \\ l_{21} & \dots & l_{2d} & u_1 \\ \vdots & \vdots & \vdots & \vdots \\ l_{d1} & \dots & l_{dd} & u_{d-1} \\ u_0 & \dots & u_{d-1} & 0 \end{pmatrix} = 0. \tag{4.10}$$

Proof The *bordered determinant*(4.10) is equal to $-\sum A_{ij}u_iv_j$, where A_{ij} is the (ij)-entry of the adjugate matrix $\mathrm{adj}(A)$. For any $x \in C$, the rank of the adjugate matrix $\mathrm{adj}(A(x))$ is equal to 1. Thus the above equation defines a bilinear form of rank 1 in the space $U^\vee \otimes V^\vee$ of bilinear forms on $U \times V$. We can write it in the form $(\sum a_iv_i)(\sum b_ju_j)$, where $\mathfrak{l}(x) = [a_0, \dots, a_{d-1}], \mathfrak{r}(x) = [b_0, \dots, b_{d-1}]$. The hyperplane $V(\sum a_iv_i)$ (resp. $V(\sum b_iu_i)$) in $|U|$ (resp.$|V|$) defines a divisor $D_1 \in |\mathcal{L}|$ (resp. $|\mathcal{M}|$) such that $x \in \langle D_1, D_2\rangle$. This checks the assertion. □

Next we use the following determinant identity which is due to O. Hesse [291].

Lemma 4.1.10 *Let $A = (a_{ij})$ be a square matrix of size k. Let*

$$D(A; u, v) := \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} & u_1 \\ a_{21} & a_{22} & \dots & a_{2k} & u_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} & u_k \\ v_1 & v_2 & \dots & v_k & 0 \end{vmatrix}.$$

Then

$$D(A; u, u)D(A; v, v) - D(A; u, v)D(A, v, u) = P \det(A), \tag{4.11}$$

where $P = P(a_{11}, \dots, a_{kk}; u_1, \dots, u_k; v_1, \dots, v_k)$ is a polynomial of degree k in variables a_{ij} and of degree 2 in variables u_i and v_j.

Proof Consider $D(A;u,v)$ as a bilinear function in u,v satisfying $D(A;u,v) = D({}^tA;v,u)$. We have $D(A;u,v) = -\sum A_{ij}u_iv_j$, where A_{ij} is the (ij)-entry of adj(A). This gives

$$D(A;u,u)D(A;v,v) - D(A;u,v)D(A;v,u)$$

$$= (\sum A_{ij}u_iu_j)(\sum A_{ij}v_iv_j) - (\sum A_{ij}u_iv_j)(\sum A_{ji}u_iv_j)$$

$$= \sum u_au_bv_cv_d(A_{ab}A_{dc} - A_{ac}A_{db}).$$

Observe that $A_{ab}A_{dc} - A_{ac}A_{db}$ is equal to a 2×2-minor of adj(A). Thus, if $\det A = 0$, all these minors are equal to zero, and the left-hand side in (4.11) is equal to zero. This shows that $\det A$, considered as a polynomial in variables a_{ij}, divides the left-hand side of (4.11). Comparing the degrees of the expression in the variables a_{ij},u_i,v_j, we get the assertion about the polynomial P. □

Let us see a geometric meaning of the previous Lemma. The curve $T_u = V(D(A;u,u))$ intersects the curve $C = V(\det A)$ at $d(d-1)$ points which can be written as a sum of two divisors $D_v \in |\mathcal{L}|$ and $D'_v \in |\mathcal{M}|$ cut out by the curve $V(D(A;u,v))$ and $V(D(A;v,u))$, where $[v] \in \mathbb{P}(V)$. Similarly, the curve $T_v = V(D(A;v,v))$ intersects the curve $C = V(\det A)$ at $d(d-1)$ points which can be written as a sum of two divisors $D_u \in |\mathcal{L}|$ and $D'_u \in |\mathcal{M}|$ cut out by the curve $V(D(A;u,v))$ and $V(D(A;v,u))$, where $[u] \in \mathbb{P}(U)$.

Now let us specialize, assuming that we are in the case when the matrix A is symmetric. Then $U = V$, and (4.11) becomes

$$D(A;u,v)^2 - D(A;u,u)D(A;v,v) = P\det A. \tag{4.12}$$

This time the curve $T_u = V(D(A;u,u))$ cuts out in C the divisor $2D_v$, where $D_v \in |\mathcal{L}|$, i.e. it touches C at $d(d-1)/2$ points. The curve $V(D(A;u,v)$ cuts out in C the divisor $D_v + D_u$, where $2D_u$ is cut out by the curve $T_v = V(D(A;v,v))$. We obtain that a choice of a symmetric determinantal representation $C = V(\det A)$ defines an algebraic system of *contact curves* $T_u, [u] \in \mathbb{P}(U)$. By definition, a contact curve of an irreducible plane curve C is a curve T such that

$$\mathcal{O}_C(T) \cong \mathcal{L}^{\otimes 2}$$

for some invertible sheaf $\mathcal{L}$ on C with $h^0(\mathcal{L}) > 0$. Up to a projective transformation of U, the number of such families of contact curves is equal to the number of non-effective even theta characteristics on the curve C.

Note that not every contact curve T of C belongs to one of the these $d-1$-dimensional algebraic systems. In fact, T cuts out a divisor D such that $2D \in$

$|\mathcal{O}_C(d-1)|$. Then $\theta = \mathcal{O}_C(D)(-1)$ is a theta characteristic, not necessarily non-effective. Assume that θ is non-effective theta characteristic. Next, we find a symmetric determinantal representation of C corresponding to θ and a curve $V(D(A;u,u))$ which cuts out the same divisor D in C. Since the degrees of the curves T and $V(D(A;u,u))$ are less than the degree of C, they must coincide.

The algebraic systems of contact curves $V(D(A;u,u))$ are not linear systems of curves, they depend quadratically on the parameters $[u] \in \mathbb{P}(U)$. This implies that a general point in the plane is contained in a subfamily of the system isomorphic to a quadric in $\mathbb{P}(U)$, not a hyperplane as it would be in the case of linear systems. The universal family of an algebraic system of contact curves is a hypersurface $\mathcal{T}$ in $|E| \times \mathbb{P}(U)$ of type $(d-1,2)$. It is given by the equation

$$\sum A_{ij}(t_0,t_1,t_2)u_iu_j = 0,$$

where (A_{ij}) is the adjugate matrix of A. Its projection to $\mathbb{P}(E)$ is a *quadric bundle* with discriminant curve given by equation $\det \mathrm{adj}(A) = |A|^{d-1}$. The reduced curve is equal to C. The projection of $\mathcal{T}$ to $\mathbb{P}(U)$ is a fibration in curves of degree $d-1$.

One can also see the contact curves as follows. Let $[\xi] = [\xi_0,\dots,\xi_{d-1}]$ be a point in $|U^\vee|$ and let $H_\xi = V(\sum \xi_i t_i)$ be the corresponding hyperplane in $|U|$. The restriction of the net of quadrics defined by ϑ to H_ξ defines a net of quadrics $N(\xi)$ in H_ξ parameterized by the plane E. The discriminant curve of this net of quadrics is a contact curve of C. In fact, a quadric $Q_x|H_\xi$ in $N(\xi)$ is singular if and only if the hyperplane is tangent to Q_x. Or, by duality, the point $[\xi]$ belongs to the dual quadric $Q_x^\vee = V(D(A(x);\xi,\xi))$. This is the equation of the contact curve corresponding to the parameter ξ.

Consider the bordered determinant identity (4.12). It is clear that P is symmetric in u,v and vanishes for $u=v$. This implies that P can be expressed as a polynomial of degree 2 in Plücker coordinates of lines in $\mathbb{P}^{d-1} = |\theta(1)|$. Thus $P=0$ represents a family of *quadratic line complexes* of lines in $\mathbb{P}^{d-1}$ parameterized by points in the plane.

Proposition 4.1.11 *Let $\phi : |E| \to |S^2(U)^\vee|$ be the net of quadrics in $|U|$ defined by the theta characteristic ϑ. For any $x \in |E|$ the quadratic line complex $V(U(u,v;x))$ consists of lines in $|U^\vee|$ such that the dual subspace of codimension 2 in $|U|$ is tangent to the quadric $Q_x = \phi(x)$.*

Proof Note that the dual assertion is that the line is tangent to the dual quadric $Q_x^\vee$. The equation of the dual quadric is given by $D(A(x);u,u) = 0$. A line spanned by the points $[\xi] = [\xi_0,\dots,\xi_{d-1}]$ and $[\eta] = [\eta_0,\dots,\eta_{d-1}]$ is tangent

to this quadric if and only if the restriction of this quadric to the line is given by a singular binary form in coordinates on the line. The discriminant of this quadratic form is $D(A(x);\xi,\xi)D(A(x);\eta,\eta) - D(A(x);\xi,\eta)^2$. We assume that the point x is a general point in the plane, in particular, it does not belong to C. Thus this expression vanishes if and only if $P(\xi,\eta) = 0$. □

4.1.5 First examples

Take $d = 2$. Then there is only one isomorphism class of $\mathcal{L}$ with $\deg \mathcal{L} = 1$. Since $\deg \mathcal{L}(-1) = -1$, $h^0(C, \mathcal{L}(-1)) = 0$, so $\mathcal{L} \cong \mathcal{M}$, and C admits a unique equivalence class of determinantal representations which can be chosen to be symmetric. For example, if $C = V(t_0t_1 - t_2^2)$, we can choose

$$A = \begin{pmatrix} t_0 & t_2 \\ t_2 & t_1 \end{pmatrix}.$$

We have $\mathbb{P}(U) \cong \mathbb{P}^1$, and $\mathfrak{r} = \mathfrak{l}$ maps C isomorphically to $\mathbb{P}^1$. There is only one family of contact curves of degree 1. It is the system of tangents to C. It is parameterized by the conic in the dual plane, the dual conic of C. Thus, there is a natural identification of the dual plane with $\mathbb{P}(S^2U)$.

Take $d = 3$. Then $\mathrm{Pic}^{g-1}(C) = \mathrm{Pic}^0(C)$ and $\Theta = \mathrm{Pic}^0(C) \setminus \{\mathcal{O}_C\}$. Thus the equivalence classes of determinantal representations are parameterized by the curve itself minus one point. There are three systems of contact conics. Let T be a contact conic cutting out a divisor $2(p_1 + p_2 + p_3)$. If we fix a group law on C defined by an inflection point $\mathfrak{o}$, then the points p_i add up to a nonzero 2-torsion point ϵ. We have $p_1 + p_2 + p_3 \sim 2\mathfrak{o} + \epsilon$. This implies that $\mathcal{L} \cong \mathcal{O}_C(2\mathfrak{o} + \epsilon)$. The contact conic that cuts out the divisor $2(2\mathfrak{o} + \epsilon)$ is equal to the union of the inflection tangent line at $\mathfrak{o}$ and the tangent line at ϵ (which passes through $\mathfrak{o}$). We know that each nonsingular curve can be written as the Hessian curve in three essentially different ways. This gives the three ways to write C as a symmetric determinant and also write explicitly the three algebraic systems of contact conics.

Let $(\mathcal{L}, \mathcal{M})$ define a determinantal representation of C. Let $\mathfrak{l} : C \hookrightarrow \mathbb{P}(U)$ be the reembedding of C in $\mathbb{P}(U)$ given by the linear system $|\mathcal{L}|$. For any $D_0 \in |\mathcal{M}|$, there exists $D \in |\mathcal{L}|$ such that $D_0 + D$ is cut out by a conic. Thus we can identify the linear system $|\mathcal{L}|$ with the linear system of conics through D_0. This linear system defines a birational map $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}(U)$ with indeterminacy points in D_0. The map $\mathfrak{l} : C \to \mathbb{P}(U)$ coincides with the restriction of σ to C.

Consider the map

$$(\mathfrak{l}, \mathfrak{r}) : C \to \mathbb{P}(U) \times \mathbb{P}(V) \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

Proposition 4.1.12 *The image of* $(\mathfrak{l}, \mathfrak{r})$ *is a complete intersection of three hyperplane sections in the Segre embedding of the product.*

Proof Consider the restriction map (4.9)

$$U \times V = H^0(\mathbb{P}(U) \times \mathbb{P}(V), \mathcal{O}_{\mathbb{P}(U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)) \to H^0(X, \mathcal{O}_X(1)),$$

where X is the image of C in $\mathbb{P}(U \otimes V)$ under the composition of the map $(\mathfrak{l}, \mathfrak{r})$ and the Segre map. Here we identify the spaces $H^0(C, \mathcal{L} \otimes \mathcal{M})$ and $H^0(X, \mathcal{O}_X(1))$. Since the map (4.9) is surjective, and its target space is of dimension 6, the kernel is of dimension 3. So the image X of C in $\mathbb{P}(U) \times \mathbb{P}(V)$ is contained in the complete intersection of three hypersurfaces of type $(1, 1)$. By the adjunction formula, the intersection is a curve X' of arithmetic genus 1. Choose coordinates (u_0, u_1, u_2) in $U^\vee$ and coordinates (v_0, v_1, v_2) in V to be able to write the three hypersurfaces by equations

$$\sum_{0 \le i,j \le 2} a_{ij}^{(k)} u_i v_j = 0, \quad k = 1, 2, 3.$$

The projection of X to the first factor is equal to the locus of points $[u_0, u_1, u_2]$ such that the system

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} u_i v_j = \sum_{j=0}^{2} (\sum_{i=0}^{2} a_{ij}^{(k)} u_i) v_j = 0, \quad k = 1, 2, 3,$$

has a nontrivial solution (v_0, v_1, v_2). The condition for this is

$$\begin{pmatrix} \sum_{i=0}^{2} a_{i0}^{(1)} u_i & \sum_{i=0}^{2} a_{i1}^{(1)} u_i & \sum_{i=0}^{2} a_{i2}^{(1)} u_i \\ \sum_{i=0}^{2} a_{i0}^{(2)} u_i & \sum_{i=0}^{2} a_{i1}^{(2)} u_i & \sum_{i=0}^{2} a_{i2}^{(2)} u_i \\ \sum_{i=0}^{2} a_{i0}^{(3)} u_i & \sum_{i=0}^{2} a_{i1}^{(3)} u_i & \sum_{i=0}^{2} a_{i2}^{(3)} u_i \end{pmatrix} = 0. \tag{4.13}$$

This checks that the projection of X' to the factor $\mathbb{P}(U)$ is a cubic curve, the same as the projection of X. Repeating the argument, replacing the first factor with the second one, we obtain that the projections of X' and X to each factor coincide. This implies that $X = X'$. □

Recall that a determinantal representation of C is defined by a linear map

$\phi : E \to U \otimes V$. Let us show that its image is the kernel of the restriction map. We identify its target space $H^0(X, \mathcal{O}_X(1))$ with $H^0(C, \mathcal{O}_C(d-1)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))$. In coordinates, the map ϕ is defined by $[x] \mapsto \sum a_{ij}(x)u_j \otimes v_i$, where $C = V(\det(a_{ij}))$. The restriction map is defined by the map $u_i \otimes v_j \mapsto \bar{A}_{ij}$, where A_{ij} is a (ij)-cofactor of the adjugate matrix of (a_{ij}) and the bar means the restriction to C. The composition is given by

$$x \mapsto \sum a_{ji}A_{ij} = \det(a_{ij}) \quad \text{restricted to } C.$$

Since the restriction of the determinant to C is zero, we see that E can be identified with the linear system of hyperplane sections of $\mathbb{P}(U) \times \mathbb{P}(V)$ defining the curve $(\mathfrak{l}, \mathfrak{r})(C)$.

Note that the determinant (4.13) gives a determinantal representation of the plane cubic C reembedded in the plane by the linear system $|\mathcal{L}|$. It is given by a linear map $U^\vee \to E^\vee \otimes V$ obtained from the tensor $\tau \in E^\vee \otimes U \otimes V$ which defines the linear map $\phi : E \to U \otimes V$.

4.1.6 The moduli space

Let us consider the moduli space of pairs (C, A), where C is a nonsingular plane curve of degree d and A is a matrix of linear forms such that $C = V(\det A)$. To make everything coordinate-free and match our previous notations, we let $\mathbb{P}^2 = |E|$ and consider A as a linear map $\phi : E \to U \otimes V = \mathrm{Hom}(U^\vee, V)$. Our equivalence relation on such pairs is defined by the natural action of the group $\mathrm{GL}(U) \times \mathrm{GL}(V)$ on $U \otimes V$. The composition of ϕ with the determinant map $U \otimes V \to \mathrm{Hom}(\bigwedge^d U^\vee, \bigwedge^d V) \cong \mathbb{C}$ is an element of $S^d(E^\vee)$. It corresponds to the determinant of the matrix A. Under the action of $(g, h) \in \mathrm{GL}(U) \times \mathrm{GL}(V)$, it is multiplied by $\det g \det h$, and hence represents a projective invariant of the action. Consider ϕ as an element of $E^\vee \otimes U \otimes V$, and let

$$\det : E^\vee \otimes U \otimes V / \mathrm{GL}(U) \times \mathrm{GL}(V) \to |S^d E^\vee|$$

be the map of the set of orbits defined by taking the determinant. We consider this map as a map of sets since there is a serious issue here as to whether the orbit space exists as an algebraic variety. However, we are interested only in the restriction of the determinant map on the open subset $(E^\vee \otimes U \otimes V)^o$ defining nonsingular determinantal curves. One can show that the quotient of this subset is an algebraic variety.

We know that the fiber of the map $\det$ over a nonsingular curve C is bijective to $\mathrm{Pic}^{g-1}(C) \setminus \Theta$. Let be the universal family of nonsingular plane curves of

degree d (and genus g). It defines a family

$$\tilde{\pi} : \mathcal{P}ic_d^{g-1} \to |S^d(E^\vee)|$$

whose fiber over a curve C is isomorphic to $\mathrm{Pic}^{g-1}(C)$. It is the relative Picard scheme of π. It comes with a divisor $\mathcal{T}$ such that its intersection with $\tilde{\pi}^{-1}(C)$ is equal to the divisor Θ. It follows from the previous sections that there is an isomorphism of algebraic varieties

$$(E^\vee \otimes U \otimes V)^o/\mathrm{GL}(U) \times \mathrm{GL}(V) \cong \mathcal{P}ic_d^{g-1} \setminus \mathcal{T}.$$

This shows that the relative Picard scheme $\mathcal{P}ic_d^{g-1}$ is a unirational variety. An easy computation shows that its dimension is equal to $d^2 + 1$.

It is a very difficult question to decide whether the variety $\mathcal{P}ic_d^{g-1}$ is rational. It is obviously rational if $d = 2$. It is known that it is rational for $d = 3$ and $d = 4$ [223]. Let us sketch a beautiful proof of the rationality in the case $d = 3$ due to M. Van den Bergh [584].

Theorem 4.1.13 *Assume $d = 3$. Then $\mathcal{P}ic_3^0$ is a rational variety.*

Proof A point of $\mathcal{P}ic^0$ is a pair $(C, \mathcal{L})$, where C is a nonsingular plane cubic and $\mathcal{L}$ is the isomorphism class of an invertible sheaf of degree 0. Let D be a divisor of degree 0 such that $\mathcal{O}_C(D) \cong \mathcal{L}$. Choose a line ℓ and let $H = \ell \cap C = p_1 + p_2 + p_3$. Let $p_i + D \sim q_i, i = 1, 2, 3$, where q_i is a point. Since $p_i - q_i \sim p_j - q_j$, we have $p_i + q_j \sim p_j + q_i$. This shows that the lines $\langle p_i, q_j\rangle$ and $\langle p_j, q_i\rangle$ intersect at the same point r_{ij} on C. Since, $p_i + q_j + r_{ij} \sim H$, it is immediately checked that

$$p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + r_{12} + r_{23} + r_{13} \sim 3H.$$

This easily implies that there is a cubic curve which intersects C at the nine points. Together with C they generate a pencil of cubics with the nine points as the set of its base points. Let $X = \ell^3 \times (\mathbb{P}^2)^3/\mathfrak{S}_3$, where $\mathfrak{S}_3$ acts by

$$\sigma : \big((p_1, p_2, p_3), (q_1, q_2, q_3)\big) = \big((p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}), (q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})\big).$$

The variety X is easily seen to be rational. The projection to $\ell^3/\mathfrak{S}_3 \cong \mathbb{P}^3$ defines a birational isomorphism between the product of $\mathbb{P}^3$ and $(\mathbb{P}^2)^3$. For each $x = (\mathcal{P}, \mathcal{Q}) \in X$, let $c(x)$ be the pencil of cubics through the points $p_1, p_2, p_3, q_1, q_2, q_3$ and the points $r_{ij} = \langle p_i, q_j\rangle$, where $(ij) = (12), (23), (13)$. Consider the set U' of pairs $(x, C), C \in c(x)$. The projection $(u, C) \mapsto u$ has fibres isomorphic to $\mathbb{P}^1$. Thus the field of rational functions on X' is isomorphic to the field of rational functions on a conic over the field $\mathbb{C}(X)$. But this conic has a rational point. It is defined by fixing a point in $\mathbb{P}^2$ and choosing

a member of the pencil passing through this point. Thus the conic is isomorphic to $\mathbb{P}^1$ and $\mathbb{C}(X')$ is a purely trancendental extension of $\mathbb{C}(X)$. Now we define a birational map from $\mathcal{P}ic^0_3$ to X'. Each $(C, \mathcal{L})$ defines a point of U' by ordering the set $\ell \cap C$, then defining q_1, q_2, q_3 as above. The member of the corresponding pencil through p_i's, q_i's and r_{ij}'s is the curve C. Conversely, a point $(x, C) \in X'$ defines a point $(C, \mathcal{L})$ in $\mathcal{P}ic^0_3$. We define $\mathcal{L}$ to be the invertible sheaf corresponding to the divisor $q_1 + q_2 + q_3$. It is easy that these map are inverse to each other. □

Remark 4.1.14 If we choose a basis in each space E, U, V, then a map $\phi : E \to \mathrm{Hom}(U, V)$ is determined by three matrices $A_i = \phi(e_i)$. Our moduli space becomes the space of triples (A_1, A_2, A_3) of $d \times d$ matrices up to the action of the group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ simultaneously by left and right multiplication

$$(\sigma_1, \sigma_2) \cdot (A_1, A_2, A_3) = (\sigma_1 A_1 \sigma_2^{-1}, \sigma_1 A_2 \sigma_2^{-1}, \sigma_1 A_3 \sigma_2^{-1}).$$

Consider an open subset of maps ϕ such that A_1 is an invertible matrix. Taking $(\sigma_1, \sigma_2) = (1, A_1^{-1})$, we may assume that $A_1 = I_d$ is the identity matrix. The stabilizer subgroup of (I_d, A_2, A_3) is the subgroup of (σ_1, σ_2) such that $\sigma_1 \sigma_2 = 1$. Thus our orbit space is equal to the orbit space of pairs of matrices (A, B) up to simultaneous conjugation. The rationality of this space is a notoriously very difficult problem.

4.2 Determinantal equations for hypersurfaces

4.2.1 Determinantal varieties

Let $\mathrm{Mat}_{m,k} = \mathbb{C}^{m \times k}$ be the space of complex $m \times k$ matrices with natural basis e_{ij} and coordinates t_{ij}. The coordinate ring $\mathbb{C}[\mathbb{C}^{m \times n}]$ is isomorphic to the polynomial ring $\mathbb{C}[(t_{ij})]$ in mk variables. For any vector spaces U, V of dimensions k, m, respectively, a choice of a basis (u_i) in U and a basis (v_j) in V identifies $U \otimes V$ with $\mathrm{Mat}_{m,k}$ by sending $u_i \otimes v_j$ to e_{ij}. An element $\sigma \in U \otimes V$ can be viewed as a linear map $U^\vee \to V$, or as a bilinear form on $U^\vee \otimes V^\vee$. Under the natural isomorphism $U \otimes V \to V \otimes U$, the map σ changes to the transpose map ${}^t\sigma$.

We denote by ${}^lN(\sigma)$ (resp. ${}^rN(\sigma)$) the left (resp. the right) kernel of σ considered as a bilinear map. These are subspaces of $U^\vee$ and $V^\vee$, respectively. Equivalently, ${}^lN(\sigma) = \mathrm{Ker}(\sigma)$ (resp. ${}^rN(\sigma) = \mathrm{Ker}({}^t\sigma) = \sigma(U)^\perp$) if σ is considered as a linear map. For any r in the range $0 < r \leq \min\{m, k\}$, we set

$$(U \otimes V)_r := \{\sigma \in U \otimes V : \mathrm{rank}\ \sigma \leq r\}$$

and denote by $|U \otimes V|_r$ its image in the projective space $|U \otimes V|$. The varieties $|U \otimes V|_r$ are closed subvarieties of the projective space $|U \otimes V|$, called the *determinant varieties*. Under isomorphism $|U \otimes V| \cong |\mathbb{C}^{m\times k}| = \mathbb{P}^{mk-1}$, the variety $|U \otimes V|_r$ becomes isomorphic to the closed subvariety of $\mathbb{P}^{mk-1}$ defined by $(r+1) \times (r+1)$ minors of a $m \times k$ matrix with entries t_{ij}.

Let $G(r, V)$ be the Grassmann variety of r-dimensional linear subspaces of V and let

$$\widetilde{|U \otimes V|}_r = \{(\phi, L) \in |U \otimes V| \times G(r, V) : \phi(U^\vee) \subset L\}.$$

The projection to $G(r, V)$ exhibits $\widetilde{|U \otimes V|}_r$ as a projective vector bundle of relative dimension kr and implies that $\widetilde{|U \otimes V|}_r$ is a smooth variety of dimension $mk - (m-k)(k-r)$. The projection to $|U \otimes V|_r$ is a proper map which is an isomorphism over $|U \otimes V|_r \setminus |U \otimes V|_{r-1}$. It defines a resolution of singularities

$$\sigma : \widetilde{|U \otimes V|}_r \to |U \otimes V|_r.$$

It identifies the tangent space $\mathbb{T}_{[\sigma]}(|U \otimes V|_r)$ at a point $[\sigma] \in |U \otimes V|_r$ with the projective space of maps $\tau : U^\vee \to V$ such that $\tau(\mathrm{Ker}(\sigma)) \subset \sigma(U^\vee)$. If we view σ as a bilinear form on $U^\vee \otimes V^\vee$, then the tangent space consists of bilinear forms $\tau \in U \otimes V$ such that $\tau(u^* \otimes v^*) = 0$ for all $u^* \in {}^lN(\sigma), v^* \in {}^rN(\sigma)$.

Here are some known properties of the determinantal varieties (see [10], Chapter II, §5).

Theorem 4.2.1 *Let* $\mathrm{Mat}_{m,k}(r) \subset \mathbb{C}^{m\times k}, m \le n$, *be the subvariety of matrices of rank* $\le r < m$. *Then*

- $\mathrm{Mat}_{m,k}(r)$ *is an irreducible Cohen-Macaulay variety of codimension* $(m-r)(k-r)$*;*
- $\mathrm{Sing}(\mathrm{Mat}_{m,k}(r)) = \mathrm{Mat}_{m,k}(r-1)$*;*
- *the multiplicity of* $\mathrm{Mat}_{m,k}(r)$ *at a point* A *of rank* $s \le r$ *is equal to*

$$\mathrm{mult}_A \mathrm{Mat}_{m,k}(r) = \prod_{j=0}^{m-r-1} \frac{(n-s+j)!j!}{(r-s+j)!(n-r+j)!},$$

in particular,

- *the degree of* $\mathrm{Mat}_{m,k}(r)$ *is equal to*

$$\deg \mathrm{Mat}_{m,k}(r) = \mathrm{mult}_0 \mathrm{Mat}_{m,k}(r) = \prod_{j=0}^{m-r-1} \frac{(n+j)!j!}{(r+j)!(n-r+j)!}.$$

Let $\phi : E \to U \otimes V$ be an injective linear map and $|\phi| : |E| \hookrightarrow |U \otimes V|$ be the corresponding closed embedding morphism. Let

$$D_r(\phi) = |\phi|^{-1}(|U \otimes V|_r) \cong |\phi|(|E|) \cap |U \otimes V|_r.$$

We say that $\phi : E \to U \otimes V$ is *proper*, if

$$\mathrm{codim} D_r(\phi) = \mathrm{codim} |U \otimes V|_r = (m-r)(k-r).$$

In particular, this implies that $D_r(\phi)$ is a Cohen-Macaulay variety of dimension $n-(m-r)(k-r)$ in $|E|$. We also say that ϕ is *transversal* if

$$\mathrm{Sing}(|U \otimes V|_r) = |U \otimes V|_{r-1}, \quad r < \min\{m,k\}.$$

Using the description of the tangent space of $|U \otimes V|_r$ at its nonsingular point, we obtain the following.

Proposition 4.2.2 *Assume ϕ is proper. A point $[x] \in D_r(\phi) \setminus D_{r-1}(\phi)$ is nonsingular if and only if*

$$\dim\{y \in E : \phi(y)(\mathrm{Ker}(\phi(x)) \otimes \mathrm{Ker}({}^t\phi(x))) = 0\} = n+1-(m-r)(k-r).$$

For example, suppose $k = m = d$ and $r = d-1$. Let $[x] \in D_{d-1}(\phi) \setminus D_{d-2}(\phi)$. Then $\mathrm{Ker}(\phi(x))$ and $\mathrm{Ker}({}^t\phi(x))$ are 1-dimensional subspaces. Let u^*, v^* be their respective bases. Then $[x]$ is a nonsingular point on $D_{d-1}(\phi)$ if and only if the tensor $u^* \otimes v^*$ is not contained in the kernel of the map ${}^t\phi : U^\vee \otimes V^\vee \to E^\vee$.

For any vector space F we denote by $\underline{F}$ the trivial vector bundles $F \otimes \mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$ with a fixed isomorphism from F to its space of global sections. Since

$$\mathrm{Hom}(\underline{U}^\vee(-1), \underline{V}) \cong H^0(\mathbb{P}^n, \underline{U}(1), \underline{V}) \cong E^\vee \otimes U \otimes V,$$

a linear map $\phi : E \to U \otimes V$ defines a homomorphism of vector bundles $\underline{U}^\vee(-1) \to \underline{V}$. For any point $[x] \in \mathbb{P}^n$, the fiber $(\underline{U}^\vee(-1))(x)$ is canonically identified with $U^\vee \otimes \mathbb{C}x$ and the map of fibres $\underline{U}^\vee(-1)(x) \to \underline{V}(x)$ is the map $u \otimes x \mapsto \phi(x)(u)$.

Assume that $k \geq m$ and $\phi(x)$ is of maximal rank for a general point $[x] \in \mathbb{P}^n$. Since a locally free sheaf has no nontrivial torsion subsheaves, the homomorphism $\underline{U}^\vee(-1) \to \underline{V}(1)$ is injective, and we obtain an exact sequence

$$0 \to \underline{U}^\vee(-1) \xrightarrow{\phi} \underline{V} \to \mathcal{F} \to 0. \tag{4.14}$$

Recall that the *fibre* $\mathcal{F}(x)$ of a sheaf $\mathcal{F}$ over a point x is the vector space $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ over the residue field $\mathcal{O}_x/\mathfrak{m}_x$ of x. A sheaf over a reduced scheme

is locally free of rank r if and only if all its fibres are vector spaces of dimension r. Passing to the fibres of the sheaves in the exact sequence, we obtain

$$\dim \mathcal{F}(x) = m - \text{rank } \phi(x). \tag{4.15}$$

In particular, if $k > m$, then $\mathcal{F}$ is locally free of rank $m - k$ outside $D_{m-1}(\phi)$ of rank $m - k$. It has singularities on $D_{m-2}(\phi)$.

Assume $m = k$. Let X denote the set-theoretical support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ and let X_s denote the scheme-theoretical support of $\mathcal{F}$ defined by the determinant of ϕ. It is known that the annihilator ideal $\text{Ann}(\mathcal{F})$ of the sheaf $\mathcal{F}$ is equal to $(\text{Fitt}_1(\mathcal{F}) : \text{Fitt}_0(\mathcal{F}))$, where $\text{Fitt}_i(\mathcal{F})$ denote the *Fitting ideals* of $\mathcal{F}$ generated by $k - i \times k - i$ minors of the matrix defining ϕ (see [208], p. 511). We will often consider $\mathcal{F}$ as a coherent sheaf on X_s. Note that $X = (X_s)_{\text{red}}$, and, in general, $X \neq X_s$.

Let $r = \max\{s : D_s(\phi) \neq D_k(\sigma)\}$. Assume $X = X_s$. It follows from (4.15) that $\mathcal{F}_{\text{red}}$ is locally free on X outside D_s. For example, when the matrix of ϕ is skew-symmetric, we expect that $\mathcal{F}_{\text{red}}$ is of rank 2 outside $D_{k-2}(\phi)$.

Remark 4.2.3 The homomorphism $\phi : \underline{U}^\vee \to \underline{V}$ of vector bundles is a special case of a homomorphism of vector bundles on a variety X. The rank degeneracy loci of such homomorphisms are studied in detail in Fulton's book [231].

Remark 4.2.4 In view of classical geometry, determinantal varieties represent a special class of varieties. Let us elaborate. Let $A = (a_{ij})$ be a $m \times k$ matrix, where a_{ij} are linear forms in variables $t_0, \ldots, t_n$. Consider each entry as a hyperplane in $\mathbb{P}^n$. Assume that the linear forms $a_{1j}, \ldots, a_{mj}$ in each j-th column are linearly independent. Let B_j be their common zeros. These are projective subspaces in $\mathbb{P}^n$ of codimension m. A linear form $\sum_{i=1}^m u_i a_{ij}$ defines a hyperplane $H_j(u)$ containing B_j. Varying $u_1, \ldots, u_m$, we obtain a $(n-m)$-dimensional subspace of hyperplanes containing B_j. In classical language this is the *star* $]B_j[$ of hyperplanes (a pencil if $m = 2$, a net if $m = 3$, a web if $m = 4$ of hyperplanes). It can be considered as a projective subspace of dimension $m - 1$ in the dual space $(\mathbb{P}^n)^\vee$. Now, the matrix defines k stars $[B_j[$ with uniform coordinates $(u_1, \ldots, u_m)$. In classical language, k collinear $m - 1$-dimensional subspaces of the dual space.

Consider the subvariety of $\mathbb{P}^n$

$$X = \{x \in \mathbb{P}^n : x \in H_1(u) \cap \ldots \cap H_k(u), \text{ for some } u \in \mathbb{C}^m\}.$$

It is clear that

$$X = \{x \in \mathbb{P}^n : \text{rank} A(x) < m\}.$$

If $k < m$, we have $X = \mathbb{P}^n$, so we assume that $m \leq k$. If not, we replace A

with its transpose matrix. In this way we obtain a proper subvariety X of $\mathbb{P}^n$, a hypersurface, if $m = k$, with linear determinantal representation $X = \det A$. For any $x \in X$ let

$$^lN(x) := \{u \in \mathbb{C}^m : x \in H_1(u) \cap \ldots \cap H_k(u)\}.$$

Then the subvariety X_r of X

$$X_r = \{x \in X : \dim {}^lN(x) \geq m - r\}, \quad r \leq m - 1,$$

is the determinantal subvariety of $\mathbb{P}^n$ given by the condition rank$A(x) \leq r$. We have a regular map

$$\mathfrak{l} : X \setminus X_{m-2} \to \mathbb{P}^{m-1}, \quad x \mapsto |^lN(x)|.$$

The image is the subvariety of $\mathbb{P}^{m-1}$ given by

$$\text{rank } L(u_1, \ldots, u_m) \leq n,$$

where L is the $k \times (n+1)$ matrix with js-th entry equal to $\sum_{i=1}^m a_{ij}^{(s)} u_i$. If $k \leq n$, the map $\mathfrak{l}$ is dominant, and if $k = n$, it is birational.

4.2.2 Arithmetically Cohen-Macaulay sheaves

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and

$$\Gamma_*(\mathcal{F}) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{F}(k)).$$

It is a graded module over the graded ring

$$S = \Gamma_*(\mathcal{O}_{\mathbb{P}^n}) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \mathbb{C}[t_0, \ldots, t_n].$$

We say that $\mathcal{F}$ is an *arithmetically Cohen-Macaulay sheaf* (*aCM sheaf* for brevity) if $M = \Gamma_*(\mathcal{F})$ is a graded Cohen-Macaulay module over S. Recall that this means that every localization of M is a *Cohen-Macaulay module*, i.e. its depth is equal to its dimension. Let us identify M with the coherent sheaf on Spec A. The associated sheaf $\tilde{M}$ on Proj S is isomorphic to $\mathcal{F}$. Let $U = \text{Spec } S \setminus \mathfrak{m}_0$, where $\mathfrak{m}_0 = (t_0, \ldots, t_n)$ is the irrelevant maximal ideal of the graded ring S. Since the projection $U \to \text{Proj } S = \mathbb{P}^n$ is a smooth morphism, the localizations of M at every maximal ideal different from $\mathfrak{m}$ are Cohen-Macaulay modules if and only if

- $\mathcal{F}_x$ is a Cohen-Macaulay module over $\mathcal{O}_{\mathbb{P}^n,x}$ for all $x \in \mathbb{P}^n$.

The condition that the localization of $M = \Gamma_*(\mathcal{F})$ at $\mathfrak{m}_0$ is Cohen-Macaulay is satisfied if and only if the local cohomology $H^i_{\mathfrak{m}_0}(M)$ vanish for all i with $0 \leq i < \dim M$. We have $H^i(U, M) = \oplus_{k\in\mathbb{Z}} H^i(\mathbb{P}^n, \tilde{M}(k))$. The exact sequence of local cohomology gives an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0(U, M) \to H^1_{\mathfrak{m}}(M) \to 0,$$

and isomorphisms

$$H^{i+1}_{\mathfrak{m}}(M) \cong H^i(U, M), \quad i > 0.$$

In the case $M = \Gamma_*(\mathcal{F})$, the map $M \to H^0(U, M) = \Gamma_*(\tilde{M})$ is an isomorphism, hence $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$. Since the canonical homomorphism $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is bijective, the conditions $H^i_{\mathfrak{m}}(M) = 0, i > 1$, become equivalent to the conditions

- $H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0, \quad 1 \leq i < \dim \mathrm{Supp}(\mathcal{F}),\ k \in \mathbb{Z}.$

Finally, let us recall that for any finitely generated module M over a regular Noetherian local ring R of dimension n, we have

$$\mathrm{depth}\ M = n - \mathrm{pd}\ M,$$

where pd denotes the projective dimension of M, the minimal length of a projective resolution of M.

We apply this to the sheaf $\mathcal{F}$ from the exact sequence (4.14), where we assume that $k = m$.

Exact sequence (4.14) gives us that $\mathrm{pd}\ \mathcal{F}_x = 1$ for all $x \in X = \mathrm{Supp}(\mathcal{F})$. This implies that $\mathrm{depth}\ \mathcal{F}_x = n-1$ for all $x \in X$. In particular, X is hypersurface in $\mathbb{P}^n$ and the stalks of $\mathcal{F}_x$ are Cohen-Macaulay modules over $\mathcal{O}_{\mathbb{P}^n,x}$. The scheme-theoretical support X_s of $\mathcal{F}$ is a hypersurface of degree $d = k = m$.

A Cohen-Macaulay sheaf of rank 1 is defined by a Weil divisor on X, not necessarily a Cartier divisor. Recall the definitions. Let X be a Noetherian integral scheme of dimension ≥ 1 and $X^{(1)}$ be its set of points of codimension 1 (i.e. points $x \in X$ with $\dim \mathcal{O}_{X,x} = 1$). We assume that X is regular in codimension 1, i.e. all local rings of points from $X^{(1)}$ are regular. In this case we can define *Weil divisors* on X as elements of the free abelian group $\mathrm{WDiv}(X) = \mathbb{Z}^{X^{(1)}}$ and also define linear equivalence of Weil divisors and the group $\mathbf{Cl}(X)$ of linear equivalence classes of Weil divisors (see [282], Chapter 2, §6).

We identify a point $x \in X^{(1)}$ with its closure E in X. We call it an *irreducible divisor*. Any irreducible reduced closed subscheme E of codimension 1 is an irreducible divisor, the closure of its generic point.

For any Weil divisor D let $\mathcal{O}_X(D)$ be the sheaf whose section on an open

affine subset U consists of functions from the quotient field $Q(\mathcal{O}(U))$ such that $\mathrm{div}(\Phi) + D \geq 0$.

It follows from the definition that $\mathcal{O}_X(D)$ is torsion-free and, for any open subset $j : U \hookrightarrow X$ which contains all points of codimension 1, the canonical homomorphism of sheaves

$$\mathcal{O}_X(D) \to j_* j^* \mathcal{O}_X(D) \tag{4.16}$$

is an isomorphism. These two conditions characterize *reflexive sheaves* on any normal integral scheme X. It follows from the theory of local cohomology that the latter condition is equivalent to the condition that for any point $x \in X$ with $\dim \mathcal{O}_{X,x} \geq 2$ the depth of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is greater than or equal to 2. By equivalent definition, a reflexive sheaf $\mathcal{F}$ is a coherent sheaf such that the canonical homomorphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is an isomorphism. The sheaves $\mathcal{O}_X(D)$ are reflexive sheaves of rank 1. Conversely, a reflexive sheaf $\mathcal{F}$ of rank 1 on a normal integral scheme is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D. In fact, we restrict $\mathcal{F}$ to some open subset $j : U \hookrightarrow X$ with complement of codimension ≥ 2 such that $j^*\mathcal{F}$ is locally free of rank 1. Thus it corresponds to a Cartier divisor on U. Taking the closure of the corresponding Weil divisor in X, we get a Weil divisor D on X and it is clear that $\mathcal{F} = j_* j^* \mathcal{F} \cong \mathcal{O}_X(D)$. In particular, we see that any reflexive sheaf of rank 1 on a regular scheme is invertible. It is not true for reflexive sheaves of rank > 1. They are locally free outside of a closed subset of codimension ≥ 3 (see [283]).

Reflexive sheaves of rank 1 form a group with respect to the operation

$$\mathcal{L} \cdot \mathcal{G} = ((\mathcal{L} \otimes \mathcal{G})^\vee)^\vee, \ \mathcal{L}^{-1} = \mathcal{L}^\vee.$$

For any reflexive sheaf $\mathcal{L}$ and an integer n we set

$$\mathcal{L}^{[n]} = ((\mathcal{L}^{\otimes n})^\vee)^\vee.$$

One checks that

$$\mathcal{O}_X(D + D') = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$$

and the map $D \mapsto \mathcal{O}_X(D)$ defines an isomorphism from the group $\mathbf{Cl}(X)$ to the group of isomorphism classes of reflexive sheaves of rank 1.

Next, we look at the exact sequence of cohomology for (4.14). Using that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) = 0$ for $i \neq 0, n$ and all $j \in \mathbb{Z}$,

$$H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0, \quad 1 \leq i < n - 1 = \dim \mathrm{Supp}(\mathcal{F}), \ k \in \mathbb{Z}.$$

Thus $\mathcal{F}$ satisfies the two conditions from the above to be an aCM sheaf. For future use, observe also that

$$V \cong H^0(\mathbb{P}^n, \underline{V}) \cong H^0(\mathbb{P}^n, \mathcal{F}). \tag{4.17}$$

Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(-, \mathcal{O}_{\mathbb{P}^n}(-1))$ to (4.14), we obtain an exact sequence

$$0 \to \underline{V}^\vee \to \underline{U} \to \mathcal{G} \to 0,$$

where

$$\mathcal{G} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}(-1)). \tag{4.18}$$

The sheaf $\mathcal{G}$ plays the role of $\mathcal{F}$ when we interchange the roles of U and V. In the following we use some standard facts from the Grothendieck-Serre Duality (see [269]). We have

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}(-1)) \cong \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{X_s}, \mathcal{O}_{\mathbb{P}^n}(-1)))$$

$$\cong \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{X_s}, \omega_{\mathbb{P}^n}))(n) \cong \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \omega_{X_s})(n)$$

$$\cong \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \mathcal{O}_{X_s}(d-n-1))(n) \cong \mathcal{F}^\vee(d-1),$$

where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \mathcal{O}_{X_s})$. Thus (4.18) becomes

$$\mathcal{G} \cong \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \omega_{X_s})(n) \cong \mathcal{F}^\vee(d-1). \tag{4.19}$$

This agrees with the theory from the previous Subsection.

Suppose $\mathcal{F}$ is of rank 1 and X is a normal variety. Then $\mathcal{F} \cong \mathcal{O}_X(D)$ for some Weil divisor D, and

$$\mathcal{G} \cong \mathcal{O}_X(-D)(d-1).$$

We have seen how a determinantal representation of a hypersurface in $\mathbb{P}^n$ leads to an aCM sheaf on $\mathbb{P}^n$. Now let us see the reverse construction. Let $\mathcal{F}$ be an aCM sheaf on $\mathbb{P}^n$ supported on a hypersurface X. Since $M = \Gamma_*(\mathcal{F})$ is a Cohen-Macaulay module over $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$ of depth $n-1$, its projective dimension is equal to 1. Since any graded projective module over the polynomial ring is isomorphic to the direct sum of free modules of rank 1, we obtain a resolution

$$0 \to \bigoplus_{i=1}^m S[-b_i] \to \bigoplus_{i=1}^m S[-a_i] \to \Gamma_*(\mathcal{F}) \to 0,$$

for some sequences of integers (a_i) and (b_i). Passing to the associated sheaves on the projective space, it gives a projective resolution of $\mathcal{F}$:

$$0 \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(-b_i) \xrightarrow{\phi} \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{F} \to 0. \tag{4.20}$$

The homomorphism of sheaves ϕ is given by a square matrix A of size m. Its

ij-th entry is a polynomial of degree $b_j - a_i$. The support X of $\mathcal{F}$ is equal to $V(\det A)_{\text{red}}$. The degree of $Y = V(\det A)$ is equal to

$$d = (b_1 + \cdots + b_m) - (a_1 + \cdots + a_m). \tag{4.21}$$

We assume that the resolution is minimal, i.e. $b_j < a_i$ for all i, j. This can be always achieved by dropping the isomorphic summands in the first and the second module. The case we considered before is a special case when $\mathcal{F}$ is an aCM sheaf for which

$$a_1 = \ldots = a_m = 0, \quad b_1 = \ldots = b_m = -1. \tag{4.22}$$

In this case A is a matrix of linear forms and $d = m$.

Proposition 4.2.5 *Let $\mathcal{F}$ be an aCM sheaf on $\mathbb{P}^n$ supported on a reduced hypersurface X and let (4.20) be its projective resolution. Then (4.22) holds if and only if*

$$H^0(\mathbb{P}^n, \mathcal{F}(-1)) = 0, \quad H^{n-1}(\mathbb{P}^n, \mathcal{F}(1-n)) \cong H^0(\mathbb{P}^n, \mathcal{G}(-1)) = 0. \tag{4.23}$$

Proof By duality,

$$H^{n-1}(\mathbb{P}^n, \mathcal{F}(1-n)) = H^{n-1}(X, \mathcal{F}(1-n))$$

$$\cong H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}(1-n), \omega_X)) \cong H^0(X, \mathcal{G}(-1)) = 0.$$

Taking global sections in the exact sequence (4.20), we immediately get that all a_i are non-positive. Taking higher cohomology, we obtain

$$H^{n-1}(\mathbb{P}^n, \mathcal{F}(1-n)) = \bigoplus_{i=1}^{m} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-b_i + 1 - n))$$

$$= \bigoplus_{i=1}^{m} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b_i - 2)) = 0.$$

Since $b_i < a_i \leq 0$, this implies that all $b_i = -1$. □

Let $\mathcal{F}$ be an aCM sheaf defining a linear determinantal representation of a normal hypersurface X. We assume that rank $\mathcal{F} = 1$. We have a rational map

$$\mathfrak{r} : X \dashrightarrow \mathbb{P}(U), \quad x \mapsto |\text{Ker}(\phi(x))|.$$

The map is defined on the complement of the open set X' where $\mathcal{F}$ is locally free. We know that $\mathcal{F} \cong \mathcal{O}_X(D)$ for some effective Weil divisor D. The sheaf $\mathcal{F}^\vee \cong \mathcal{O}_X(-D)$ is an ideal sheaf on X. Let $b : \tilde{X} \to X$ be the blow-up of

the ideal sheaf $\mathcal{J}_Z$. It resolves the map $\mathfrak{r}$ in the sense that there exists a regular map

$$\tilde{\mathfrak{r}} : \tilde{X} \to \mathbb{P}(U)$$

such that $\mathfrak{r} = \tilde{\mathfrak{r}} \circ \pi^{-1}$ (as rational maps). We will explain this in more detail in Chapter 7.

4.2.3 Symmetric and skew-symmetric aCM sheaves

Let $\mathcal{F}$ be an aCM sheaf on $\mathbb{P}^n$ whose scheme-theoretical support is a hypersurface X_s of degree d. Suppose we have a homomorphism of coherent sheaves on X_s

$$\alpha : \mathcal{F} \to \mathcal{F}^\vee(N) \tag{4.24}$$

for some integer N. Passing to duals, we get a homomorphism $(\mathcal{F}^\vee)^\vee(-N) \to \mathcal{F}^\vee$. After twisting by r, we get a homomorphism $(\mathcal{F}^\vee)^\vee \to \mathcal{F}(N)$. Composing it with the natural homomorphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$, we get a homomorphism

$${}^t\alpha : \mathcal{F} \to \mathcal{F}^\vee(N),$$

which we call the *transpose* of α.

We call the pair $(\mathcal{F}, \alpha)$ as in the above a *ϵ-symmetric sheaf* if α is an isomorphism and ${}^t\alpha = \epsilon\alpha$, where $\epsilon = \pm 1$. We say it is symmetric if $\epsilon = 1$ and skew-symmetric otherwise.

We refer for the proof of the following result to [66] or to [37], Theorem B.

Theorem 4.2.6 *Let $(\mathcal{F}, \alpha)$ be an ϵ-symmetric aCM sheaf. Assume that $X_s = X$. Then it admits a resolution of the form (4.20), where*

$$(a_1, \ldots, a_m) = (b_1 + N - d, \ldots, b_m + N - d),$$

and the map ϕ is defined by a symmetric matrix if $\epsilon = 1$ and a skew-symmetric matrix if $\epsilon = -1$.

Corollary 4.2.7 *Suppose $(\mathcal{F}, \alpha)$ is a symmetric sheaf with $N = d-1$ satisfying the vanishing conditions from (4.23). Then $\mathcal{F}$ admits a projective resolution*

$$0 \to \underline{U}^\vee(-1) \xrightarrow{\phi} \underline{U} \to \mathcal{F} \to 0,$$

where $U = H^0(\mathbb{P}^n, \mathcal{F})$ and ϕ is defined by a symmetric matrix with linear entries.

Note that the isomorphism α defines an isomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ and an isomorphism $V = H^0(\mathbb{P}^n, \mathcal{F}) \to U = H^0(\mathbb{P}^n, \mathcal{G})$. Suppose n is even. Twisting the isomorphism $\mathcal{F} \to \mathcal{G} = \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}, \omega_{X_s})(n)$ by $-\frac{1}{2}n$, we obtain an isomorphism

$$\mathcal{F}(-\tfrac{1}{2}n) \to \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{F}(-n), \omega_{X_s}).$$

Definition 4.2.8 *A rank 1 torsion-free coherent sheaf θ on a reduced variety Y with canonical sheaf ω_Y is called a* theta characteristic *if there exists an isomorphism*

$$\alpha : \theta \to \mathcal{H}om_{\mathcal{O}_Y}(\theta, \omega_Y).$$

Note that in the case when a theta characteristic θ is an invertible sheaf, we obtain

$$\theta^{\otimes 2} \cong \omega_Y,$$

which agrees with our previous definition of a theta characteristic on a nonsingular curve. If X is a normal variety, and θ is a reflexive sheaf (e.g. a Cohen-Macaulay sheaf), we know that $\theta \cong \mathcal{O}_X(D)$ for some Weil divisor D. Then θ must satisfy $\mathcal{O}_X(2D) \cong \omega_X$. In particular, if ω_X is an invertible sheaf, D is a $\mathbb{Q}$-Cartier divisor.

Since α and ${}^t\alpha$ differ by an automorphism of θ, and any automorphism of a rank 1 torsion-free sheaf is defined by a nonzero scalar multiplication, we can always choose an isomorphism α defining a structure of a symmetric sheaf on θ.

Let X be a reduced hypersurface of degree d in $\mathbb{P}^n$ and let θ be a theta characteristic on X. Assume $n = 2k$ is even. Then $\mathcal{F} = \theta(k)$ satisfies $\mathcal{F}(k) \cong \mathcal{F}(k)^\vee(d-1)$ and hence has a structure of a symmetric sheaf with $N = d-1$. Assume also that θ, considered as a coherent sheaf on $\mathbb{P}^n$, is an aCM sheaf. Applying Corollary 4.2.7 , we obtain that $\mathcal{F}$ admits a resolution

$$0 \to \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^n}(-a_i - 1) \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{F} \to 0.$$

From (4.19), we obtain that $\mathcal{G} \cong \mathcal{F}^\vee(d-1) \cong \mathcal{F}$. The vanishing conditions from Proposition 4.2.5 translate into one condition:

$$H^0(X, \theta(k-1)) = 0. \tag{4.25}$$

If $n = 2$, this matches the condition that θ is a non-effective theta characteristic. If this condition is satisfied, we obtain a representation of X as a determinant with linear entries. The number of isomorphism classes of such

representations is equal to the number of theta characteristics on X satisfying condition (4.25)

4.2.4 Singular plane curves

Assume $n = 2$, and let C be a reduced irreducible curve of degree d. Let $\mathcal{F}$ be a coherent torsion-free sheaf on C. Since $\dim C = 1$, $\mathcal{F}$ is a Cohen-Macaulay sheaf. Also, the cohomological condition for an aCM sheaf are vacuous, hence $\mathcal{F}$ is an aCM sheaf. In general, a Cohen-Macaulay module M over a local Noetherian ring R admits a dualizing R-module D, and

$$\text{depth } M + \max\{q : \text{Ext}^q_R(M, D) \neq 0\} = \dim R$$

(see [208]). In our case, the global dualizing sheaf is

$$\omega_C = \omega_{\mathbb{P}^2}(C) \cong \mathcal{O}_C(d-3),$$

the previous equality implies that $\mathcal{E}xt^q_{\mathcal{O}_C}(\mathcal{F}, \omega_C) = 0, \quad q > 0$, and

$$\mathcal{F} \to D(\mathcal{F}) := \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \cong \mathcal{F}^\vee \otimes \omega_C$$

is the duality, i.e. $\mathcal{F} \to D(D(\mathcal{F}))$ is an isomorphism.

If $\mathcal{F}$ satisfies the conditions from Proposition 4.2.5

$$H^0(C, \mathcal{F}(-1)) = H^1(C, D(\mathcal{F})(-1)) = 0, \tag{4.26}$$

we obtain a determinantal representation $C = V(\det A)$ with linear entries (4.14). For a general point x on C, the corank of the matrix $A(x)$ is equal to the rank of $\mathcal{F}$. We shall assume that

$$\text{rank } \mathcal{F} = 1.$$

In this case $\mathcal{F}$ is isomorphic to a subsheaf of the constant sheaf of rational functions on C. It follows from the resolution of $\mathcal{F}$ that

$$\chi(\mathcal{F}(-1)) = 0, \quad \chi(\mathcal{F}) = d.$$

Thus

$$\deg \mathcal{F}(-1) := \chi(\mathcal{F}(-1)) + p_a(C) - 1 = p_a(C) - 1.$$

Also,

$$\deg \mathcal{F} = \deg D(\mathcal{F}) = d + p_a(C) - 1 = d(d-1)/2.$$

Suppose x is a singular point of C. Then either rank$A(x) < d - 1$, or the image of the map $\phi : \mathbb{P}^2 \to |U \times V|_{d-1}$ is tangent to $|U \times V|_{d-1}$ at a point $\phi(x) \notin |U \times V|_{d-2}$. The sheaf $\mathcal{F}$ is not invertible at x only in the former case.

It is known that the isomorphism classes of rank 1 torsion-free sheaves of fixed degree d on an irreducible reduced algebraic curve C admit a moduli space which is a projective variety that contains an irreducible component which compactifies the generalized Jacobian variety $\mathrm{Jac}^d(C)$ of C (see [8], [456]). In the case of plane curves (and, by [456], only in this case), the moduli space is irreducible. Its dimension is equal to $p_a(C)$. We denote the moduli space by $\overline{\mathrm{Jac}}^d(C)$.

Let us describe in more detail rank 1 torsion-free sheaves $\mathcal{F}$ on C.

Let $p : \bar{C} \to C$ be the normalization morphism. Its main invariant is the *conductor ideal* $\mathfrak{c}$, the annihilator ideal of the sheaf $p_*\mathcal{O}_{\bar{C}}$. Obviously, it can be considered as an ideal sheaf in $\bar{C}$ equal to $p^{-1}(\mathfrak{c})$ (the image of $p^*(\mathfrak{c})$ in $\mathcal{O}_{\bar{C}}$ under the multiplication map, or, equivalently, $p^*(\mathfrak{c})$/torsion). For any $x \in C$, $\mathfrak{c}_x$ is the conductor ideal of the normalization $\bar{R}$ of the ring $R = \mathcal{O}_{C,x}$ equal to $\prod_{y \to x} \mathcal{O}_{\bar{C},y}$. Let

$$\delta_x = \text{length } \bar{R}/R.$$

Since, in our case, R is a Gorenstein local ring, we have

$$\dim_{\mathbb{C}} \bar{R}/\mathfrak{c}_x = 2 \dim_{\mathbb{C}} \bar{R}/\mathfrak{c}_x = 2\delta$$

(see [528], Chapter 4, n.11).

Suppose R is isomorphic to the localization of $\mathbb{C}[[u,v]]/(f(u,v))$ at the origin. One can compute δ_x, using the following *Jung-Milnor formula* (see [325], [390], §10).

$$\deg \mathfrak{c}_x = \dim_{\mathbb{C}} R/J_f + r_x - 1, \tag{4.27}$$

where J_f is the ideal generated by partial derivatives of f, and r_x is the number of analytic branches of C at the point x.

Let F be the cokernel of the canonical injection of sheaves $\mathcal{O}_C \to p_*(\mathcal{O}_{\bar{C}})$. Applying cohomology to the exact sequence

$$0 \to \mathcal{O}_C \to p_*\mathcal{O}_{\bar{C}} \to \mathsf{F} \to 0, \tag{4.28}$$

we obtain the *genus formula*

$$\chi(p_*(\mathcal{O}_{\bar{C}})) = \chi(\mathcal{O}_{\bar{C}}) = \chi(\mathcal{O}_C) + \sum_{x \in C} \delta_x. \tag{4.29}$$

Consider the sheaf of algebras $\mathcal{E}nd(\mathcal{F}) = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \mathcal{F})$. Since $\mathcal{E}nd(\mathcal{F})$ embeds into $\mathcal{E}nd(\mathcal{F}_\eta)$, where η is a generic point of C, and the latter is isomorphic to the field of rational functions on C, we see that $\mathcal{E}nd(\mathcal{F})$ is a coherent $\mathcal{O}_C$-algebra. It is finitely generated as a $\mathcal{O}_C$-module, and hence it is finite and

birational over C. We set $C' = \operatorname{Spec} \mathcal{E}nd(\mathcal{F})$ and let

$$\pi = \pi_{\mathcal{F}} : \mathcal{C}' \to C$$

be the canonical projection. The normalization map $\bar{C} \to C$ factors through the map π. For this reason, π is called the *partial normalization* of C. Note that $C' = C$ if $\mathcal{F}$ is an invertible sheaf. The algebra $\mathcal{E}nd(\mathcal{F})$ acts naturally on $\mathcal{F}$ equipping it with a structure of an $\mathcal{O}_{C'}$-module, which we denote by $\mathcal{F}'$. We have

$$\mathcal{F} \cong \pi_* \mathcal{F}'.$$

Recall that for any finite morphism $f : X \to Y$ of Noetherian schemes there is a functor $f^!$ from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_X$-modules defined by

$$f^! \mathcal{M} = \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{M}),$$

considered as a $\mathcal{O}_X$-module. The functor $f^!$ is the right adjoint of the functor f_* (recall that f^* is the left adjoint functor of f_*), i.e.

$$f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, f^! \mathcal{M}) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{N}, \mathcal{M}), \tag{4.30}$$

as bi-functors in $\mathcal{M}, \mathcal{N}$. If X and Y admit dualizing sheaves, we also have

$$f^! \omega_Y \cong \omega_X$$

(see [282], Chapter III, Exercises 6.10 and 7.2).

Applying this to our map $\pi : C' \to C$, and taking $\mathcal{N} = \mathcal{O}_{C'}$, we obtain

$$\mathcal{F} \cong \pi_* \pi^! \mathcal{F}.$$

It is known that, for any torsion-free sheaves $\mathcal{A}$ and $\mathcal{B}$ on C', a morphism $\pi_* \mathcal{A} \to \pi_* \mathcal{B}$ is $\pi_* \mathcal{O}_{C'}$-linear (see, for example, [36], Lemma 3.1). This implies that the natural homomorphism

$$\operatorname{Hom}_{\mathcal{C}'}(\mathcal{A}, \mathcal{B}) \cong \operatorname{Hom}_{\mathcal{C}'}(\pi_* \mathcal{A}, \pi_* \mathcal{B}) \tag{4.31}$$

is bijective. This gives

$$\mathcal{F}' \cong \pi^! \mathcal{F}.$$

For any $\mathcal{F}' \in \overline{\operatorname{Jac}}^d(C')$,

$$\chi(\mathcal{F}') = d' + \chi(C')$$

(in fact, this equality is the definition of the degree d' of $\mathcal{F}'$, see [404])

$$d = \deg \pi_* \mathcal{F}' = \chi(\pi_* \mathcal{F}') - \chi(\mathcal{O}_C)$$

$$= \chi(\mathcal{F}') - \chi(\mathcal{O}_C) = d' + \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_{C'}).$$

Definition 4.2.9 *The collection of $\mathcal{O}_{C,x}$-modules $\mathcal{F}_x, x \in \mathrm{Sing}(C)$, is called the* local type *of $\mathcal{F}$ ([442]). The* global invariant *is the isomorphism class of $\mathcal{E}nd_{\mathcal{O}_C}(\mathcal{F})$.*

It follows from Lemma 1.7 in [442] that the global type of $\mathcal{F}$ determines the isomorphism class of $\mathcal{F}$, up to tensoring with an invertible sheaf. Also it is proven in the same Lemma that the global type depends only on the collection of local types.

Lemma 4.2.10 *The global types of $\mathcal{F}$ and $D(\mathcal{F})$ are isomorphic, and*

$$\pi^! D(\mathcal{F}) \cong D(\pi^! \mathcal{F}).$$

Proof The first assertion follows from the fact that the dualizing functor is an equivalence of the categories. Taking $\mathcal{M} = \omega_C$ in (4.30), we obtain that $\pi_*(D(\pi^!\mathcal{F})) \cong D(\mathcal{F})$. The second assertion follows from (4.31). □

In fact, by Lemma 3.1 from [36], the map

$$\pi_* : \overline{\mathrm{Jac}}^{d'}(C') \to \overline{\mathrm{Jac}}^{d}(C)$$

is a closed embedding of projective varieties.

It follows from the duality that $\chi(\mathcal{F}) = -\chi(D(\mathcal{F}))$. Thus the functor $\mathcal{F} \to D(\mathcal{F})$ defines an involution $D_{C'}$ on $\overline{\mathrm{Jac}}^{p_a(C')-1}(C')$ and an involution D_C on $\overline{\mathrm{Jac}}^{p_a(C)-1}(C)$. By Lemma 4.2.10, the morphism π_* commutes with the involutions.

Let us describe the isomorphism classes of the local types of $\mathcal{F}$. Let $\tilde{\mathcal{F}} = p^{-1}(\mathcal{F}) = p^*(\mathcal{F})/\mathrm{torsion}$. This is an invertible sheaf on $\tilde{C}$. The canonical map $\mathcal{F} \to p_*(p^*\mathcal{F})$ defines the exact sequence

$$0 \to \mathcal{F} \to p_*\tilde{\mathcal{F}} \to \mathsf{T} \to 0, \tag{4.32}$$

where T is a torsion sheaf whose support is contained in the set of singular points of C.

The immediate corollary of this is the following.

Lemma 4.2.11 *For any $x \in C$,*

$$\dim_{\mathbb{C}} \mathcal{F}(x) = \mathrm{mult}_x C,$$

where $\mathcal{F}(x)$ denotes the fiber of the sheaf $\mathcal{F}$ and $\mathrm{mult}_x C$ denotes the multiplicity of the point x on C.

Proof Since the cokernel of $\mathcal{F} \to p_*\tilde{\mathcal{F}}$ is a torsion sheaf, we have

$$\dim_{\mathbb{C}} \mathcal{F}(x) = \dim_{\mathbb{C}} \tilde{\mathcal{F}}(x) = \dim_{\mathbb{C}} p_*(\mathcal{O}_{\tilde{C}})(x). \tag{4.33}$$

It is clear that the dimension of the fiber of a coherent sheaf is equal to the dimension of the fiber over the closed point of the formal completion of $\mathcal{F}_x$. Let R (resp. $\bar{R}$) denote the formal completion of $\mathcal{O}_{C,x}$ (resp. its normalization). We know that $\bar{R} = \prod_{y \to x} \bar{R}_y$, where $\bar{R}_y \cong \mathbb{C}[[t]]$. Let (u, v) be local parameters in R generating the maximal ideal $\mathfrak{m}$ of R. One can choose the latter isomorphism in such a way that the composition of the map $R \to \bar{R}$ with the projection map $\bar{R} \to \bar{R}_i$ is given by

$$(u, v) \mapsto (t_i^{m_i}, \sum_{j=m_i}^{\infty} a_j t^j),$$

where m_j is the multiplicity of the analytic branch of the curve C corresponding to the point y over x. It follows that

$$\dim_{\mathbb{C}} \bar{R}/\mathfrak{m} = \dim_{\mathbb{C}} \prod_{i=1}^{r_x} \mathbb{C}[[t]]/(t^{m_i}) = \sum_{i=1}^{r_x} m_i = \text{mult}_x C.$$

Thus the last dimension in (4.33) is equal to the multiplicity, and we are done. □

Corollary 4.2.12 *Suppose $\mathcal{F}$ satisfies (4.26), and hence defines a linear determinantal representation $C = V(\det A)$. Then*

$$d - \text{rank } A(x) = \text{mult}_x C.$$

We denote by $\delta_x(\mathcal{F})$ the length of $\mathcal{T}_x$. The length $\delta_x(\mathcal{F})$ of $\mathcal{T}_x$ is the local invariant of the $\mathcal{O}_{C,x}$-module $\mathcal{F}_x$ (see [264]). Let M be a rank 1 torsion-free module over $R = \mathcal{O}_{C,x}$ and $\bar{M} = M \otimes \bar{R}$/torsion. Let Q be the fraction field of R. Since $M \otimes_R Q \cong Q$, one can find a fractional ideal isomorphic to M. It is known that the isomorphism class of M can be represented by a fractional ideal J with local invariant $\delta(M) = \dim \bar{M}/M$ contained in $\bar{R}$ and containing the ideal $\mathfrak{c}(R)$, where $\mathfrak{c}(M)$ is the conductor ideal of R. This implies that local types of $\mathcal{F}$ at x with $\delta_x(\mathcal{F}) = \delta$ are parameterized by the fixed locus of the group R^* acting on the Grassmann variety $G(\delta, \bar{R}/\mathfrak{c}_x) \cong G(\delta, 2\delta)$ (see [264], Remark 1.4, [456], Theorem 2.3 (d)). The dimension of the fixed locus is equal to δ_x. Thus local types with fixed local invariant δ are parameterized by a projective variety of dimension δ.

Example 4.2.13 Let C' be the proper transform of C under the blow-up of the plane at a singular point $x \in C$ of multiplicity m_x. Since it lies on a nonsingular surface, C' is a Gorenstein curve. The projection $\pi : C' \to C$ is a partial normalization. Let $\mathcal{F} = \pi_* \mathcal{O}_{C'}$. Then $\mathfrak{m}_{C,x}^{m_x}$ contains the conductor $\mathfrak{c}_x$ and $\mathfrak{c}(\mathcal{F}_x) = \mathfrak{m}_{C,x}^{m_x - 1}$, hence $\delta_x(\mathcal{F}) = m_x - 1$ (see [456], p. 219).

Let $\mathcal{F}$ define a linear determinantal representation $C = V(\det A)$. We know that $D(\mathcal{F})$ defines the linear representation corresponding to the transpose matrix ${}^tA$. The case when $\mathcal{F} \cong D(\mathcal{F})$ corresponds to the symmetric matrix A. We assume that rank $\mathcal{F} = 1$, i.e. $\mathcal{F}$ is a theta characteristic θ on C.

By duality, the degree of a theta characteritic θ is equal to $p_a(C) - 1$ and $\chi(\theta) = 0$. We know that each theta characteristic θ is isomorphic to $\pi_*\theta'$, where θ' is a theta characteristic on the partial normalization of C defined by θ. Since, locally, $\mathcal{E}nd(\theta') \cong \mathcal{O}_{C'}$, we obtain that θ' is an invertible sheaf on C'.

Let $\mathrm{Jac}(X)[2]$ denote the 2-torsion subgroup of the group $\mathrm{Jac}(X)$ of isomorphism classes of invertible sheaves on a curve X. Via tensor product it acts on the set $\mathrm{TChar}(C)$ of theta characteristics on C. The pull-back map p^* defines an exact sequence

$$0 \to G \to \mathrm{Jac}(C) \to \mathrm{Jac}(\bar{C}) \to 0. \tag{4.34}$$

The group $\mathrm{Jac}(\bar{C})$ is the group of points on the Jacobian variety of $\bar{C}$, an abelian variety of dimension equal to the genus g of $\bar{C}$. The group $G \cong \mathcal{O}^*_{\bar{C}}/\mathcal{O}^*_C$ has a structure of a commutative group, isomorphic to the product of additive and multiplicative groups of $\mathbb{C}$. Its dimension is equal to $\delta = \sum_x \delta_x$. It follows from the exact sequence that

$$\mathrm{Jac}(C)[2] \cong (\mathbb{Z}/2\mathbb{Z})^{2g+b}, \tag{4.35}$$

where b is equal to the dimension of the multiplicative part of G. It is easy to see that

$$b = \#p^{-1}(\mathrm{Sing}(C)) - \#\mathrm{Sing}(C) = \sum_x (r_x - 1). \tag{4.36}$$

Proposition 4.2.14 *The group* $\mathrm{Jac}(C)[2]$ *acts transitively on the set of theta characteristics with fixed global type. The order of the stabilizer subgroup of a theta characteristic* θ *is equal to the order of the* 2*-torsion subgroup of the kernel of* $\pi^* : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$.

Proof Let $\theta, \theta' \in \mathrm{TChar}(C)$ with the isomorphic global type. Since two sheaves with isomorphic global type differ by an invertible sheaf, we have $\theta' \cong \theta \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$. This implies

$$\theta' \otimes \mathcal{L} \cong \theta'^{\vee} \otimes \omega_C \cong \theta^{\vee} \otimes \mathcal{L}^{-1} \otimes \omega_C \cong \theta \otimes \mathcal{L}^{-1} \cong \theta \otimes \mathcal{L}.$$

By Lemma 2.1 from [36], $\pi^*\mathcal{F} \cong \pi^*\mathcal{F} \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Jac}(C)$ if and only if $\pi^*\mathcal{L} \cong \mathcal{O}_{C'}$. This gives $\pi^*\mathcal{L}^2 \cong \mathcal{O}_{C'}$ and hence $\pi^*(\mathcal{L}) \in \mathrm{Jac}(C')[2]$. It follows from exact sequence (4.34) (where C is replaced with C') that $\mathrm{Jac}(C')$ is a divisible group, hence the homomorphism $p^* : \mathrm{Jac}(C)[2] \to \mathrm{Jac}(C')[2]$

is surjective. This implies that there exists $\mathcal{M} \in \mathrm{Jac}(C)[2]$ such that $\pi^*(\mathcal{L} \otimes \mathcal{M}) \cong \mathcal{O}_{C'}$. Thus, we obtain

$$\theta' \otimes \mathcal{M} \cong \theta \otimes \mathcal{L} \otimes \mathcal{M} \cong \theta.$$

This proves the first assertion. The second assertion follows from the loc. cit. Lemma. □

Corollary 4.2.15 *The number of theta characteristics of global type defined by a partial normalization $\pi : C' \to C$ is equal to $2^{2g+b-b'}$, where $b' = \#\pi^{-1}(\mathrm{Sing}(C)) - \#\mathrm{Sing}(C)$.*

Recall that a theta characteristic θ defines a symmetric determinantal representation of C if and only if it satisfies $h^0(\theta) = 0$. So, we would like to know how many such theta characteritics exist. A weaker condition is that $h^0(\theta)$ is even. In this case the theta characteristic is called *even*, the remaining ones are called *odd*. The complete answer on the number of even theta characteristics on a plane curve C is not known. In the case when $\theta \in \mathrm{Jac}(C)$, the answer, in terms of some local invariants of singularities, can be found in [277] (see also [361] for a topological description of the local invariants). The complete answer is known in the case when C has *simple* (or *ADE*) singularities.

Definition 4.2.16 *A singular point $x \in C$ is called a* simple singularity *if its local ring is formally isomorphic to the local ring of the singularity at the origin of one of the following plane affine curves*

$$\begin{aligned} a_k &: x^2 + y^{k+1} = 0, \quad k \geq 1, \\ d_k &: x^2 y + y^{k-1} = 0, \quad k \geq 4 \\ e_6 &: x^3 + y^4 = 0, \\ e_7 &: x^3 + xy^3 = 0, \\ e_8 &: x^3 + y^5 = 0. \end{aligned}$$

According to [263], a simple singularity is characterized by the property that there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring. This implies that the set $\mathrm{TChar}(C)$ is finite if C is a plane curve with only simple singularities.

The number of even theta characteristics on an irreducible reduced plane curve C with only simple singularities is given in the following Theorem from [442].

Theorem 4.2.17 *The number of invertible even theta characteristics on C is*

2^{2g+b-1} *if C has an A_{4s+1}, D_{4s+2}, or E_7 singularity,*

$2^{g+b-1}(2^g+1)$ *if C has no singularities as above, and has an even number of types $A_{8s+2}, A_{8s+3}, A_{8s+4}, D_{8s+3}, D_{8s+4}, D_{8s+5}, E_6$,*

$2^{g+b-1}(2^g+1)$ *otherwise.*

The number of non-invertible even theta characteristics on a curve with simple singularities depends on their known local types. An algorithm to compute them is given in [442].

Example 4.2.18 Let C be a plane irreducible cubic curve. Suppose it has an ordinary node. This is a simple singularity of type A_1. We have $\mathrm{Jac}(C) \cong \mathbb{C}^*$ and $\mathrm{Jac}(C)[2] \cong \mathbb{Z}/2\mathbb{Z}$. The only partial normalization is the normalization map. There is one invertible theta characteristic θ_1 with $h^0(\theta_1) = 0$ and one non-invertible theta characteristic $\theta_2 \cong p_*\mathcal{O}_{\bar{C}}(-1)$ with $h^0(\theta_2) = 0$. It is isomorphic to the conductor ideal sheaf on C. Thus there are two isomorphism classes of symmetric determinant representations for C. Without loss of generality we may assume that $C = V(t_0t_2^2 + t_1^3 + t_0t_1^2)$. The theta characteristic θ_1 defines the symmetric determinantal representation

$$t_0t_2^2 + t_1^3 + t_0t_1^2 = \det \begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & -t_0 - t_1 & 0 \\ t_1 & 0 & -t_0 \end{pmatrix}.$$

Observe that rank $A(x) = 2$ for all points $x \in C$. The theta characteristic θ_2 defines the symmetric determinantal representation

$$t_0t_2^2 + t_1^3 + t_0t_1^2 = \det \begin{pmatrix} -t_0 & 0 & -t_1 \\ 0 & -t_1 & -t_2 \\ -t_1 & -t_2 & t_1 \end{pmatrix}.$$

The rank of $A(x)$ is equal to 1 for the singular point $x = [1, 0, 0]$ and equals 2 for other points on C.

Assume now that C is a cuspidal cubic with equation $V(t_0t_2^2 + t_1^3)$. There are no invertible theta characteristics and there is only one non-invertible. It is isomorphic to the conductor ideal sheaf on C. It defines the symmetric linear determinantal representation

$$t_0t_2^2 + t_1^3 = \det \begin{pmatrix} 0 & -t_2 & -t_1 \\ -t_2 & -t_1 & 0 \\ -t_1 & 0 & -t_0 \end{pmatrix}.$$

Remark 4.2.19 We restricted ourselves with irreducible curves. The case of reducible nodal curves was studied in [71].

4.2.5 Linear determinantal representations of surfaces

Let S be a normal surface of degree d in $\mathbb{P}^3$. We are looking for an an aCM sheaf $\mathcal{F}$ on $\mathbb{P}^3$ with scheme-theoretical support equal to S. We also require that $\mathcal{F}$ is of rank 1 and satisfies the additional assumption (4.23)

$$H^0(\mathbb{P}^3, \mathcal{F}(-1)) = H^2(\mathbb{P}^3, \mathcal{F}(-2)) = 0. \tag{4.37}$$

Every such $\mathcal{F}$ will define a linear determinantal representation $f = \det A$ defined by the resolution (4.14) of $\mathcal{F}$ such that rank $A(x) = d-1$ for a general point on S.

Since the exact sequence (4.14) implies that $\mathcal{F}$ is generated by its global sections, we see that $\mathcal{F} \cong \mathcal{O}_S(C)$ for some effective Weil divisor C. By taking a general section of $\mathcal{F}$ and applying Bertini's Theorem, we may assume that C is an integral curve, nonsingular outside $\mathrm{Sing}(S)$.

Recall that, as an aCM sheaf, $\mathcal{F}$ satisfies the cohomological condition

$$H^1(\mathbb{P}^3, \mathcal{F}(j)) = 0,\ j \in \mathbb{Z}. \tag{4.38}$$

Let s be a nonzero section of $\mathcal{F}$ whose zero subscheme is an integral curve such that $\mathcal{F} \cong \mathcal{O}_S(C)$. The dual of the map $\mathcal{O}_S \xrightarrow{s} \mathcal{L}$ defines an exact sequence

$$0 \to \mathcal{F}^\vee(j) \to \mathcal{O}_S(j) \to \mathcal{O}_C(j) \to 0. \tag{4.39}$$

By Serre's Duality,

$$H^1(S, \mathcal{F}^\vee(j)) \cong H^1(S, \mathcal{F}(-j) \otimes \omega_S) \cong H^1(S, \mathcal{F}(d-4-j)) = 0.$$

Applying cohomology, we obtain that the restriction map

$$H^0(S, \mathcal{O}_S(j)) \to H^0(C, \mathcal{O}_C(j)) \tag{4.40}$$

is surjective for all $j \in \mathbb{Z}$. Recall that, by definition, this means that C is *projectively normal* in $\mathbb{P}^3$. Conversely, if C is projectively normal, we obtain (4.38).

Before we state the next Theorem we have to remind ourselves some facts about the intersection theory on a normal singular surface (see [403]).

Let $\sigma : S' \to S$ be a resolution of singularities that we always assume to be minimal. Let $\mathcal{E} = \sum_{i \in I} E_i$ be its reduced exceptional locus. For any curve C on S we denote by $\sigma^{-1}(C)$ the proper transform of C and define

$$\sigma^*(C) := \pi^{-1}(C) + \sum_{i \in I} n_i E_i,$$

where n_i are rational numbers uniquely determined by the system of linear

equations

$$0 = \sigma^*(C) \cdot E_i = \cdot\pi^{-1}(C) \cdot E_j + \sum_{i \in I} n_i E_i \cdot E_j = 0,\ j \in I.$$

Now we define the intersection number $C \cdot C'$ of two curves S by

$$C \cdot C' := \sigma^*(C) \cdot \sigma^*(C').$$

This can be extended by linearity to all Weil divisors on S. It coincides with the usual intersection on the subgroup of Cartier divisors. Also it depends only on the equivalence classes of the divisors.

Recall that S admits a dualizing sheaf ω_S. It is a reflexive sheaf of rank 1, hence determines the linear equivalence class of a Weil divisors denoted by K_S (the *canonical class* of S). It is a Cartier divisor class if and only if S is Gorenstein (as it will be in our case when S is a hypersurface). We have

$$K_{S'} = \sigma^*(K_S) + \Delta,$$

where $\Delta = \sum_{i \in I} a_i E_i$ is the *discrepancy divisor*. The rational numbers a_i are uniquely determined from linear equations

$$K_{S'} \cdot R_j = \sum_{i \in I} a_i E_i \cdot E_j,\ j \in I.$$

For any reduced irreducible curve C on S define

$$A_S(C) := -\tfrac{1}{2}(\sigma^*(C) - \sigma^{-1}(C))^2 + \tfrac{1}{2}\sigma^{-1}(C) \cdot \Delta - \delta,$$

where $\delta = h^0(p^*\mathcal{O}_{\bar{C}}/\mathcal{O}_C)$ is our familiar invariant of the normalization of C. The following results can be found in [44].

Proposition 4.2.20 *For any reduced curve C on S and a Weil divisor D let $\mathcal{O}_C(D)$ be the cokernel of the natural injective map $\mathcal{O}_S(D - C) \to \mathcal{O}(D)$ extending the similar map on $S \setminus \mathrm{Sing}(S)$. Then*

(i) $C \mapsto A_S(C)$ extends to a homomorphism $\mathrm{WDiv}(S)/\mathrm{Div}(S) \to \mathbb{Q}$ which is independent of a resolution;
(ii) $\chi(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C) + C \cdot D - 2A_S(C)$;
(iii) $-2\chi(\mathcal{O}_C) = C^2 + C \cdot K_S - 2A_S(C)$.

Example 4.2.21 Assume that S has only ordinary double points. Then a minimal resolution $\sigma : S' \to S$ has the properties that $\Delta = 0$ and $\mathcal{E} = E_1 + \cdots + E_k$, where k is the number of singular points and each E_i is a

smooth rational curves with $E_i \cdot K_{S'} = 0$ (see more about this in Chapter 8). Let $\sigma^{-1}(C) \cdot E_i = m_i$. Then easy computations show that

$$\begin{aligned}
\sigma^*(C) &= \sigma^{-1}(C) + \tfrac{1}{2}\sum_{i=1}^{n} m_i E_i,\\
C^2 &= \sigma^{-1}(C)^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i^2,\\
C \cdot K_S &= \sigma^{-1}(C) \cdot K_{S'},\\
A_S(C) &= \frac{1}{4}\sum_{i=1}^{k} m_i^2 - \delta.
\end{aligned}$$

Now we are ready to state and to prove the following theorem.

Theorem 4.2.22 *Let $\mathcal{F}$ be an aCM sheaf of rank 1. Then $\mathcal{F}$ defines a linear determinantal representation of S if and only if $\mathcal{F} \cong \mathcal{O}_S(C)$ for some projectively normal integral curve C with*

$$\deg C = \tfrac{1}{2}d(d-1), \quad p_a(C) = \frac{1}{6}(d-2)(d-3)(2d+1).$$

Proof Suppose $\mathcal{F}$ defines a linear determinantal representation of S. Then it is an aCM sheaf isomorphic to $\mathcal{O}_S(C)$ for some integral projectively normal curve C, and satisfies conditions (4.37) and (4.38).

We have

$$\chi(\mathcal{F}(-1)) = h^0(\mathcal{F}(-1)) - h^1(\mathcal{F}(-1)) + h^2(\mathcal{F}(-1)).$$

By (4.37) and (4.38), the right-hand side is equal to $h^2(\mathcal{F}(-1))$. Let H be a general plane section of S and

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S \to \mathcal{O}_H \to 0 \tag{4.41}$$

be the tautological exact sequence defining the ideal sheaf of H. Tensoring it by $\mathcal{F}(-1)$, we obtain an exact sequence

$$0 \to \mathcal{F}(-2) \to \mathcal{F}(-1) \to \mathcal{F}(-1) \otimes \mathcal{O}_H \to 0.$$

It shows that the condition $h^2(\mathcal{F}(-2)) = 0$ from (4.37) implies $h^2(\mathcal{F}(-1)) = 0$, hence

$$\chi(\mathcal{F}(-1)) = 0. \tag{4.42}$$

Similar computation shows that

$$\chi(\mathcal{F}(-2)) = 0. \tag{4.43}$$

Tensoring the exact sequence (4.41) by $\mathcal{O}_S(C-H)$, we obtain an exact sequence

$$0 \to \mathcal{F}(-2) \to \mathcal{F}(-1) \to \mathcal{O}_H(C-H) \to 0.$$

Applying the Riemann-Roch Theorem to the sheaf $\mathcal{O}_H(C-H)$ on H, we get

$$\deg \mathcal{O}_H(C-H) = \deg C - d = \chi(\mathcal{O}_H(C-H)) - \chi(\mathcal{O}_H)$$

$$= \chi(\mathcal{F}(-1)) - \chi(\mathcal{F}(-2)) - \chi(\mathcal{O}_H) = -\chi(\mathcal{O}_H).$$

This gives

$$\deg C = d - \chi(\mathcal{O}_H) = d - 1 + \tfrac{1}{2}(d-1)(d-2) = \tfrac{1}{2}d(d-1),$$

as asserted.

Applying Proposition 4.2.20 (ii), we get,

$$\chi(O_C) = -C \cdot C + C \cdot H + \chi(\mathcal{O}_C(C-H)) + 2A_S(C)$$

$$= \deg C - C^2 + \chi(\mathcal{O}_C(C-H)) + 2A_S(C).$$

By Proposition 4.2.20 (iii),

$$C^2 = -C \cdot K_S - 2\chi(\mathcal{O}_C) + 2A_S(C) = -(d-4)\deg C - 2\chi(\mathcal{O}_C) + 2A_S(C),$$

hence

$$-\chi(\mathcal{O}_C) = (d-3)\deg C + \chi(\mathcal{O}_C(C-H)).$$

The exact sequence

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S(C-H) \to \mathcal{O}_C(C-H) \to 0$$

gives

$$\chi(\mathcal{O}_C(C-H)) = \chi(\mathcal{F}(-1)) - \chi(\mathcal{O}_S(-1)) = -\chi(\mathcal{O}_S(-1)).$$

Easy computations of the cohomology of projective space gives

$$\chi(\mathcal{O}_S(-1)) = \binom{d}{3}.$$

Combining all together, we obtain

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 + \tfrac{1}{2}d(d-1)(d-3) - \frac{1}{6}d(d-1)(d-2)$$

$$= \frac{1}{6}(d-2)(d-3)(2d+1),$$

as asserted. We leave it to the reader to reverse the arguments and prove the converse. □

Example 4.2.23 We will study the case of cubic surfaces in more detail in Chapter 9. Let us consider the case of quartic surfaces. Assume first that S is nonsingular. Then $\mathcal{F} \cong \mathcal{O}_S(C)$, where C is a projectively normal smooth curve of degree 6 and genus 3. The projective normality is equivalent to the condition that C is not hyperelliptic (Exercise 4.10). We also have $h^0(\mathcal{O}_X(C)) = 4$. According to Noether's Theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section. Since a plane section of a quartic surface is of degree 4, we see that a general quartic surface does not admit a determinantal equation. The condition that X contains a curve C as in the above imposes one algebraic condition on the coefficients of a quartic surface (one condition on the moduli of quartic surfaces).

Suppose now that S contains such a curve. By (4.18), the transpose determinantal representation $C = \det {}^t A$ is defined by the sheaf $\mathcal{G} \cong \mathcal{F}^\vee(3) \cong \mathcal{O}_S(3H - C)$, where H is a plane section of S. We have two maps $\mathfrak{l} : S \to \mathbb{P}^3, \mathfrak{r} : S \to \mathbb{P}^3$ defined by the complete linear systems $|C|$ and $|3H - C|$. Since $C^2 = -C \cdot K_S - 2\chi(\mathcal{O}_C) = 4$, the images are quartic surfaces. We will see later, in Chapter 7, that the two images are related by a Cremona transformation from $|U^\vee| = |C|^\vee$ to $|V^\vee| = |3H - C|^\vee$.

We will find examples with singular surface S in the next Subsection.

4.2.6 Symmetroid surfaces

These are surfaces in $\mathbb{P}^3$ which admit a linear determinantal representation $S = V(\det A)$ with symmetric matrix A. The name was coined by A. Cayley.

According to our theory the determinantal representation is given by an aCM sheaf $\mathcal{F}$ satisfying

$$\mathcal{F} \cong \mathcal{F}^\vee(d-1). \tag{4.44}$$

For example, if S is a smooth surface of degree d, we have $\mathcal{F} \cong \mathcal{O}_S(C)$ and we must have $C \sim (d-1)H - C$, where H is a plane section. Thus, numerically, $C = \frac{1}{2}(d-1)H$, and we obtain $C^2 = \frac{1}{4}d(d-1)^2, C \cdot K_S = \frac{1}{2}d(d-1)(d-4)$, and $p_a(C) = 1 + \frac{3}{8}d(d-1)(d-3)$. It is easy to see that it disagrees with the formula for $p_a(C)$ for any $d > 1$. A more obvious reason why a smooth surface cannot be a symmetroid is the following. The codimension of the locus of quadrics in $\mathbb{P}^d$ of corank ≥ 2 is equal to 3. Thus each three-dimensional linear system of quadrics intersects this locus, and hence at some point $x \in S$ we must have rank $A(x) \leq d-2$. Since our sheaf $\mathcal{F}$ is an invertible sheaf, this is impossible.

So we have to look for singular surfaces. Let us state the known analog of Theorem 4.2.1 in the symmetric case.

The proof of the following theorem can be found in [279] or [323]).

Theorem 4.2.24 *Let* Sym_m *be the space of symmetric matrices of size* m *and* $\mathrm{Sym}_m(r)$ *be the subvariety of matrices of rank* $1 \leq r < m$.

- $\mathrm{Sym}_m(r)$ *is an irreducible Cohen-Macaulay subvariety of codimension* $\frac{1}{2}(m-r)(m-r+1)$;
- $\mathrm{Sing}(\mathrm{Sym}_m(r)) = \mathrm{Sym}_m(r-1)$;
- $\deg \mathrm{Sing}(\mathrm{Sym}_m(r)) = \prod_{0 \leq i \leq m-r-1} \frac{\binom{i+m}{m-r-i}}{\binom{2i+1}{i}}$.

For example, we find that

$$\deg \mathsf{Q}_2(2) = 4, \quad \deg \mathsf{Q}_{d-1}(2) = \binom{d+1}{3}. \tag{4.45}$$

Thus, we expect that a general cubic symmetroid has four singular points, a general quartic symmetroid has 10 singular points, and a general quintic symmetroid has 20 singular points.

Note that a symmetroid surface of degree d is the Jacobian hypersurface of the web of quadrics equal to the image of the map $\phi : \mathbb{P}^3 \to \mathsf{Q}_{d-1}$ defined by the determinantal representation. We identify $|E|$ with a web W of quadrics in $\mathbb{P}(U)$. The surface S is the discriminant hypersurface $\mathsf{D}(|E|)$ of W. The left kernel map $\mathfrak{l} : S \dashrightarrow \mathbb{P}^{d-1}$ given by $|\mathcal{O}_S(C)|$ maps S onto the Jacobian surface $\mathrm{Jac}(|E|)$ in $\mathbb{P}(U)$. $|E|$ is a regular web of quadrics if $|E|$ intersects the discriminant hypersurface of quadrics in $\mathbb{P}(U)$ transversally. In this case we have the expected number of singular points on S, and all of them are ordinary nodes. The surface S admits a minimal resolution $\sigma : \tilde{S} := \tilde{\mathsf{D}}(|E|) \to S$. The map $\mathfrak{l} = \tilde{\mathfrak{l}} \circ \sigma^{-1}$, where $\tilde{\mathfrak{l}} : \tilde{S} \to \mathrm{Jac}(|E|)$. The map is given by the linear system $|\sigma^{-1}(C)|$. The Jacobian surface is a smooth surface of degree equal to $\sigma^{-1}(C)^2$.

Proposition 4.2.25 *Let* S' *be the Jacobian surface of* $|E|$*, the image of* S *under the right kernel map* $\mathfrak{r}$*. Assume that* $|E|$ *is a regular web of quadrics. Then* $\mathrm{Pic}(S')$ *contains two divisor classes* η, h *such that* $h^2 = d, \eta^2 = \binom{d}{3}$*, and*

$$2\eta = (d-1)h - \sum_{i=1}^{k} E_i,$$

where E_i *are exceptional curves of the resolution* $\sigma : \tilde{S} \to S$.

Proof We identify S' with the resolution $\tilde{S}$ by means of the map $\tilde{\mathfrak{r}}$. We take $h = \sigma^*(\mathcal{O}_{|E|}(1))$ and η to be $\tilde{\mathfrak{r}}^*(\mathcal{O}_{S'}(1))$. We follow the proof of Proposition

4.1.7 to show that, under the restriction $|\mathcal{O}_{\mathbb{P}(U)}(2)| \to |\mathcal{O}_{S'}(2)|$, the web of quadrics $|E|$ in $|\mathcal{O}_{\mathbb{P}(U)}(2)|$ is identified with the linear system of polars of S. This is a sublinear system in $|\mathcal{O}_S(d-1)|$. Its preimage in $\tilde{S}$ is contained in the linear system $|(d-1)h - \sum_{i=1}^{k} E_i|$. It is clear that $h^2 = d$. It follows from Proposition 4.2.24, that $4\eta^2 = (d-1)^2 d - 2\binom{d+1}{3}$. This easily gives the asserted value of η^2. □

Corollary 4.2.26

$$\deg S' = \eta^2 = \tbinom{d}{3}.$$

Using the adjunction formula, we find

$$2p_a(\eta) - 2 = \eta_S^2 + \eta \cdot K_{S'} = \eta^2 + \tfrac{1}{2}d(d-1)(d-4) = \tbinom{d}{3} + \tfrac{1}{2}d(d-1)(d-4)$$

$$= \frac{1}{3}d(d-1)(2d-7).$$

This agrees with the formula for $p_a(C)$ in Theorem 4.2.22.

It follows from the Proposition that the theta characteristic θ defining the symmetric determinantal representation of S is isomorphic to $\mathcal{O}_S(C)$, where $C = \sigma_*(D)$ for $D \in |\eta|$. We have $\mathcal{O}_S(C)^{\otimes 2} \cong \mathcal{O}_S(d-1)$ outside $\mathrm{Sing}(S)$. This gives $\theta^{[2]} \cong \mathcal{O}_S(d-1)$.

Remark 4.2.27 Suppose d is odd. Let

$$\xi := \tfrac{1}{2}(d-1)h - \eta.$$

Then $\sum_{i=1}^{k} E_i \sim 2\xi$. If d is even, we let

$$\xi := \tfrac{1}{2}dh - \eta.$$

The $h + \sum_{i=1}^{k} E_i \sim 2\xi$. So, the set of nodes is even in the former case and weakly even in the latter case (see [69]). The standard construction gives a double cover of S' ramified only over nodes if the set is even and over the union of nodes and a member of $|h|$ if the set is weakly even.

The bordered determinant formula (4.10) for the family of contact curves extends to the case of surfaces. It defines a $(d-1)$-dimensional family of contact surfaces of degree $d-1$. The proper transform of a contact curve in S' belongs to the linear system $|\eta|$.

Example 4.2.28 We will consider the case $d = 3$ later. Assume $d = 4$ and the determinantal representation is transversal, i.e. S has the expected number 10 of nodes. Let S' be its minimal resolution. The linear system η consists of curves of genus 3 and degree 6. It maps S' isomorphically onto a quartic

surface in $\mathbb{P}^3$, the Jacobian surface of the web of quadrics defined by the determinantal representation. The family of contact surfaces is a 3-dimensional family of cubic surfaces passing through the nodes of S and touching the surface along some curve of genus 3 and degree 6 passing through the nodes. The double cover corresponding to the divisor class ξ is a regular surface of general type with $p_g = 1$ and $c_1^2 = 2$.

Consider the linear system $|2h - E_1|$ on S'. Since $(h - E_1)^2 = 2$, it defines a degree 2 map onto $\mathbb{P}^2$. Since $(2h - E_i) \cdot E_j = 0, i > 10$, the curves $E_i, i \neq 1$, are blown down to points. The curve R_1 is mapped to a conic K on the plane. One can show that the branch curve of the cover is the union of two cubic curves and the conic K is tangent to both of the curves at each intersection point. Conversely, the double cover of the plane branched along the union of two cubics, which both everywhere are tangent to a nonsingular conic, is isomorphic to a quartic symmetroid (see [132]). We refer to Chapter 1 where we discussed the Reye varieties associated to n-dimensional linear systems of quadrics in $\mathbb{P}^n$. In the case of the quartic symmetroid parameterizing singular quadrics in a web of quadrics in $\mathbb{P}^3$, the Reye variety is an Enriques surface.

Assume $d = 5$ and S has expected number 20 of nodes. The linear system η consists of curves of genus 11 and degree 10. It maps S' isomorphically onto a surface of degree 10 in $\mathbb{P}^4$, the Jacobian surface of the web of quadrics defined by the determinantal representation. The family of contact surfaces is a 4-dimensional family of quartic surfaces passing through the nodes of S and touching the surface along some curve of genus 11 and degree 10 passing through the nodes. The double cover X of S branched over the nodes is a regular surface of general type with $p_g = 4$ and $c_1^2 = 10$. It is easy to see that the canonical linear system on X is the preimage of the canonical linear system on S. This gives an example of a surface of general type such that the canonical linear system maps the surface onto a canonically embedded normal surface, a counter-example to Babbage's conjecture (see [69]).

Exercises

4.1 Find explicitly all equivalence classes of linear determinantal representations of a nodal or a cuspidal cubic.

4.2 Show that a general binary form admits a unique equivalence class of symmetric determinantal representations.

4.3 The following problems lead to a symmetric determinantal expression of a plane rational curve [347].

(i) Show that, for any two degree d binary forms $p(u_0, u_1)$ and $q(u_0, u_1)$, there

exists a unique $d \times d$ symmetric matrix $B(p,q) = (b_{ij})$ whose entries are bilinear functions of the coefficients of p and q such that

$$p(u_0,u_1)q(v_0,v_1) - q(u_0,u_1)p(v_0,v_1) = (u_0v_1 - u_1v_0)\sum b_{ij}u_0^i u^{d-j}v_0^j v_1^{d-j}.$$

(ii) Show that the determinant of $B(p,q)$ (the *bezoutiant* of p,q) vanishes if and only if the two binary forms have a common zero.

(iii) Let p_0,p_1,p_2 be three binary forms of degree d without common zeros and C be the image of the map $\mathbb{P}^1 \to \mathbb{P}^2, [u_0,u_1] \mapsto [p_0(u_0,u_1),p_1(u_0,u_1),p_2(u_0,u_1)]$. Show that C is given by the equation $f(t_0,t_1,t_2) = |B(t_0p_1 - t_1p_0, t_0p_2 - t_2p_0)| = 0$.

(iv) Prove that $f = |t_0B(p_1,p_2) - t_1B(t_0,t_2) - t_2B(t_0,t_1)|$ and any symmetric determinantal equation of C is equivalent to this.

4.4 Let $C = V(f)$ be a nonsingular plane cubic, p_1,p_2,p_3 be three non-collinear points. Let (A_0,A_1,A_2) define a quadratic Cremona transformation with fundamental points p_1,p_2,p_3. Let q_1,q_2,q_3 be another set of three points such that the six points p_1,p_2,p_3,q_1,q_2,q_3 are cut out by a conic. Let (B_0,B_1,B_2) define a quadratic Cremona transformation with fundamental points q_1,q_2,q_3. Show that

$$F^{-3}\det \text{adj}\begin{pmatrix} A_0B_0 & A_0B_1 & A_0B_2 \\ A_1B_0 & A_1B_1 & A_1B_2 \\ A_2B_0 & A_2B_1 & A_2B_2 \end{pmatrix}$$

is a determinantal equation of C.

4.5 Find determinantal equations for a nonsingular quadric surface in $\mathbb{P}^3$.

4.6 Let $E \subset \text{Mat}_d$ be a linear subspace of dimension 3 of the space of $d \times d$ matrices. Show that the locus of points $x \in \mathbb{P}^{d-1}$ such that there exists $A \in E$ such that $Ax = 0$ is defined by $\binom{d}{3}$ equations of degree 3. In particular, for any determinantal equation of a curve C, the images of C under the maps $\mathfrak{r} : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ and $\mathfrak{l} : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ are defined by such a system of equations.

4.7 Show that the variety of nets of quadrics in $\mathbb{P}^n$ whose discriminant curve is singular is reducible.

4.8 Let $C = V(\det A)$ be a linear determinantal representation of a plane curve C of degree d defiined by a rank 1 torsion-free sheaf $\mathcal{F}$ of global type $\pi : C' \to C$. Show that the rational map $\mathfrak{l} : C \to \mathbb{P}^{d-1}, x \mapsto |N(A(x))|$ extends to a regular map $C' \to \mathbb{P}^{d-1}$.

4.9 Let C be a non-hyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}^3$.

(i) Show that the homogeneous ideal of C in $\mathbb{P}^3$ is generated by four cubic polynomials f_0,f_1,f_2,f_3.

(ii) Show that the equation of any quartic surface containing C can be written in the form $\sum l_if_i = 0$, where l_i are linear forms.

(iii) Show that (f_0,f_1,f_2,f_3) define a birational map f from $\mathbb{P}^3$ to $\mathbb{P}^3$. The image of any quartic containing C is another quartic surface.

4.10 Show that a curve of degree 6 and genus 3 in $\mathbb{P}^3$ is projectively normal if and only if it is not hyperelliptic.

4.11 Let C be a nonsingular plane curve of degree d and $\mathcal{L}_0 \in \text{Pic}^{g-1}(C)$ with $h^0(\mathcal{L}_0) \neq 0$. Show that the image of C under the map given by the complete linear system $\mathcal{L}_0(1)$ is a singular curve.

4.12 Let θ be a theta characteristic on a nonsingular plane curve of degree d with $h^0(\theta) = 1$. Show that the corresponding aCM sheaf on $\mathbb{P}^2$ defines an equation of C expressed as the determinant of a symmetric $(d-1) \times (d-1)$ matrix $(a_{ij}(t))$, where $a_{ij}(t)$ are of degree 1 for $1 \leq i, j \leq d-3$, $a_{1j}(t)$ are of degree 2, and $a_{d-1d-1}(t)$ is of degree 3 [37].

4.13 Let $S = V(\det A)$ be a linear determinantal representation of a nonsingular quartic surface in $\mathbb{P}^3$. Show that the four 3×3 minors of the matrix B obtained from A by deleting one row define the equations of a projectively normal curve of degree 6 and genus 3 lying on S.

4.14 Show that any quartic surfaces containing a line and a rational normal cubic not intersecting the line admits a determinantal representation.

4.15 Show that the Hessian hypersurface of a general cubic hypersurface in $\mathbb{P}^4$ is hypersurface of degree 5 whose singular locus is a curve of degree 20. Show that its general hyperplane section is a quintic symmetroid surface.

4.16 Let C be a curve of degree $N(d) = d(d-1)/2$ and arithmetic genus $G(d) = \frac{1}{6}(d-2)(d-3)(2d+1)$ on a smooth surface of degree d in $\mathbb{P}^3$. Show that the linear system $|\mathcal{O}_S(-C)(d)|$ consists of curves of degree $N(d+1)$ and arithmetic genus $G(d+1)$.

4.17 Let S be a general symmetroid quintic surface in $\mathbb{P}^3$ and $|L|$ be the linear system of projectively normal curves of degree 10 and genus 11 that defines a symmetric linear determinantal representation of S and let S' be the image of S under the rational map $\Phi : \mathbb{P}^3 \to \mathbb{P}^d = |\mathcal{O}_C|^\vee$. Let W be the web of quadrics defining the linear determinantal representation of S. Consider the rational map $T : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ defined by sending a point $x \in \mathbb{P}^4$ to the intersection of polar hyperplanes $P_x(Q), Q \in W$. Prove the following assertions (see [571].

(i) The fundamental locus of T (where T is not defined) is equal to S'.
(ii) The image of a general hyperplane H is a quartic hypersurface X_H.
(iii) The intersection of two such quartics X_H and $X_{H'}$ is equal to the union of the surface S' and a surface F of degree 6.
(iv) Each 4-secant line of C contained in H (there are 20 of them) is blown down under T to 20 nodes of X_H.

4.18 Let $p_1, \ldots, p_5$ be five points in $\mathbb{P}^3$ in general linear position. Prove the following assertions (see [572]).

(i) Show that one can choose a point q_{ij} on the line $\overline{p_i p_j}$ such that the lines $\overline{p_1 q_{34}}$, $\overline{p_2 q_{45}}, \overline{p_3 q_{25}}, \overline{p_4 q_{12}}, \overline{p_5 q_{23}}$ form a closed space pentagon.
(ii) Show that the union of five lines $\overline{p_i p_j}$ and five lines defined in (i) is a curve of arithmetic genus 11.
(ii) Show that the linear system of quartic surfaces containing the 10 lines maps $\mathbb{P}^3$ to a quartic hypersurface in $\mathbb{P}^4$ with 45 nodes (the *Burhardt quartic threefold*).

4.19 Show that the equivalence classes of determinantal representations of plane curve C of degree $2k$ with quadratic forms as entries correspond to aCM sheaves on C satisfying $h^0(\mathcal{F}(-1)) = 0$ and $\mathcal{F}(-\frac{1}{2}d-2)^\vee \cong \mathcal{F}(-\frac{1}{2}(d-2))$.

4.20 Show that the union of d different hyperplanes in $\mathbb{P}^n$ always admits a unique equivalence class of symmetric linear determinant representations.

Historical Notes

Apparently, O. Hesse was the first to state clearly the problem of representation of the equation of a hypersurface as a symmetric determinant of linear forms [291]. He was able to do it for plane curves of order 4 [292]. He also showed that it can be done in 36 different ways corresponding to 36 families of contact cubics. For cubic curves the representation follows from the fact that any cubic curve can be written in three ways as the Hessian curve. This fact was also proven by Hesse [288], p. 89. The fact that a general plane curve of degree d can be defined by the determinant of a symmetric $d \times d$ matrix with entries as homogeneous linear forms was first proved by A. Dixon [169]. Dixon's result was reproved later by J. Grace and A. Young[258]. Modern expositions of Dixon's theory were given by A. Beauville [33] and A. Tyurin [566], [567].

The first definition of non-invertible theta characteristics on a singular curve was given by W. Barth. It was studied for nodal planes curves by A. Beauville [33] and F. Catanese [71], and for arbitrary singular curves of degree ≤ 4, by C.T.C. Wall [594].

It was proved by L. Dickson [166] that any plane curve can be written as the determinant of not necessarily symmetric matrix with linear homogeneous forms as its entries. The relationship between linear determinantal representations of an irreducible plane curve of degree d and line bundles of degree $d(d-1)/2$ was first established in [126]. This was later elaborated by V. Vinnikov [592]. A deep connection between linear determinantal representations of real curves and the theory of colligations for pairs of commuting operators in a Hilbert space was discovered by M. Lifšic [364] and his school (see [365]).

The theory of linear determinantal representation for cubic surfaces was developed by L. Cremona [144]. Dickson proves in [166] that a general homogeneous form of degree $d > 2$ in r variables cannot be represented as a linear determinant unless $r = 3$ or $r = 4, d \leq 3$. The fact that a determinantal representations of quartic surfaces is possible only if the surface contains a projectively normal curve of genus 3 and degree 6 goes back to F. Schur [506]. However, it was A. Coble who was the first to understand the reason: by Noether's theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section [121], p. 39. The case of quartic surfaces was studied in detail in a series of papers of T. Room [475]. Quartic symmetroid surfaces were first studied by A. Cayley [89]. They appear frequently in algebraic geometry. Coble's paper [119] studies (in a disguised form) the group of birational automorphisms of such surfaces. There is a close relationship between quartic symmetroids and Enriques surfaces (see [132]. M. Artin and D. Mumford [17]

used quartic symmetroids in their celebrated constru ction of counter-examples to the Lüroth Problem. A modern theory of symmetroid surfaces can be found in papers of A. Beauville [37] and F. Catanese [69].

We refer to [37] for a comprehensive survey of modern theory of determinantal representations of hypersurfaces based on the theory of aCM sheaves. One can find numerous special examples of determinantal representations in this paper. We followed this exposition in many places.

In classical algebraic geometry, a determinantal representation was considered as a special case of a projective generation of subvarieties in a projective space. It seems that the geometric theory of determinantal varieties started from the work of H. Grassmann in 1856 [262], where he considers the projective generation of a cubic surface by three collinear nets of planes. Grassmann's construction was greatly generalized in a series of papers of T. Reye [467]. In the last paper of the series he studies curves of degree 10 and genus 11 which lead to linear determinantal representation of quintic surfaces.

Algebraic theory of determinantal varieties started from the work of F. S. Macaulay [373], where the fact that the loci of rank $\leq r$ square matrices are Cohen-Macaulay varieties can be found. The classical account of the theory of determinantal varieties is T. Room's monograph [476]. A modern treatment of determinantal varieties can be found in several books [10], [231], [267]. The book by W. Bruns and U. Vetter [58] gives a rather complete account of the recent development of the algebraic theory of determinantal ideals. The formula for the dimensions and the degrees of determinantal varieties in the general case of $m \times n$ matrices and also symmetric matrices goes back to C. Segre [522] and G. Giambelli [244], [245].

5

Theta characteristics

5.1 Odd and even theta characteristics

5.1.1 First definitions and examples

We have already dealt with theta characteristics on a plane curves in the previous Chapter. Here we will study theta characteristics on any nonsingular projective curve in more detail.

It follows from the definition that two theta characteristics, considered as divisor classes of degree $g-1$, differ by a 2-torsion divisor class. Since the 2-torsion subgroup $\mathrm{Jac}(C)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, there are 2^{2g} theta characteristics. However, in general, there is no canonical identification between the set $\mathrm{TChar}(C)$ of theta characteristics on C and the set $\mathrm{Jac}(C)[2]$. One can say only that $\mathrm{TChar}(C)$ is an affine space over the vector space of $\mathrm{Jac}(C)[2] \cong \mathbb{F}_2^{2g}$.

There is one more structure on $\mathrm{TChar}(C)$ besides being an affine space over $\mathrm{Jac}(C)[2]$. Recall that the subgroup of 2-torsion points $\mathrm{Jac}(C)[2]$ is equipped with a natural symmetric bilinear form over $\mathbb{F}_2$, called the *Weil pairing*. It is defined as follows (see [10], Appendix B). Let ϵ, ϵ' be two 2-torsion divisor classes. Choose their representatives D, D' with disjoint supports. Write $\mathrm{div}(\phi) = 2D, \mathrm{div}(\phi') = 2D'$ for some rational functions ϕ and ϕ'. Then $\frac{\phi(D')}{\phi'(D)} = \pm 1$. Here, for any rational function ϕ defined at points x_i, $\phi(\sum_i x_i) = \prod_i \phi(x_i)$. Now we set

$$\langle \epsilon, \epsilon' \rangle = \begin{cases} 1 & \text{if } \phi(D')/\phi'(D) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Weil pairing is a symplectic form, i.e. satisfies $\langle \epsilon, \epsilon \rangle = 0$. One can show that it is a nondegenerate symplectic form (see [406]).

For any $\vartheta \in \mathrm{TChar}(C)$, define the function

$$q_\vartheta : \mathrm{Jac}(C)[2] \to \mathbb{F}_2,\ \epsilon \mapsto h^0(\vartheta+\epsilon) + h^0(\vartheta).$$

The proof of the following Theorem can be found in [10], p. 290).

Theorem 5.1.1 (Riemann-Mumford Relation) *The function q_ϑ is a quadratic form on* $\mathrm{Jac}(C)[2]$ *whose associated symmetric bilinear form is equal to the Weil pairing.*

Later we shall see that there are two types of quadratic forms associated to a fixed nondegenerate symplectic form: even and odd. They agree with our definition of an even and odd theta characteristic. The number of even (odd) theta characteristics is equal to $2^{g-1}(2^g+1)$ $(2^{g-1}(2^g-1))$.

An odd theta characteristic ϑ is obviously effective, i.e. $h^0(\vartheta) > 0$. If C is a canonical curve, then divisor $D \in |\vartheta|$ satisfies the property that $2D$ is cut out by a hyperplane H in the space $|K_C|^\vee$, where C is embedded. Such a hyperplane is called a *contact hyperplane*. It follows from the above that a canonical curve either has $2^{g-1}(2^g-1)$ contact hyperplanes or infinitely many. The latter case happens if and only if there exists a theta characteristic ϑ with $h^0(\vartheta) > 1$. Such a theta characteristic is called a *vanishing theta characteristic*. An example of a vanishing odd theta characteristic is the divisor class of a line section of a plane quintic curve. An example of a vanishing even theta characteristic is the unique g^1_3 on a canonical curve of genus 4 lying on a singular quadric.

The geometric interpretation of an even theta characteristic is more subtle. In the previous Chapter we related theta characteristics, both even and odd, to determinantal representations of plane curves. The only known geometrical construction related to space curves that I know is the Scorza construction of a quartic hypersurface associated to a canonical curve and a non-effective theta characteristic. We will discuss this construction in Section 5.5.

5.1.2 Quadratic forms over a field of characteristic 2

Recall that a quadratic form on a vector space V over a field $\mathbb{K}$ is a map $q : V \to \mathbb{K}$ such that $q(av) = a^2 q(v)$ for any $a \in \mathbb{K}$ and any $v \in V$, and the map

$$b_q : V \times V \to \mathbb{K}, \quad (v,w) \mapsto q(v+w) - q(v) - q(w)$$

is bilinear (it is called the *polar bilinear form*). We have $b_q(v,v) = 2q(v)$ for any $v \in V$. In particular, q can be reconstructed from b_q if $\mathrm{char}(\mathbb{K}) \neq 2$. In the case when $\mathrm{char}(\mathbb{K}) = 2$, we get $b_q(v,v) \equiv 0$, hence b_q is a symplectic bilinear form. Two quadratic forms q, q' have the same polar bilinear form if and only if $q - q' = l$, where $l(v+w) = l(v) + l(w), l(av) = a^2 l(v)$ for any

$v, w \in V, a \in \mathbb{K}$. If $\mathbb{K}$ is a finite field of characteristic 2, $\sqrt{l}$ is a linear form on V, and we obtain

$$b_q = b_{q'} \Longleftrightarrow q = q' + \ell^2 \tag{5.1}$$

for a unique linear form $\ell : V \to \mathbb{K}$.

Let $e_1, \ldots, e_n$ be a basis in V and $A = (a_{ij}) = (b_q(e_i, e_j))$ be the matrix of the bilinear form b_q. It is a symmetric matrix with zeros on the diagonal if $\mathrm{char}(\mathbb{K}) = 2$. It follows from the definition that

$$q(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i^2 q(e_i) + \sum_{1 \le i < j \le n} x_i x_j a_{ij}.$$

The *rank* of a quadratic form is the rank of the matrix A of the polar bilinear form. A quadratic form is called *nondegenerate* if the rank is equal to $\dim V$. In coordinate-free way this is the rank of the linear map $V \to V^\vee$ defined by b_q. The kernel of this map is called the *radical* of b_q. The restriction of q to the radical is identically zero. The quadratic form q arises from a nondegenerate quadratic form on the quotient space. In the following we assume that q is nondegenerate.

A subspace L of V is called *singular* if $q|L \equiv 0$. Each singular subspace is an *isotropic subspace* with respect to b_q, i.e. $b_q(v, w) = 0$ for any $v, w \in E$. The converse is true only if $\mathrm{char}(\mathbb{K}) \neq 2$.

Assume $\mathrm{char}(\mathbb{K}) = 2$. Since b_q is a nondegenerate symplectic form, $n = 2k$, and there exists a basis $e_1, \ldots, e_n$ in V such that the matrix of b_q is equal to

$$J_k = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}. \tag{5.2}$$

We call such a basis a *standard symplectic basis*. In this basis

$$q(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i^2 q(e_i) + \sum_{i=1}^{k} x_i x_{i+k}.$$

Assume, additionally, that $\mathbb{K}^* = \mathbb{K}^{*2}$, i.e., each element in $\mathbb{K}$ is a square (e.g. $\mathbb{K}$ is a finite or algebraically closed field). Then, we can further reduce q to the form

$$q(\sum_{i=1}^{2k} x_i e_i) = (\sum_{i=1}^{n} \alpha_i x_i)^2 + \sum_{i=1}^{k} x_i x_{i+k}, \tag{5.3}$$

where $q(e_i) = \alpha_i^2,\ i = 1, \ldots, n$. This makes (5.1) more explicit. Fix a nondegenerate symplectic form $\langle , \rangle : V \times V \to \mathbb{K}$. Each linear function on V is

given by $\ell(v) = \langle v, \eta\rangle$ for a unique $\eta \in V$. By (5.1), two quadratic forms q, q' with the polar bilinear form equal to $\langle , \rangle$ satisfy

$$q(v) = q'(v) + \langle v, \eta\rangle^2$$

for a unique $\eta \in V$. Choose a standard symplectic basis. The quadratic form defined by

$$q_0(\sum_{I=1}^{2k} x_i e_i) = \sum_{i=1}^{k} x_i x_{i+k}$$

has the polar bilinear form equal to the standard symplectic form. Any other form with the same polar bilinear form is defined by

$$q(v) = q_0(v) + \langle v, \eta_q\rangle^2,$$

where

$$\eta_q = \sum_{i=1}^{2k} \sqrt{q(e_i)} e_i.$$

From now on, $\mathbb{K} = \mathbb{F}_2$, the field of two elements. In this case $a^2 = a$ for any $a \in \mathbb{F}_2$. Formula (5.1) shows that the set $Q(V)$ of quadratic forms associated to the standard symplectic form is an affine space over V with addition $q+\eta, q \in Q(V), \eta \in V$, defined by

$$(q + \eta)(v) = q(v) + \langle v, \eta\rangle = q(v + \eta) + q(\eta). \tag{5.4}$$

The number

$$\text{Arf}(q) = \sum_{i=1}^{k} q(e_i)q(e_{i+k}) \tag{5.5}$$

is called the *Arf invariant* of q. One can show that it is independent of the choice of a standard symplectic basis (see [268], Proposition 1.11). A quadratic form $q \in Q(V)$ is called *even* (resp. *odd*) if $\text{Arf}(q) = 0$ (resp. $\text{Arf}(q) = 1$).

If we choose a standard symplectic basis for b_q and write q in the form $q_0 + \eta_q$, then we obtain

$$\text{Arf}(q) = \sum_{i=1}^{k} \alpha_i \alpha_{i+k} = q_0(\eta_q) = q(\eta_q). \tag{5.6}$$

In particular, if $q' = q + v = q_0 + \eta_q + v$,

$$\text{Arf}(q') + \text{Arf}(q) = q_0(\eta_q + v) + q_0(\eta_q) = q_0(v) + \langle v, \eta_q\rangle = q(v). \tag{5.7}$$

It follows from (5.6) that the number of even (resp. odd) quadratic forms is equal to the cardinality of the set $q_0^{-1}(0)$ (resp. $q_0^{-1}(1)$). We have

$$|q_0^{-1}(0)| = 2^{k-1}(2^k+1), \quad |q_0^{-1}(1)| = 2^{k-1}(2^k-1). \tag{5.8}$$

This is easy to prove by using induction on k.

Let $\mathrm{Sp}(V)$ be the group of linear automorphisms of the symplectic space V. If we choose a standard symplectic basis then

$$\mathrm{Sp}(V) \cong \mathrm{Sp}(2k, \mathbb{F}_2) = \{X \in \mathrm{GL}(2k)(\mathbb{F}_2) : {}^tX \cdot J_k \cdot X = J_k\}.$$

It is easy to see by induction on k that

$$|\mathrm{Sp}(2k, \mathbb{F}_2)| = 2^{k^2}(2^{2k}-1)(2^{2k-2}-1)\cdots(2^2-1). \tag{5.9}$$

The group $\mathrm{Sp}(V)$ has two orbits in $Q(V)$, the set of even and the set of odd quadratic forms. An even quadratic form is equivalent to the form q_0 and an odd quadratic form is equivalent to the form

$$q_1 = q_0 + e_k + e_{2k},$$

where $(e_1, \ldots, e_{2k})$ is the standard symplectic basis. Explicitly,

$$q_1(\sum_{i=1}^{2k} x_i e_i) = \sum_{i=1}^{k} x_i x_{i+k} + x_k^2 + x_{2k}^2.$$

The stabilizer subgroup $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) of an even quadratic form (resp. an odd quadratic form) is a subgroup of $\mathrm{Sp}(V)$ of index $2^{k-1}(2^k+1)$ (resp. $2^{k-1}(2^k-1)$). If $V = \mathbb{F}_2^{2k}$ with the symplectic form defined by the matrix J_k, then $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) is denoted by $\mathrm{O}(2k, \mathbb{F}_2)^+$ (resp. $\mathrm{O}(2k, \mathbb{F}_2)^-$).

5.2 Hyperelliptic curves

5.2.1 Equations of hyperelliptic curves

Let us first describe explicitly theta characteristics on hyperelliptic curves. Recall that a hyperelliptic curve of genus g is a nonsingular projective curve X of genus $g > 1$ admitting a degree 2 map $\varphi : C \to \mathbb{P}^1$. By Hurwitz formula, there are $2g+2$ branch points $p_1, \ldots, p_{2g+2}$ in $\mathbb{P}^1$. Let $f_{2g+2}(t_0, t_1)$ be a binary form of degree $2g+2$ whose zeros are the branch points. The equation of C in the weighted projective plane $\mathbb{P}(1, 1, g+1)$ is

$$t_2^2 + f_{2g+2}(t_0, t_1) = 0. \tag{5.10}$$

Recall that a weighted projective space $\mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \ldots, q_n)$ is defined as the quotient $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by

$$t : [z_0, \ldots, z_n] \mapsto [t^{q_0} z_0, \ldots, t^{q_n} z_n].$$

A more general definition of $\mathbb{P}(\mathbf{q})$ which works over $\mathbb{Z}$ is

$$\mathbb{P}(\mathbf{q}) = \text{Proj } \mathbb{Z}[T_0, \ldots, T_n],$$

where the grading is defined by setting $\deg T_i = q_i$. Here $\mathbf{q} = (q_0, \ldots, q_n)$ are integers ≥ 1. We refer to [175] or [311] for the theory of weighted projective spaces and their subvarieties. Note that a hypersurface in $\mathbb{P}(\mathbf{q})$ is defined by a homogeneous polynomial where the unknowns are homogeneous of degree q_i. Thus Equation (5.10) defines a hypersurface of degree $2g + 2$. Although, in general, $\mathbb{P}(\mathbf{q})$ is a singular variety, it admits a canonical sheaf

$$\omega_{\mathbb{P}(\mathbf{q})} = \mathcal{O}_{\mathbb{P}(\mathbf{q})}(-|\mathbf{q}|),$$

where $|\mathbf{q}| = q_0 + \cdots + q_n$. Here the Serre sheaves are understood in the sense of theory of projective spectrums of graded algebras. There is also the adjunction formula for a hypersurface $X \subset \mathbb{P}(\mathbf{q})$ of degree d

$$\omega_X = \mathcal{O}_X(d - |\mathbf{q}|). \tag{5.11}$$

In the case of a hyperelliptic curve, we have

$$\omega_C = \mathcal{O}_C(g - 1).$$

The morphism $\varphi : C \to \mathbb{P}^1$ corresponds to the projection $[t_0, t_1, t_2] \mapsto [t_0, t_1]$ and we obtain that

$$\omega_C = \varphi^* \mathcal{O}_{\mathbb{P}^1}(g - 1).$$

The weighted projective space $\mathbb{P}(1, 1, g + 1)$ is isomorphic to the projective cone in $\mathbb{P}^{g+2}$ over the Veronese curve $v_{g+1}(\mathbb{P}^1) \subset \mathbb{P}^{g+1}$. The hyperelliptic curve is isomorphic to the intersection of this cone and a quadric hypersurface in $\mathbb{P}^{g+1}$ not passing through the vertex of the cone. The projection from the vertex to the Veronese curve is the double cover $\varphi : C \to \mathbb{P}^1$. The canonical linear system $|K_C|$ maps C to $\mathbb{P}^g$ with the image equal to the Veronese curve $v_{g-1}(\mathbb{P}^1)$.

5.2.2 2-torsion points on a hyperelliptic curve

Let $c_1, \ldots, c_{2g+2}$ be the ramification points of the map φ. We assume that $\varphi(c_i) = p_i$. Obviously, $2c_i - 2c_j \sim 0$, hence the divisor class of $c_i - c_j$ is of

order 2 in $\mathrm{Pic}(C)$. Also, for any subset I of the set $B_g = \{1, \dots, 2g+2\}$,

$$\alpha_I = \sum_{i \in I} c_i - \#I c_{2g+2} = \sum_{i \in I} (c_i - c_{2g+2}) \in \mathrm{Pic}(C)[2].$$

Now observe that

$$\alpha_{B_g} = \sum_{i \in B_g} c_i - (2g+2) c_{2g+2} = \mathrm{div}(\phi) \sim 0, \tag{5.12}$$

where $\phi = t_2/(bt_0 - at_1)^{g+1}$ and $p_{2g+2} = [a, b]$ (we consider the fraction modulo Equation (5.10) defining C). Thus

$$c_i - c_j \sim 2c_i + \sum_{k \in B_g \setminus \{j\}} c_k - (2g+2) c_{2g+2} \sim \alpha_{B_g \setminus \{i,j\}}.$$

Adding to α_I the zero divisor $c_{2g+2} - c_{2g+2}$, we can always assume that $\#S$ is even. Also adding the principal divisor α_{B_g}, we obtain that $\alpha_I = \alpha_{\bar{I}}$, where $\bar{I}$ denotes $B_g \setminus I$.

Let $\mathbb{F}_2^{B_g} \cong \mathbb{F}_2^{2g+2}$ be the $\mathbb{F}_2$-vector space of functions $B_g \to \mathbb{F}_2$, or, equivalently, subsets of B_g. The sum is defined by the symmetric sum of subsets

$$I + J = I \cup J \setminus (I \cap J).$$

The subsets of even cardinality form a hyperplane. It contains the subsets $\emptyset$ and B_g as a subspace of dimension 1. Let E_g denote the quotient space. Elements of E_g are represented by subsets of even cardinality up to the complementary set (*bifid maps* in terminology of A. Cayley). We have

$$E_g \cong \mathbb{F}_2^{2g},$$

hence the correspondence $I \mapsto \alpha_I$ defines an isomorphism

$$E_g \cong \mathrm{Pic}(C)[2]. \tag{5.13}$$

Note that E_g carries a natural symmetric bilinear form

$$e : E_g \times E_g \to \mathbb{F}_2, \quad e(I, J) = \#I \cap J \mod 2. \tag{5.14}$$

This form is symplectic (i.e. $e(I, I) = 0$ for any I) and nondegenerate. The subsets

$$A_i = \{2i-1, 2i\}, \quad B_i = \{2i, 2i+1\}, \quad i = 1, \dots, g, \tag{5.15}$$

form a standard symplectic basis.

Under isomorphism (5.13), this bilinear form corresponds to the Weil pairing on 2-torsion points of the Jacobian variety of C.

Remark 5.2.1 The symmetric group $\mathfrak{S}_{2g+2}$ acts on E_g via its action on B_g and preserves the symplectic form e. This defines a homomorphism

$$s_g : \mathfrak{S}_{2g+2} \to \mathrm{Sp}(2g, \mathbb{F}_2).$$

If $g = 1$, $\mathrm{Sp}(2, \mathbb{F}_2) \cong \mathfrak{S}_3$, and the homomorphism s_1 has the kernel isomorphic to the group $(\mathbb{Z}/2\mathbb{Z})^2$. If $g = 2$, the homomorphism s_2 is an isomorphism. If $g > 2$, the homomorphism s_g is injective but not surjective.

5.2.3 Theta characteristics on a hyperelliptic curve

For any subset T of B_g set

$$\vartheta_T = \sum_{i \in T} c_i + (g - 1 - \#T c_{2g+2}) = \alpha_T + (g-1)c_{2g+2}.$$

We have

$$2\vartheta_T \sim 2\alpha_T + (2g-2)c_{2g+2} \sim (2g-2)c_{2g+2}.$$

It follows from the proof of the Hurwitz formula that

$$K_C = \varphi^*(K_{\mathbb{P}^1}) + \sum_{i \in B_g} c_i.$$

Choose a representative of $K_{\mathbb{P}^1}$ equal to $-2p_{2g+2}$ and use (5.12) to obtain

$$K_C \sim (2g-2)c_{2g+2}.$$

This shows that ϑ_T is a theta characteristic. Again adding and subtracting c_{2g+2} we may assume that $\#T \equiv g + 1 \mod 2$. Since T and $\bar{T}$ define the same theta characteristic, we will consider the subsets up to taking the complementary set. We obtain a set Q_g which has a natural structure of an affine space over E_g, the addition is defined by

$$\vartheta_T + \alpha_I = \vartheta_{T+I}.$$

Thus all theta characteristics are uniquely represented by the divisor classes ϑ_T, where $T \in Q_g$.

An example of an affine space over $V = \mathbb{F}_2^{2g}$ is the space of quadratic forms $q : \mathbb{F}_2^{2g} \to \mathbb{F}_2$ whose associated symmetric bilinear form b_q coincides with the standard symplectic form defined by (5.2). We identify V with its dual $V^\vee$ by means of b_0 and set $q + l = q + l^2$ for any $l \in V^\vee$.

For any $T \in Q_g$, we define the quadratic form q_T on E_g by

$$q_T(I) = \tfrac{1}{2}(\#(T + I) - \#T) = \#T \cap I + \tfrac{1}{2}\#I = \tfrac{1}{2}\#I + e(I, T) \mod 2.$$

We have (all equalities are modulo 2)

$$q_T(I+J)+q_T(I)+q_T(J)$$

$$=\tfrac{1}{2}(\#(I+J)+\#I+\#J)+e(I+J,T)+e(I,T)+e(J,T)=\#I\cap J.$$

Thus each theta characteristic can be identified with an element of the space $Q_g=Q(E_g)$ of quadratic forms on E_g with polar form e.

Also notice that

$$(q_T+\alpha_I)(J)=q_T(J)+e(I,J)=\tfrac{1}{2}\#J+e(T,J)+e(I,J)$$

$$=\tfrac{1}{2}\#J+e(T+I,J)=q_{T+I}(J).$$

Lemma 5.2.2 *Let ϑ_T be a theta characteristic on a hyperelliptic curve C of genus g identified with a quadratic form on E_g. Then the following properties are equivalent:*

(i) $\#T\equiv g+1 \mod 4$;
(ii) $h^0(\vartheta_T)\equiv 0 \mod 2$;
(iii) q_T *is even.*

Proof Without loss of generality, we may assume that p_{2g+2} is the point $(0,1)$ at infinity in $\mathbb{P}^1$. Then the field of rational functions on C is generated by the functions $y=t_2/t_0$ and $x=t_1/t_0$. We have

$$\vartheta_T=\sum_{i\in T}c_i+(g-1-\#T)c_{2g+2}\sim(g-1+\#T)c_{2g+2}-\sum_{i\in T}c_i.$$

Any function ϕ from the space $L(\vartheta_T)=\{\phi:\operatorname{div}(\phi)+\vartheta_T\geq 0\}$ has a unique pole at c_{2g+2} of order $<2g+1$. Since the function y has a pole of order $2g+1$ at c_{2g+2}, we see that $\phi=\varphi^*(p(x))$, where $p(x)$ is a polynomial of degree $\leq\frac{1}{2}(g-1+\#T)$ in x. Thus $L(\vartheta_T)$ is isomorphic to the linear space of polynomials $p(x)$ of degree $\leq\frac{1}{2}(g-1+\#T)$ with zeros at $p_i, i\in T$. The dimension of this space is equal to $\frac{1}{2}(g+1-\#T)$. This proves the equivalence of (i) and (ii).

Let

$$U=\{1,3,\dots,2g+1\}\subset B_g \tag{5.16}$$

be the subset of odd numbers in B_g. If we take the standard symplectic basis in E_g defined in (5.15), then we obtain that $q_U=q_0$ is the standard quadratic form associated to the standard symplectic basis. It follows from (5.6) that q_T is an even quadratic form if and only if $T=U+I$, where $q_U(I)=0$. Let I consists of k even numbers and s odd numbers. Then $q_U(I)=\#U\cap I+\frac{1}{2}\#I=m+\frac{1}{2}(k+m)=0 \mod 2$. Thus $\#T=\#(U+S)=\#U+\#I-$

$2\#U \cap S = (g+1) + (k+m) - 2m = g + 1 + k - m$. Then $m + \frac{1}{2}(k+m)$ is even, hence $3m + k \equiv 0 \mod 4$. This implies that $k - m \equiv 0 \mod 4$ and $\#T \equiv g + 1 \mod 4$. Conversely, if $\#T \equiv g + 1 \mod 4$, then $k - m \equiv 0 \mod 4$ and $q_U(I) = 0$. This proves the assertion. □

5.2.4 Families of curves with odd or even theta characteristic

Let $\mathcal{X} \to S$ be a smooth projective morphism whose fiber X_s over a point $s \in S$ is a curve of genus $g > 0$ over the residue field $\kappa(s)$ of s. Let $\mathbf{Pic}^n_{\mathcal{X}/S} \to S$ be the *relative Picard scheme* of $\mathcal{X}/S$. It represents the sheaf in étale topology on S associated to the functor on the category of S-schemes defined by assigning to a S-scheme T the group $\mathrm{Pic}^d(X \times_S T)$ of isomorphism classes of invertible sheaves on $X \times_S T$ of relative degree n over T modulo tensor product with invertible sheaves coming from T. The S-scheme $\mathbf{Pic}^n_{\mathcal{X}/S} \to S$ is a smooth projective scheme over S. Its fiber over a point $s \in S$ is isomorphic to the Picard variety $\mathbf{Pic}^n_{\mathcal{X}_s/\kappa(s)}$ over the field $\kappa(s)$. The relative Picard scheme comes with a universal invertible sheaf $\mathcal{U}$ on $\mathcal{X} \times_S \mathbf{Pic}^n_{\mathcal{X}/S}$ (locally in étale topology). For any point $y \in \mathbf{Pic}^n_{\mathcal{X}/S}$ over a point $s \in S$, the restriction of $\mathcal{U}$ to the fiber of the second projection over y is an invertible sheaf $\mathcal{U}_y$ on $X_s \otimes_{\kappa(s)} \kappa(y)$ representing a point in $\mathrm{Pic}^n(\mathcal{X}_s \otimes \kappa(y))$ defined by y.

For any integer m, raising a relative invertible sheaf into m-th power defines a morphism

$$[m] : \mathbf{Pic}^n_{\mathcal{X}/S} \to \mathbf{Pic}^{mn}_{\mathcal{X}/S}.$$

Taking $n = 2g - 2$ and $m = 2$, the preimage of the section defined by the relative canonical class $\omega_{\mathcal{X}/S}$ is a closed subscheme of $\mathbf{Pic}^{g-1}_{\mathcal{X}/S}$. It defines a finite cover

$$\mathcal{TC}_{\mathcal{X}/S} \to S$$

of degree 2^{2g}. The pull-back of $\mathcal{U}$ to $\mathcal{TC}_{\mathcal{X}/S}$ defines an invertible sheaf $\mathcal{T}$ over $\mathcal{P} = \mathcal{X} \times_S \mathcal{TC}_{\mathcal{X}/S}$ satisfying $\mathcal{T}^{\otimes 2} \cong \omega_{\mathcal{P}/\mathcal{TC}_{\mathcal{X}/S}}$. By a theorem of Mumford [406], the parity of a theta characteristic is preserved in an algebraic family, thus the function $\mathcal{TC}_{\mathcal{X}/S} \to \mathbb{Z}/2\mathbb{Z}$ defined by $y \mapsto \dim H^0(U_y, \mathcal{T}_y) \mod 2$ is constant on each connected component of $\mathcal{TC}_{\mathcal{X}/S}$. Let $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S}$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S}$) be the closed subset of $\mathcal{TC}_{\mathcal{X}/S}$, where this function takes the value 0 (resp. 1). The projection $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S} \to S$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S} \to S$) is a finite cover of degree $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$).

It follows from the above that $\mathcal{TC}_{\mathcal{X}/S}$ has at least two connected components.

Now take $S = |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ to be the space of nonsingular plane curves C

of degree d and $\mathcal{X} \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{\text{ns}}$ be the universal family of curves defined by $\{(x, C) : x \in C\}$. We set

$$\mathcal{TC}_d = \mathcal{TC}_{\mathcal{X}/S},\ \mathcal{TC}_d^{\text{ev/odd}} = \mathcal{TC}_{\mathcal{X}/S}^{\text{ev/odd}}.$$

The proof of the following Proposition can be found in [34].

Proposition 5.2.3 *If d is even or $d = 3$, $\mathcal{TC}_d$ consists of two irreducible components $\mathcal{TC}_d^{ev}$ and $\mathcal{TC}_d^{odd}$. If $d \equiv 1 \mod 4$, then $\mathcal{TC}_d^{ev}$ is irreducible but $\mathcal{TC}_d^{odd}$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$. If $d \equiv 3 \mod 4$, then $\mathcal{TC}_d^{odd}$ is irreducible but $\mathcal{TC}_d^{ev}$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$.*

Let $\mathcal{TC}_d^0$ be the open subset of $\mathcal{TC}_d^{\text{ev}}$ corresponding to the pairs (C, ϑ) with $h^0(\vartheta) = 0$. It follows from the theory of symmetric determinantal representations of plane curves that $\mathcal{TC}_d^0/\text{PGL}(3)$ is an irreducible variety covered by an open subset of a Grassmannian. Since the algebraic group $\text{PGL}(3)$ is connected and acts freely on a Zariski open subset of $\mathcal{TC}_d^0$, we obtain that $\mathcal{TC}_d^0$ is irreducible. It follows from the previous Proposition that

$$\mathcal{TC}_d^0 = \mathcal{TC}_d^{\text{ev}} \quad \text{if } d \not\equiv 3 \mod 4. \tag{5.17}$$

Note that there exist coarse moduli spaces $\mathcal{M}_g^{ev}$ and $\mathcal{M}_g^{\text{odd}}$ of curves of genus g together with an even (odd) theta characteristic. We refer to [130] for the proof of irreducibility of these varieties and for construction of a certain compactifications of these spaces.

5.3 Theta functions

5.3.1 Jacobian variety

Recall the definition of the Jacobian variety of a nonsingular projective curve C of genus g over $\mathbb{C}$. We consider C as a compact oriented 2-dimensional manifold of genus g. We view the linear space $H^0(C, K_C)$ as the space of holomorphic 1-forms on C. By integration over 1-dimensional cycles, we get a homomorphism of $\mathbb{Z}$-modules

$$\iota : H_1(C, \mathbb{Z}) \to H^0(C, K_C)^\vee, \quad \iota(\gamma)(\omega) = \int_\gamma \omega.$$

The image of this map is a lattice Λ of rank $2g$ in $H^0(C, K_C)^\vee$. The quotient by this lattice

$$\text{Jac}(C) = H^0(C, K_C)^\vee/\Lambda$$

is a complex g-dimensional torus. It is called the *Jacobian variety* of C.

Recall that the cap product

$$\cap : H_1(C,\mathbb{Z}) \times H_1(C,\mathbb{Z}) \to H_2(C,\mathbb{Z}) \cong \mathbb{Z}$$

defines a nondegenerate symplectic form on the group $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Let $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ be a standard symplectic basis. We choose a basis $\omega_1,\dots,\omega_g$ of holomorphic 1-differentials on C such that

$$\int_{\alpha_i} \omega_j = \delta_{ij}. \tag{5.18}$$

Let

$$\tau_{ij} = \int_{\beta_i} \omega_i.$$

The complex matrix $\tau = (\tau_{ij})$ is called the *period matrix*. The basis $\omega_1,\dots,\omega_g$ identifies $H^0(C,K_C)^\vee$ with $\mathbb{C}^g$ and the period matrix identifies the lattice Λ with the lattice $\Lambda_\tau = [\tau\ I_g]\mathbb{Z}^{2g}$, where $[\tau\ I_g]$ denotes the block-matrix of size $g \times 2g$. The period matrix $\tau = \Re(\tau) + \sqrt{-1}\Im(\tau)$ satisfies

$${}^t\tau = \tau, \quad \Im(\tau) > 0.$$

As is well-known (see [267]) this implies that $\mathrm{Jac}(C)$ is a projective algebraic group, i.e. an abelian variety. It is isomorphic to the Picard scheme $\mathbf{Pic}^0_{C/\mathbb{C}}$.

We consider any divisor $D = \sum n_x x$ on C as a 0-cycle on C. The divisors of degree 0 are boundaries, i.e. $D = \partial\gamma$ for some 1-chain β. By integrating over β we get a linear function on $H^0(C,K_C)$ whose coset modulo $\Lambda = \iota(H_1(C,\mathbb{Z}))$ does not depend on the choice of β. This defines a homomorphism of groups $p : \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$. The *Abel-Jacobi Theorem* asserts that p is zero on principal divisors (Abel's part), and surjective (Jacobi's part). This defines an isomorphism of abelian groups

$$\mathfrak{a} : \mathrm{Pic}^0(C) \to \mathrm{Jac}(C) \tag{5.19}$$

which is called the *Abel-Jacobi map*. For any positive integer d let $\mathrm{Pic}^d(C)$ denote the set of divisor classes of degree d. The group $\mathrm{Pic}^0(C)$ acts simply transitively on $\mathrm{Pic}^d(C)$ via addition of divisors. There is a canonical map

$$u_d : C^{(d)} \to \mathrm{Pic}^d(C),\ D \mapsto [D],$$

where we identify the symmetric product with the set of effective divisors of degree d. One can show that $\mathrm{Pic}^d(C)$ can be equipped with a structure of a projective algebraic variety (isomorphic to the Picard scheme $\mathbf{Pic}^d_{C/\mathbb{C}}$) such that the map u_d is a morphism of algebraic varieties. Its fibres are projective

spaces, the complete linear systems corresponding to the divisor classes of degree d. The action of $\mathrm{Pic}^0(C) = \mathrm{Jac}(C)$ on $\mathrm{Pic}^d(C)$ is an algebraic action equipping $\mathrm{Pic}^d(C)$ with a structure of a torsor over the Jacobian variety.

Let

$$W^r_{g-1} = \{[D] \in \mathrm{Pic}^{g-1}(C) : h^0(D) \geq r+1\}.$$

In particular, W^0_{g-1} was denoted by Θ in Theorem 4.1.4, where we showed that the invertible sheaves $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ defining a determinantal equation of a plane curve of genus g belong to the set $\mathrm{Pic}^{g-1}(C) \setminus W^0_{g-1}$. The fundamental property of the loci W^r_{g-1} is given by the following *Riemann-Kempf Theorem*.

Theorem 5.3.1

$$W^r_{g-1} = \{x \in W^0_{g-1} : \mathrm{mult}_x W^0_{g-1} \geq r+1\}.$$

Here mult_x *denotes the multiplicity of a hypersurface at the point* x.

In particular, we get

$$W^1_{g-1} = \mathrm{Sing}(W^0_{g-1}).$$

From now on we will identify $\mathrm{Pic}^0(C)$ with the set of points on the Jacobian variety $\mathrm{Jac}(C)$ by means of the Abel-Jacobi map. For any theta characteristic ϑ the subset

$$\Theta = W^0_{g-1} - \vartheta \subset \mathrm{Jac}(C)$$

is a hypersurface in $\mathrm{Jac}(C)$. It has the property that

$$h^0(\Theta) = 1, \quad [-1]^*(\Theta) = \Theta, \tag{5.20}$$

where $[m]$ is the multiplication by an integer m in the group variety $\mathrm{Jac}(C)$. Conversely, any divisor on $\mathrm{Jac}(C)$ satisfying these properties is equal to W^0_{g-1} translated by a theta characteristic. This follows from the fact that a divisor D on an abelian variety A satisfying $h^0(D) = 1$ defines a bijective map $A \to \mathrm{Pic}^0(A)$ by sending a point $x \in A$ to the divisor $t_x^* D - D$, where t_x is the translation map $a \mapsto a + x$ in the group variety, and $\mathrm{Pic}^0(A)$ is the group of divisor classes algebraically equivalent to zero. This fact implies that any two divisors satisfying properties (5.20) differ by translation by a 2-torsion point.

We call a divisor satisfying (5.20) a *symmetric theta divisor*. An abelian variety that contains such a divisor is called a *principally polarized abelian variety*.

Let $\Theta = W^0_{g-1} - \theta$ be a symmetric theta divisor on $\mathrm{Jac}(C)$. Applying Theorem 5.3.1 we obtain that, for any 2-torsion point $\epsilon \in \mathrm{Jac}(C)$, we have

$$\mathrm{mult}_\epsilon \Theta = h^0(\vartheta + \epsilon). \tag{5.21}$$

In particular, $\epsilon \in \Theta$ if and only if $\theta + \epsilon$ is an effective theta characteristic. According to ϑ, the symmetric theta divisors are divided into two groups: even and odd theta divisors.

5.3.2 Theta functions

The preimage of Θ under the quotient map $\mathrm{Jac}(C) = H^0(C, K_C)^\vee/\Lambda$ is a hypersurface in the complex linear space $V = H^0(C, K_C)^*$ equal to the zero set of some holomorphic function $\phi : V \to \mathbb{C}$. This function ϕ is not invariant with respect to translations by Λ. However, it has the property that, for any $v \in V$ and any $\gamma \in \Lambda$,

$$\phi(v + \gamma) = e_\gamma(v)\phi(v), \tag{5.22}$$

where e_γ is an invertible holomorphic function on V. A holomorphic function ϕ satisfying (5.22) is called a *theta function* with *theta factor* $\{e_\gamma\}$. The set of zeros of ϕ does not change if we replace ϕ with $\phi\alpha$, where α is an invertible holomorphic function on V. The function $e_\gamma(v)$ will change into the function $e_{\gamma'}(v) = e_\gamma(v)\phi(v + \gamma)\phi(v)^{-1}$. One can show that, after choosing an appropriate α, one may assume that

$$e_\gamma(v) = \exp(2\pi i(a_\gamma(v) + b_\gamma)),$$

where a_γ is a linear function and b_γ is a constant (see [405], Chapter 1, §1). We will assume that such a choice has been made.

It turns out that the theta function corresponding to a symmetric theta divisor Θ from (5.20) can be given in coordinates defined by a choice of a normalized basis (5.18) by the following expression

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau) = \sum_{r\in\mathbb{Z}^g} \exp \pi i\big[(\mathbf{r}+\tfrac{1}{2}\boldsymbol{\epsilon})\cdot\tau\cdot(\mathbf{r}+\tfrac{1}{2}\boldsymbol{\epsilon})+2(\mathbf{z}+\tfrac{1}{2}\boldsymbol{\eta})\cdot(\mathbf{r}+\tfrac{1}{2}\boldsymbol{\epsilon})\big], \tag{5.23}$$

where $\boldsymbol{\epsilon}, \boldsymbol{\eta} \in \{0, 1\}^g$ considered as column or raw vectors from $\mathbb{F}_2^g$. The function defined by this expression is called a *theta function with characteristic*. The theta factor $e_\lambda(z_1, \ldots, z_g)$ for such a function is given by the expression

$$e_\gamma(\mathbf{z}) = \exp -\pi i(\mathbf{m}\cdot\tau\cdot\mathbf{m} - 2\mathbf{z}\cdot\mathbf{m} - \boldsymbol{\epsilon}\cdot\mathbf{n} + \boldsymbol{\eta}\cdot\mathbf{m}),$$

where we write $\gamma = \tau\cdot\mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$.One can check that

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](-\mathbf{z};\tau) = \exp(\pi i\boldsymbol{\epsilon}\cdot\boldsymbol{\eta})\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau). \tag{5.24}$$

This shows that $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](-\mathbf{z};\tau)$ is an odd (resp. even) function if and only if $\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} = 1$ (resp. 0). In particular, $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](0;\tau) = 0$ if the function is odd. It

follows from (5.21) that $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](0;\tau) = 0$ if θ is an odd theta characteristic or an effective even theta characteristic.

Taking $\boldsymbol{\epsilon}, \boldsymbol{\eta} = 0$, we obtain the *Riemann theta function*

$$\theta(\mathbf{z};\tau) = \sum_{\mathbf{r}\in\mathbb{Z}^g} \exp \pi i(\mathbf{r}\cdot\tau\cdot\mathbf{r} + 2\mathbf{z}\cdot\mathbf{r}).$$

All other theta functions with characteristic are obtained from $\theta(\mathbf{z};\tau)$ by a translate

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau) = \exp \pi i(\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} + \boldsymbol{\epsilon}\cdot\tau\cdot\boldsymbol{\epsilon})\theta(\mathbf{z} + \tfrac{1}{2}\tau\cdot\boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\epsilon};\tau).$$

In this way points on $\mathbb{C}^g$ of the form $\frac{1}{2}\tau\cdot\boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\eta}$ are identified with elements of the 2-torsion group $\frac{1}{2}\Lambda/\Lambda$ of $\mathrm{Jac}(C)$. The theta divisor corresponding to the Riemann theta function is equal to W^0_{g-1} translated by a certain theta characteristic κ called the *Riemann constant*. Of course, there is no distinguished theta characteristic; the definition of κ depends on the choice of a symplectic basis in $H_1(C,\mathbb{Z})$.

The multiplicity m of a point on a theta divisor $\Theta = W^0_{g-1} - \vartheta$ is equal to the multiplicity of the corresponding theta function defined by vanishing partial derivatives up to order $m-1$. Thus the quadratic form defined by θ can be redefined in terms of the corresponding theta function as

$$q_\vartheta(\tfrac{1}{2}\tau\cdot\boldsymbol{\epsilon}' + \tfrac{1}{2}\boldsymbol{\eta}') = \mathrm{mult}_0\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}'\\ \boldsymbol{\eta}+\boldsymbol{\eta}'\end{smallmatrix}\right](\mathbf{z},\tau) + \mathrm{mult}_0\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z},\tau).$$

It follows from (5.24) that this number is equal to

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\eta}' + \boldsymbol{\eta}\cdot\boldsymbol{\eta}' + \boldsymbol{\eta}'\cdot\boldsymbol{\eta}'. \tag{5.25}$$

A choice of a symplectic basis in $H_1(C,\mathbb{Z})$ defines a standard symplectic basis in $H_1(C,\mathbb{F}_2) \cong \frac{1}{2}\Lambda/\Lambda = \mathrm{Jac}(C)[2]$. Thus we can identify 2-torsion points $\frac{1}{2}\tau\cdot\boldsymbol{\epsilon}' + \frac{1}{2}\boldsymbol{\eta}'$ with vectors $(\boldsymbol{\epsilon}',\boldsymbol{\eta}') \in \mathbb{F}_2^{2g}$. The quadratic form corresponding to the Riemann theta function is the standard one

$$q_0((\boldsymbol{\epsilon}',\boldsymbol{\eta}')) = \boldsymbol{\epsilon}'\cdot\boldsymbol{\eta}'.$$

The quadratic form corresponding to $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ is given by (5.25). The Arf invariant of this quadratic form is equal to

$$\mathrm{Arf}(q_\vartheta) = \boldsymbol{\epsilon}\cdot\boldsymbol{\eta}.$$

5.3.3 Hyperelliptic curves again

In this case we can compute the Riemann constant explicitly. Recall that we identify 2-torsion points with subsets of even cardinality of the set $B_g =$

$\{1, \dots, 2g+2\}$ which we can identify with the set of ramification or branch points. Let us define a standard symplectic basis in C by choosing the 1-cycle α_i to be the path which goes from c_{2i-1} to c_{2i} along one sheet of the Riemann surface C and returns to c_{2i-1} along the other sheet. Similarly, we define the 1-cycle β_i by choosing the points c_{2i} and c_{2i+1}. Choose g holomorphic forms ω_j normalized by the condition (5.18). Let τ be the corresponding period matrix. Notice that each holomorphic 1-form changes sign when we switch the sheets. This gives

$$\begin{aligned}\tfrac{1}{2}\delta_{ij} &= \tfrac{1}{2}\int_{\alpha_i}\omega_j = \int_{c_{2i-1}}^{c_{2i}}\omega_j = \int_{c_{2i-1}}^{c_{2g+2}}\omega_j - \int_{c_{2i}}^{c_{2g+2}}\omega_j \\ &= \int_{c_{2i-1}}^{c_{2g+2}}\omega_j + \int_{c_{2i}}^{c_{2g+2}}\omega_j - 2\int_{c_{2i}}^{c_{2g+2}}\omega_j.\end{aligned}$$

Since

$$2\Big(\int_{c_{2i}}^{c_{2g+2}}\omega_1, \dots, \int_{c_{2i}}^{c_{2g+2}}\omega_g\Big) = \mathfrak{a}(2c_{2i} - 2c_{2g+2}) = 0,$$

we obtain

$$\iota(c_{2i-1} + c_{2i} - 2c_{2g+2}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau,$$

where, as usual, $\mathbf{e}_i$ denotes the i-th unit vector. Let A_i, B_i be defined as in (5.15). We obtain that

$$\mathfrak{a}(\alpha_{A_i}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau.$$

Similarly, we find that

$$\mathfrak{a}(\alpha_{B_i}) = \tfrac{1}{2}\tau\cdot\mathbf{e}_i \mod \Lambda_\tau.$$

Now we can match the set Q_g with the set of theta functions with characteristics. Recall that the set $U = \{1, 3, \dots, 2g+1\}$ plays the role of the standard quadratic form. We have

$$q_U(A_i) = q_U(B_i) = 0, \quad i = 1, \dots, g.$$

Comparing it with (5.25), we see that the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ corresponding to ϑ_U must coincide with the function $\theta(\mathbf{z};\tau)$. This shows that

$$\iota_{c_{2g+2}}^{g-1}(\vartheta_U) = \iota_{c_{2g+2}}(\vartheta_U - k_{c_{2g+2}}) = 0.$$

Thus the Riemann constant κ corresponds to the theta characteristic ϑ_U. This allows one to match theta characteristics with theta functions with theta characteristics.

Write any subset I of E_g in the form

$$I = \sum_{i=1}^{g} \epsilon_i A_i + \sum_{i=1}^{g} \eta_i B_i,$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_g)$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_g)$ are binary vectors. Then

$$\vartheta_{U+I} \longleftrightarrow \theta \left[\begin{smallmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{smallmatrix} \right] (\mathbf{z}; \tau).$$

In particular,

$$\vartheta_{U+I} \in \mathrm{TChar}(C)^{\mathrm{ev}} \Longleftrightarrow \epsilon \cdot \eta = 0 \mod 2.$$

Example 5.3.2 We give the list of theta characteristics for small genus. We also list 2-torsion points at which the corresponding theta function vanishes.

$g = 1$

3 even "thetas":

$$\vartheta_{12} = \theta \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] \quad (\alpha_{12}),$$

$$\vartheta_{13} = \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \quad (\alpha_{13}),$$

$$\vartheta_{14} = \theta \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \quad (\alpha_{14}).$$

1 odd theta:

$$\vartheta_{\emptyset} = \theta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}).$$

$g = 2$

10 even thetas:

$$\vartheta_{123} = \theta \left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{23}, \alpha_{13}, \alpha_{45}, \alpha_{46}, \alpha_{56}),$$

$$\vartheta_{124} = \theta \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{24}, \alpha_{14}, \alpha_{35}, \alpha_{36}, \alpha_{56}),$$

$$\vartheta_{125} = \theta \left[\begin{smallmatrix} 00 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{25}, \alpha_{15}, \alpha_{34}, \alpha_{36}, \alpha_{46}),$$

$$\vartheta_{126} = \theta \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{16}, \alpha_{26}, \alpha_{34}, \alpha_{35}, \alpha_{45}),$$

$$\vartheta_{234} = \theta \left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{34}, \alpha_{24}, \alpha_{15}, \alpha_{56}, \alpha_{16}),$$

$$\vartheta_{235} = \theta \left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{25}, \alpha_{35}, \alpha_{14}, \alpha_{16}, \alpha_{46}),$$

$$\vartheta_{236} = \theta \left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{26}, \alpha_{36}, \alpha_{14}, \alpha_{45}, \alpha_{15}),$$

$$\vartheta_{245} = \theta \left[\begin{smallmatrix} 11 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{24}, \alpha_{25}, \alpha_{13}, \alpha_{45}, \alpha_{16}, \alpha_{36}),$$

$$\vartheta_{246} = \theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] \quad (\alpha_{26}, \alpha_{24}, \alpha_{13}, \alpha_{35}, \alpha_{46}, \alpha_{15}),$$

$$\vartheta_{256} = \theta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] \quad (\alpha_{26}, \alpha_{25}, \alpha_{13}, \alpha_{14}, \alpha_{34}, \alpha_{56}).$$

6 odd thetas:

$$\vartheta_1 = \theta\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}),$$

$$\vartheta_2 = \theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{12}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}),$$

$$\vartheta_3 = \theta\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{13}, \alpha_{23}, \alpha_{34}, \alpha_{35}, \alpha_{36}),$$

$$\vartheta_4 = \theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{14}, \alpha_{24}, \alpha_{34}, \alpha_{45}, \alpha_{46}),$$

$$\vartheta_5 = \theta\left[\begin{smallmatrix}10\\11\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{15}, \alpha_{35}, \alpha_{45}, \alpha_{25}, \alpha_{56}),$$

$$\vartheta_6 = \theta\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] \quad (\alpha_\emptyset, \alpha_{16}, \alpha_{26}, \alpha_{36}, \alpha_{46}, \alpha_{56}).$$

$g = 3$
36 even thetas $\vartheta_\emptyset, \vartheta_{ijkl}$,
28 odd thetas ϑ_{ij}.

$g = 4$
136 even thetas $\vartheta_i, \vartheta_{ijklm}$,
120 odd thetas ϑ_{ijk}.

5.4 Odd theta characteristics

5.4.1 Syzygetic triads

We have already remarked that effective theta characteristics on a canonical curve $C \subset \mathbb{P}^{g-1}$ correspond to *contact hyperplanes*, i.e. hyperplanes everywhere tangent to C. They are also called *bitangent hyperplanes* (not to be confused with hyperplanes tangent at ≥ 2 points).

An odd theta characteristic is effective and determines a contact hyperplane, a unique one if it is nonvanishing. In this Section we will study the configuration of contact hyperplanes to a canonical curve. Let us note here that a general canonical curve is determined uniquely by the configuration of its contact hyperplanes [63].

From now on we fix a nondegenerate symplectic space (V, ω) of dimension

$2g$ over $\mathbb{F}_2$. Let $Q(V)$ be the affine space of quadratic forms with associated symmetric bilinear form equal to ω. The Arf invariant divides $Q(V)$ into the union of two sets $Q(V)_+$ and $Q(V)_-$, of even or odd quadratic forms. Recall that $Q(V)_-$ is interpreted as the set of odd theta characteristics when $V = \mathrm{Pic}(C)$ and ω is the Weil pairing. For any $q \in Q(V)$ and $v \in V$, we have

$$q(v) = \mathrm{Arf}(q+v) + \mathrm{Arf}(q).$$

Thus the function Arf is a symplectic analog of the function $h^0(\vartheta) \mod 2$ for theta characteristics.

The set $\tilde{V} = V \coprod Q(V)$ is equipped with a structure of a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{F}_2$. It combines the addition on V (the 0-th graded piece) and the structure of an affine space on $Q(V)$ (the 1-th graded piece) by setting $q + q' := v$, where $q' = q + v$. One can also extend the symplectic form on V to $\tilde{V}$ by setting

$$\omega(q, q') = q(q+q'), \quad \omega(q, v) = \omega(v, q) = q(v).$$

Definition 5.4.1 *A set of three elements q_1, q_2, q_3 in $Q(V)$ is called a* syzygetic triad *(resp.* azygetic triad*) if*

$$\mathrm{Arf}(q_1) + \mathrm{Arf}(q_2) + \mathrm{Arf}(q_3) + \mathrm{Arf}(q_1+q_2+q_3) = 0 \ (\textit{resp.} \ = 1).$$

A subset of $k \geq 3$ elements in $Q(V)$ is called an azygetic set *if any subset of three elements is azygetic.*

Note that a syzygetic triad defines a set of four quadrics in $Q(V)$ that add up to zero. Such a set is called a *syzygetic tetrad*. Obviously, any subset of three elements in a syzygetic tetrad is a syzygetic triad.

Another observation is that three elements in $Q(V)_-$ form an azygetic triad if their sum is an element in $Q(V)_+$.

For any odd theta characteristic ϑ, any divisor $D_\eta \in |\vartheta|$ is of degree $g-1$. The condition that four odd theta characteristics ϑ_i form a syzygetic tetrad means that the sum of divisors D_{ϑ_i} are cut out by a quadric in $\mathbb{P}^{g-1}$. The converse is true if C does not have a vanishing even theta characteristic.

Let us now compute the number of syzygetic tetrads.

Lemma 5.4.2 *Let q_1, q_2, q_3 be a set of three elemenst in $Q(V)$. The following properties are equivalent:*

(i) q_1, q_2, q_3 *is a syzygetic triad;*
(ii) $q_1(q_2+q_3) = \mathrm{Arf}(q_2) + \mathrm{Arf}(q_3)$*;*
(iii) $\omega(q_1+q_2, q_1+q_3) = 0$.

Proof The equivalence of (i) and (ii) follows immediately from the identity

$$q_1(q_2+q_3) = \text{Arf}(q_1) + \text{Arf}(q_1+q_2+q_3).$$

We have

$$\omega(q_1+q_2, q_1+q_3) = q_1(q_1+q_3) + q_2(q_1+q_3)$$

$$= \text{Arf}(q_1) + \text{Arf}(q_3) + \text{Arf}(q_2) + \text{Arf}(q_1+q_2+q_3).$$

This shows the equivalence of (ii) and (iii). □

Proposition 5.4.3 *Let $q_1, q_2 \in Q(V)_-$. The number of ways in which the pair can be extended to a syzygetic triad of odd theta characteristics is equal to $2(2^{g-1}+1)(2^{g-2}-1)$.*

Proof Assume that q_1, q_2, q_3 is a syzygetic triad in $Q(V)_-$. By the previous lemma, $q_1(q_2+q_3) = 0$. Also, we have $q_2(q_2+q_3) = \text{Arf}(q_3) + \text{Arf}(q_2) = 0$. Thus q_1 and q_2 vanish at $v_0 = q_2 + q_3$. Conversely, assume $v \in V$ satisfies $q_1(v) = q_2(v) = 0$ and $v \neq q_1 + q_2$ so that $q_3 = q_2 + v \neq q_1, q_2$. We have $\text{Arf}(q_3) = \text{Arf}(q_2) + q_2(v) = 1$, hence $q_3 \in Q(V)_-$. Since $q_1(v) = q_1(q_2+q_3) = 0$, by the previous Lemma q_1, q_2, q_3 is a syzygetic triad.

Thus the number of the ways in which we can extend q_1, q_2 to a syzygetic triad q_1, q_2, q_3 is equal to the cardinality of the set

$$Z = q_1^{-1}(0) \cap q_2^{-1}(0) \setminus \{0, v_0\},$$

where $v_0 = q_1 + q_2$. It follows from (5.6) that $v \in Z$ satisfies $\omega(v, v_0) = q_2(v) + q_1(v) = 0$. Thus any $v \in Z$ is a representative of a nonzero element in $W = v_0^\perp / v_0 \cong \mathbb{F}_2^{2g-2}$ on which q_1 and q_2 vanish. It is clear that q_1 and q_2 induce the same quadratic form q on W. It is an odd quadratic form. Indeed, we can choose a symplectic basis in V by taking as a first vector the vector v_0. Then computing the Arf invariant of q_1 we see that it is equal to the Arf invariant of the quadratic form q. Thus we get

$$\#Z = 2(\#Q(W)_- - 1) = 2(2^{g-2}(2^{g-1}-1) - 1) = 2(2^{g-1}+1)(2^{g-2}-1).$$

□

Corollary 5.4.4 *Let t_g be the the number of syzygetic tetrads of odd theta characteristics on a nonsingular curve of genus g. Then*

$$t_g = \frac{1}{3} 2^{g-3}(2^{2g}-1)(2^{2g-2}-1)(2^{g-2}-1).$$

Proof Let I be the set of triples (q_1, q_2, T), where $q_1, q_2 \in Q(V)_-$ and T is a syzygetic tetrad containing q_1, q_2. We count $\#I$ in two ways by projecting

I to the set $\mathcal{P}$ of unordered pairs of distinct elements $Q(V)_-$ and to the set of syzygetic tetrads. Since each tetrad contains 6 pairs from the set $\mathcal{P}$, and each pair can be extended in $(2^{g-1}+1)(2^{g-2}-1)$ ways to a syzygetic tetrad, we get

$$\#I = (2^{g-1}+1)(2^{g-2}-1)\binom{2^{g-1}(2^g-1)}{2} = 6t_g.$$

This gives

$$t_g = \frac{1}{3}2^{g-3}(2^{2g}-1)(2^{2g-2}-1)(2^{g-2}-1).$$

□

Let V be a vector space with a symplectic or symmetric bilinear form. Recall that a linear subspace L is called isotropic if the restriction of the bilinear form to L is identically zero.

Corollary 5.4.5 *Let $\{q_1, q_2, q_3, q_4\}$ be a syzygetic tetrad in $Q(V)_-$. Then $P = \{q_1 + q_i, \ldots, q_4 + q_i\}$ is an isotropic 2-dimensional subspace in (V, ω) that does not depend on the choice of q_i.*

Proof It follows from Lemma 5.4.2 (iii) that P is an isotropic subspace. The equality $q_1 + \cdots + q_4 = 0$ gives

$$q_k + q_l = q_i + q_j, \tag{5.26}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. This shows that the subspace P of V formed by the vectors $q_j + q_i, j = 1, \ldots, 4$, is independent on the choice of i. One of its bases is the set $(q_1 + q_4, q_2 + q_4)$. □

5.4.2 Steiner complexes

Let $\mathcal{P}$ be the set of unordered pairs of distinct elements in $Q(V)_-$. The addition map in $Q(V)_- \times Q(V)_- \to V$ defines a map

$$s : \mathcal{P} \to V \setminus \{0\}.$$

Definition 5.4.6 *The union of pairs from the same fiber $s^{-1}(v)$ of the map s is called a* Steiner compex. *It is denoted by $\Sigma(v)$.*

It follows from (5.26) that any two pairs from a syzygetic tetrad belong to the same Steiner complex. Conversely, let $\{q_1, q_1'\}, \{q_2, q_2'\}$ be two pairs from $\Sigma(v)$. We have $(q_1 + q_1') + (q_2 + q_2') = v + v = 0$, showing that the tetrad (q_1, q_1', q_2, q_2') is syzygetic.

Proposition 5.4.7 *There are $2^{2g}-1$ Steiner complexes. Each Steiner complex consists of $2^{g-1}(2^{g-1}-1)$ elements paired by translation $q \mapsto q+v$. An odd quadratic form q belongs to a Steiner complex $\Sigma(v)$ if and only if $q(v)=0$.*

Proof Since $2^{2g}-1 = \#(V\setminus\{0\})$, it suffices to show that the map $s : \mathcal{P} \to V\setminus\{0\}$ is surjective. The symplectic group $\mathrm{Sp}(V,\omega)$ acts transitively on $V\setminus\{0\}$ and on $\mathcal{P}$, and the map s is obviously equivariant. Thus its image is a non-empty G-invariant subset of $V\setminus\{0\}$. It must coincide with the whole set.

By (5.7), we have $q(v) = \mathrm{Arf}(q+v) + \mathrm{Arf}(q)$. If $q \in \Sigma(v)$, then $q+v \in Q(V)_-$, hence $\mathrm{Arf}(q+v) = \mathrm{Arf}(q) = 1$ and we get $q(v)=0$. Conversely, if $q(v)=0$ and $q\in\Sigma(v)$, we get $q+v \in Q(V)_-$ and hence $q\in\Sigma(v)$. This proves the last assertion. □

Lemma 5.4.8 *Let $\Sigma(v), \Sigma(v')$ be two Steiner complexes. Then*

$$\#\Sigma(v)\cap\Sigma(v') = \begin{cases} 2^{g-1}(2^{g-2}-1) & \text{if } \omega(v,v')=0,\\ 2^{g-2}(2^{g-1}-1) & \text{if } \omega(v,v')\neq 0.\end{cases}$$

Proof Let $q\in\Sigma(v)\cap\Sigma(v')$. Then we have $q+q'=v, q+q''=v'$ for some $q'\in\Sigma(v), q''\in\Sigma(v')$. This implies that

$$q(v)=q(v')=0. \tag{5.27}$$

Conversely, if these equalities hold, then $q+v, q+v' \in Q(V)_-$, $q, q'\in\Sigma(v)$, and $q, q''\in\Sigma(v')$. Thus we have reduced our problem to linear algebra. We want to show that the number of elements in $Q(V)_-$ which vanish at two nonzero vectors $v, v' \in V$ is equal to $2^{g-1}(2^{g-2}-1)$ or $2^{g-2}(2^{g-1}-1)$ depending on whether $\omega(v,v')=0$ or 1. Let q be one such quadratic form. Suppose we have another q' with this property. Write $q'=q+v_0$ for some v_0. We have $q(v_0)=0$ since q' is odd and

$$\omega(v_0,v)=\omega(v_0,v')=0.$$

Let L be the plane spanned by v, v'. Assume $\omega(v,v')=1$, then we can include v, v' in a standard symplectic basis. Computing the Arf invariant, we find that the restriction of q to $L^\perp$ is an odd quadratic form. Thus it has $2^{g-2}(2^{g-1}-1)$ zeros. Each zero gives us a solution for v_0. Assume $\omega(v,v')=0$. Then L is a singular plane for q since $q(v)=q(v')=q(v+v')=0$. Consider $W = L^\perp/L \cong \mathbb{F}_2^{2g-4}$. The form q has $2^{g-3}(2^{g-2}-1)$ zeros in W. Any representative v_0 of these zeros defines the quadratic form $q+v_0$ vanishing at v, v'. Any quadratic form we are looking for is obtained in this way. The number of such representatives is equal to $2^{g-1}(2^{g-2}-1)$.

□

Definition 5.4.9 *Two Steiner complexes $\Sigma(v)$ and $\Sigma(v')$ are called* syzygetic *(resp.* azygetic*) if $\omega(v, v') = 0$ (resp. $\omega(v, v') = 1$).*

Theorem 5.4.10 *The union of three mutually syzygetic Steiner complexes $\Sigma(v)$, $\Sigma(v')$ and $\Sigma(v + v')$ is equal to $Q(V)_-$.*

Proof Since

$$\omega(v + v', v) = \omega(v + v', v') = 0,$$

we obtain that the Steiner complex $\Sigma(v + v')$ is syzygetic to $\Sigma(v)$ and $\Sigma(v')$. Suppose $q \in \Sigma(v) \cap \Sigma(v')$. Then $q(v + v') = q(v) + q(v') + \omega(v, v') = 0$. This implies that $\Sigma(v) \cap \Sigma(v') \subset \Sigma(v + v')$ and hence $\Sigma(v), \Sigma(v'), \Sigma(v + v')$ share the same set of $2^{g-1}(2^{g-2} - 1)$ elements. This gives

$$\#\Sigma(v) \cup \Sigma(v') \cup \Sigma(v + v') = 6 \cdot 2^{g-2}(2^{g-1} - 1) - 2 \cdot 2^{g-1}(2^{g-2} - 1)$$

$$= 2^{g-1}(2^g - 1) = \#Q(V)_-.$$

□

Definition 5.4.11 *A set of three mutually syzygetic Steiner complexes is called a* syzygetic triad *of Steiner complexes. A set of three Steiner complexes corresponding to vectors forming a non-isotropic plane is called* azygetic triad *of Steiner complexes.*

Let $\Sigma(v_i), i = 1, 2, 3$ be a azygetic triad of Steiner complexes. Then

$$\#\Sigma(v_1) \cap \Sigma(v_2) = 2^{g-2}(2^{g-1} - 1).$$

Each set $\Sigma(v_1) \setminus (\Sigma(v_1) \cap \Sigma(v_2))$ and $\Sigma(v_2) \setminus (\Sigma(v_1) \cap \Sigma(v_2))$ consists of $2^{g-2}(2^{g-1} - 1)$ elements. The union of these sets forms the Steiner complex $\Sigma(v_3)$. The number of azygetic triads of Steiner complexes is equal to $\frac{1}{3}2^{2g-2}(2^{2g} - 1)$ (= the number of non-isotropic planes). We leave the proofs to the reader.

Let $\mathcal{S}_4(V)$ denote the set of syzygetic tetrads. By Corollary 5.4.5, each $T \in \mathcal{S}_4(V)$ defines an isotropic plane P_T in V. Let $\mathrm{Iso}_k(V)$ denote the set of k-dimensional isotropic subspaces in V.

Proposition 5.4.12 *Let $\mathcal{S}_4(V)$ be the set of syzygetic tetrads. For each tetrad T let P_T, denote the corresponding isotropic plane. The map*

$$\mathcal{S}_4(V) \to \mathrm{Iso}_2(V), \quad T \mapsto P_T,$$

is surjective. The fiber over a plane T consists of $2^{g-3}(2^{g-2} - 1)$ tetrads forming a partition of the intersection of the Steiner complexes $\Sigma(v)$, where $v \in P \setminus \{0\}$.

Proof The surjectivity of this map is proved along the same lines as we proved Proposition 5.4.7. We use the fact that the symplectic group $\mathrm{Sp}(V,\omega)$ acts transitively on the set of isotropic subspaces of the same dimension. Let $T = \{q_1, \dots, q_4\} \in \mathcal{S}_4(V)$. By definition, $P_T \setminus \{0\} = \{q_1+q_2, q_1+q_3, q_1+q_4\}$. Suppose we have another tetrad $T' = \{q_1', \dots, q_4'\}$ with $P_T = P_{T'}$. Suppose $T \cap T' \neq \emptyset$. Without loss of generality, we may assume that $q_1' = q_1$. Then, after reindexing, we get $q_1 + q_i = q_1 + q_i'$, hence $q_i = q_i'$ and $T = T'$. Thus the tetrads T with $P_T = P$ are disjoint. Obviously, any $q \in T$ belongs to the intersection of the Steiner complexes $\Sigma(v),\ v \in P \setminus \{0\}$. It remains for us to apply Lemma 5.4.8. □

A closer look at the proof of Lemma 5.4.8 shows that the fiber over P can be identified with the set $Q(P^\perp/P)_-$.

Combining Proposition 5.4.12 with the computation of the number t_g of syzygetic tetrads, we obtain the number of isotropic planes in V:

$$\#\mathrm{Iso}_2(V) = \frac{1}{3}(2^{2g}-1)(2^{2g-2}-1). \tag{5.28}$$

Let $\mathrm{Iso}_2(v)$ be the set of isotropic planes containing a nonzero vector $v \in V$. The set $\mathrm{Iso}_2(v)$ is naturally identified with nonzero elements in the symplectic space $(v^\perp/v, \omega')$, where ω' is defined by the restriction of ω to $v^\perp$. We can transfer the symplectic form ω' to $\mathrm{Iso}_2(v)$. We obtain $\omega'(P,Q) = 0$ if and only if $P+Q$ is an isotropic 3-subspace.

Let us consider the set $\mathcal{S}_4(V,v) = \alpha^{-1}(\mathrm{Iso}_2(v))$. It consists of syzygetic tetrads that are invariant with respect to the translation by v. In particular, each tetrad from $\mathcal{S}_4(V,v)$ is contained in $\Sigma(v)$. We can identify the set $\mathcal{S}_4(V,v)$ with the set of cardinality 2 subsets of $\Sigma(v)/\langle v\rangle$.

There is a natural pairing on $\mathcal{S}_4(V,v)$ defined by

$$\langle T, T'\rangle = \tfrac{1}{2}\#T \cap T' \mod 2. \tag{5.29}$$

Proposition 5.4.13 *For any $T, T' \in \mathcal{S}_4(V,v)$,*

$$\omega'(P_T, P_{T'}) = \langle T, T'\rangle.$$

Proof Let $X = \{\{T,T'\} \subset \mathcal{S}_4(V) : \alpha_v(T) \neq \alpha_v(T')\}$, $Y = \{\{P,P'\} \subset \mathrm{Iso}_2(v)\}$. We have a natural map $\tilde{\alpha}_v : X \to Y$ induced by α_v. The pairing ω' defines a function $\phi : Y \to \mathbb{F}_2$. The corresponding partition of Y consists of two orbits of the stabilizer group $G = \mathrm{Sp}(V,\omega)_v$ on Y. Suppose $\{T_1, T_2\}$ and $\{T_1', T_2'\}$ are mapped to the same subset $\{P, P'\}$. Without loss of generality,

we may assume that T_1, T_1' are mapped to P. Thus

$$\langle T_1 + T_2', T_2 + T_1' \rangle = \langle T_1, T_2 \rangle + \langle T_1', T_2' \rangle + \langle T_1, T_1' \rangle + \langle T_2, T_2'$$

$$= \langle T_1, T_2 \rangle + \langle T_1', T_2' \rangle.$$

This shows that the function $X \to \mathbb{F}_2$ defined by the pairing (5.29) is constant on fibres of $\tilde{\alpha}_v$. Thus it defines a map $\phi' : Y \to \mathbb{F}_2$. Both functions are invariant with respect to the group G. This immediately implies that their two level sets either coincide or are switched. However, $\#\mathrm{Iso}_2(v) = 2^{2g-2} - 1$ and hence the cardinality of Y is equal to $(2^{2g-2} - 1)(2^{2g-3} - 1)$. Since this number is odd, the two orbits are of different cardinalities. Since the map $\tilde{\alpha}_v$ is G-equivariant, the level sets must coincide. □

5.4.3 Fundamental sets

Suppose we have an ordered set S of $2g+1$ vectors $(u_1, \dots, u_{2g+1})$ satisfying $\omega(u_i, u_j) = 1$ unless $i = j$. It defines a standard symplectic basis by setting

$$v_i = u_1 + \cdots + u_{2i-2} + u_{2i-1},\ v_{i+g} = u_1 + \cdots + u_{2i-2} + u_{2i},\ i = 1, \dots, g.$$

Conversely, we can solve the u_i's from the v_i's uniquely to reconstruct the set S from a standard symplectic basis.

Definition 5.4.14 *A set of $2g+1$ vectors $(u_1, \dots, u_{2g+1})$ with $\omega(u_i, u_j) = 1$ unless $i = j$ is called a* normal system *in (V, ω).*

We have established a bijective correspondence between normal systems and standard symplectic bases.

Recall that a symplectic form ω defines a nondegenerate null-system in V, i.e. a bijective linear map $f : V \to V^\vee$ such that $f(v)(v) = 0$ for all $v \in V$. Fix a basis $(e_1, \dots, e_{2g})$ in V and the dual basis $(t_1, \dots, t_{2g})$ in $V^\vee$ and consider vectors $u_i = e_1 + \cdots + e_{2g} - e_i, i = 1, \dots, 2g$ and $u_{2g+1} = e_1 + \cdots + e_{2g}$. Then there exists a unique null-system $V \to V^\vee$ that sends u_i to t_i and u_{2g+1} to $t_{2g+1} = t_1 + \cdots + t_{2g}$. The vectors $u_1, \dots, u_{2g+1}$ form a normal system in the corresponding symplectic space.

Let $(u_1, \dots, u_{2g+1})$ be a normal system. We will identify nonzero vectors in V with points in the projective space $|V|$. Denote the points corresponding to the vectors u_i by ϵ_{i2g+2}. For any $i, j \neq 2g+2$, consider the line spanned by ϵ_{i2g+2} and ϵ_{j2g+2}. Let ϵ_{ij} be the third nonzero point in this line. Now do the same with points ϵ_{ij} and ϵ_{kl} with the disjoint sets of indices. Denote this point by ϵ_{ijkl}. Note that the residual point on the line spanned by ϵ_{ij} and ϵ_{jk} is equal to ϵ_{ik}. Continuing in this way, we will be able to index all points

in $|V|$ with subsets of even cardinality (up to complementary sets) of the set $B_g = \{1, \ldots, 2g+2\}$. This notation will agree with the notation of 2-torsion divisor classes for hyperelliptic curves of genus g. For example, we have

$$\omega(p_I, p_J) = \#I \cap J \mod 2.$$

It is easy to compute the number of normal systems. It is equal to the number of standard symplectic bases in (V, ω). The group $\mathrm{Sp}(V, \omega)$ acts simply transitively on such bases, so their number is equal to

$$\#\mathrm{Sp}(2g, \mathbb{F}_2) = 2^{g^2}(2^{2g}-1)(2^{2g-2}-1)\cdots(2^2-1). \tag{5.30}$$

Now we introduce the analog of a normal system for quadratic forms in $Q(V)$.

Definition 5.4.15 *A* fundamental set *in $Q(V)$ is an ordered azygetic set of $2g+2$ elements in $Q(V)$.*

The number $2g+2$ is the largest possible cardinality of a set such that any three elements are azygetic. This follows from the following immediate corollary of Lemma 5.4.2.

Lemma 5.4.16 *Let $B = (q_1, \ldots, q_k)$ be an azygetic set. Then the set $(q_1 + q_2, \ldots, q_1 + q_k)$ is a normal system in the symplectic subspace of dimension $k-2$ spanned by these vectors.*

The Lemma shows that any fundamental set in $Q(V)$ defines a normal system in V, and hence a standard symplectic basis. Conversely, starting from a normal system $(u_1, \ldots, u_{2g+1})$ and any $q \in Q(V)$ we can define a fundamental set $(q_1, \ldots, q_{2g+2})$ by

$$q_1 = q, q_2 = q + u_1, \ldots, q_{2g+2} = q + u_{2g+1}.$$

Since elements in a fundamental system add up to zero, we get that the elements of a fundamental set also add up to zero.

Proposition 5.4.17 *There exists a fundamental set with all or all but one quadratic forms are even or odd. The number of odd quadratic forms in such a set is congruent to g modulo 4.*

Proof Let $(u_1, \ldots, u_{2g+1})$ be a normal system and $(t_1, \ldots, t_{2g+1})$ be its image under the map $V \to V^\vee$ defined by ω. Consider the quadratic form

$$q = \sum_{1 \le i < j \le 2g+1} t_i t_j.$$

It is immediately checked that

$$q(u_k) \equiv \binom{2g}{2} = g(2g-1) \equiv g \mod 4.$$

Passing to the associated symplectic basis, we can compute the Arf invariant of q to get

$$\mathrm{Arf}(q) = \begin{cases} 1 & \text{if } g \equiv 1 \mod 2 \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\mathrm{Arf}(q + t_k^2) = \mathrm{Arf}(q) + q(u_k) = \begin{cases} 0 & \text{if } g \equiv 0, 3 \mod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the fundamental set of quadrics $q, q + t_k^2, k = 1, \ldots, 2g+1$. If $g \equiv 0 \mod 4$, the set consists of all even quadratic forms. If $g \equiv 1 \mod 4$, the quadratic form q is odd, all other quadratic forms are even. If $g \equiv 2 \mod 4$, all quadratic forms are odd. Finally, if $g \equiv 3 \mod 4$, then q is even, all other quadratic forms are odd. □

Definition 5.4.18 *A fundamental set with all or all but one quadratic forms are even or odd is called a* normal fundamental set.

One can show (see [116], p. 271) that any normal fundamental set is obtained as in the proof of the previous proposition.

Choose a normal fundamental set $(q_1, \ldots, q_{2g+2})$ such that the first $2g+1$ quadrics are of the same type. Any quadratic form $q \in Q(V)$ can be written in the form

$$q_{2g+2} + \sum_{i \in I} t_i^2 = q + \sum_{i \in I} t_i^2,$$

where I is a subset of $[1, 2g+1] := \{1, \ldots, 2g+1\}$. We denote such a quadratic form by q_S, where $S = I \cup \{2g+2\}$ considered as a subset of $[1, 2g+2]$ modulo the complementary set. We can and will always assume that

$$\#S \equiv g + 1 \mod 2.$$

The quadratic form q_S can be characterized by the property that it vanishes on points p_{ij}, where $i \in S$ and $j \in \{1, \ldots, 2g+2\}$.

The following properties can be checked.

Proposition 5.4.19
- $q_S + q_T = \epsilon_{S+T}$;
- $q_S + \epsilon_I = q_{S+I}$;
- $q_S(\epsilon_T) = 0$ *if and only if* $\#S \cap T + \frac{1}{2}\#S \equiv 0 \mod 2$;

- $q_S \in Q(V)_+$ *if and only if* $\#S \equiv g+1 \mod 4$.

Again we see that a choice of a fundamental set defines the notation of quadratic forms that agrees with the notation of theta characteristics for hyperelliptic curves.

Since fundamental sets are in a bijective correspondence with normal systems their number is given by (5.30).

5.5 Scorza correspondence

5.5.1 Correspondences on an algebraic curve

A *correspondence* of degree d between nonsingular curves C_1 and C_2 is a non-constant morphism T from C_1 to the d-th symmetric product $C_2^{(d)}$ of C_2.

$$\Gamma_T = \{(x,y) \in C_1 \times C_2 : y \in T(x)\}.$$

We have

$$T(x) = \Gamma_T \cap (\{x\} \times C_2), \tag{5.31}$$

where the intersection is scheme-theoretical.

One can extend the map (5.31) to any divisors on C_1 by setting $T(D) = p_1^*(D) \cap \Gamma_T$. It is clear that a principal divisor goes to a principal divisor. Taking divisors of degree 0, we obtain a homomorphism of the Jacobian varieties

$$\phi_T : \mathrm{Jac}(C_1) \to \mathrm{Jac}(C_2).$$

The projection $\Gamma_T \to C_1$ is a finite map of degree d. Since T is not constant, the projection to C_2 is a finite map of degree d'. It defines a correspondence $C_2 \to C_1^{(d')}$ which is denoted by T^{-1} and is called the *inverse correspondence*. Its graph is equal to the image of T under the switch map $C_1 \times C_2 \to C_2 \times C_1$.

We will be dealing mostly with correspondences $T : C \to C^{(d)}$ and will identify T with its graph Γ_T. If d is the degree of T and d' is the degree of T^{-1} we say that T is the correspondence of type (d, d'). A correspondence is *symmetric* if $T = T^{-1}$. We assume that T does not contain the diagonal Δ of $C \times C$. A *united point* of a correspondence is a common point with the diagonal. It comes with the multiplicity.

A correspondence $T : C \to C^{(d)}$ has *valence* ν if the divisor class of $T(x) + \nu x$ does not depend on x.

Proposition 5.5.1 *The following properties are equivalent:*

(i) T has valence ν;

(ii) the cohomology class $[T]$ *in* $H^2(C \times C, \mathbb{Z})$ *is equal to*

$$[T] = (d' + \nu)[\{x\} \times C] + (d + \nu)[C \times \{x\}] - \nu[\Delta],$$

where x *is any point on* C*;*

(iii) the homomorphism ϕ_T *is equal to homomorphism* $[-\nu] : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ *of the multiplication by* $-\nu$*.*

Proof (i) $\Rightarrow$ (ii). We know that there exists a divisor D on C such that the restriction $T + \nu\Delta - p_2^*(D)$ to any fiber of p_1 is linearly equivalent to zero. By the seesaw principle ([405] Chapter 2, Corollary 6), $T + \nu\Delta - p_2^*(D) \sim p_1^*(D')$ for some divisor D' on C. This implies that $[T] = \deg D'[\{x\} \times C] + \deg D[C \times \{x\}] - \nu[\Delta]$. Taking the intersections with fibres of the projections, we find that $d' = \deg D' - \nu$ and $d = \deg D - \nu$.

(ii) $\Rightarrow$ (i) Let $p_1, p_2 : C \times C \to C$ be the projections. We use the well-known fact that the natural homomorphism of the Picard varieties

$$p_1^*(\mathrm{Pic}^0(C)) \oplus p_2^*(\mathrm{Pic}^0(C)) \to \mathrm{Pic}^0(C \times C)$$

is an isomorphism (see [282], Chapter 3, Exercise 12.6). Fix a point $x_0 \in C$ and consider the divisor $T + \nu\Delta - (d' + \nu)(\{x_0\} \times C) - (d + \nu)(C \times \{x_0\})$. By assumption, it is algebraically equivalent to zero. Thus

$$T + \nu\Delta \sim p_1^*(D_1) + p_2^*(D_2)$$

for some divisors D_1, D_2 on C. Thus the divisor class $T(x) + \nu x$ is equal to the divisor class of the restriction of $p_2^*(D_2)$ to $\{x\} \times C$. Obviously, it is equal to the divisor class of D_2, hence is independent on x.

(i) $\Leftrightarrow$ (iii) This follows from the definition of the homomorphism ϕ_T.

□

Note that for a general curve C of genus $g > 2$

$$\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$$

(see [342]), so any correspondence has valence. An example of a correspondence without valence is the graph of an automorphism of order > 2 of C.

Observe that the proof of the Proposition shows that for a correspondence R with valence ν

$$T \sim p_1^*(D') + p_2^*(D) - \nu\Delta, \tag{5.32}$$

where D is the divisor class of $T(x) + \nu x$ and D' is the divisor class of $T^{-1}(x) + \nu x$. It follows from the Proposition that the correspondence T^{-1} has valence ν.

The next result is known as the *Cayley-Brill formula*.

Corollary 5.5.2 *Let T be a correspondence of type (a, b) on a nonsingular projective curve C of genus g. Assume that T has valence equal to ν. Then the number of united points of T is equal to*

$$d + d' + 2\nu g.$$

This immediately follows from (5.32) and the formula $\Delta \cdot \Delta = 2 - 2g$.

Example 5.5.3 Let C be a nonsingular complete intersection of a nonsingular quadric Q and a cubic in $\mathbb{P}^3$. In other words, C is a canonical curve of genus 4 curve without vanishing even theta characteristic. For any point $x \in C$, the tangent plane $\mathbb{T}_x(Q)$ cuts out the divisor $2x + D_1 + D_2$, where $|x + D_1|$ and $|x + D_2|$ are the two g^1_3's on C defined by the two rulings of the quadric. Consider the correspondence T on $C \times C$ defined by $T(x) = D_1 + D_2$. This is a symmetric correspondence of type $(4, 4)$ with valence 2. Its 24 united points correspond to the ramification points of the two g^1_3's.

For any two correspondences T_1 and T_2 on C one defines the *composition of correspondences* by considering $C \times C \times C$ with the projections $p_{ij} : C \times C \times C \to C \times C$ onto two factors and setting

$$T_1 \circ T_2 = (p_{13})_*\big(p_{12}^*(T_1) \cap p_{23}^*(T_2)\big).$$

Set-theoretically

$$T_1 \circ T_2 = \{(x, y) \in C \times C : \exists z \in C : (x, z) \in T_1, (z, y) \in T_2\}.$$

Also $T_1 \circ T_2(x) = T_1(T_2(x))$. Note that if $T_1 = T_2^{-1}$ and T_2 is of type (d, d') we have $T_1(T_2(x)) - dx > 0$. Thus the graph of $T_1 \circ T_2$ contains $d\Delta$. We modify the definition of the composition by setting $T_1 \diamondsuit T_2 = T_1 \circ T_2 - s\Delta$, where s is the largest positive multiple of the diagonal component of $T_1 \circ T_2$.

Proposition 5.5.4 *Let $T_1 \circ T_2 = T_1 \diamondsuit T_2 + s\Delta$. Suppose that T_i is of type (d_i, d_i') and valence ν_i. Then $T_1 \diamondsuit T_2$ is of type $(d_1 d_2 - s, d_1' d_2' - s)$ and valence $-\nu_1\nu_2 + s$.*

Proof Applying Proposition 5.5.1, we can write

$$[T_1] = (d_1' + \nu_1)[\{x\} \times C] + (d_1 + \nu_1)[C \times \{x\}] - \nu_1[\Delta],$$

$$[T_2] = (d_2' + \nu_2)[\{x\} \times C] + (d_2 + \nu_2)[C \times \{x\}] - \nu_2[\Delta].$$

Easy computation with intersections gives

$$[T_1 \diamondsuit T_2] = (d_1' d_2' - \nu_1\nu_2)[\{x\} \times C] + (d_1 d_2 - \nu_1\nu_2)[C \times \{x\}] + (\nu_1\nu_2 - s)[\Delta]$$

$$= (d_1' d_2' - s + \nu)[\{x\} \times C] + (d_1 d_2 - s + \nu)[C \times \{x\}] + \nu[\Delta],$$

where $\nu = -\nu_1\nu_2 + s$. This proves the assertion. □

Example 5.5.5 In Baker's book [21], vol. 6, p. 11, the symmetric correspondence $T\diamondsuit T^{-1}$ is called the *direct lateral correspondence*. If (r, s) is the type of T and γ is its valence, then it is easy to see that $T \circ T = T\diamondsuit T^{-1} + s\Delta$, and we obtain that the type of $T\diamondsuit T^{-1}$ is equal to $(s(r-1), s(r-1))$ and valence $s - \gamma^2$. This agrees with Baker's formula.

Here is one application of a direct lateral correspondence. Consider a correspondence of valence 2 on a plane nonsingular curve C of degree d such that $T(x) = \mathbb{T}_c(C) \cap C - 2x$. In other words, $T(x)$ is equal to the set of the remaining $d-2$ intersection points of the tangent at x with C. For any point $y \in C$ the inverse correspondence assigns to y the divisor $P_y(C) - 2y$, where $P_y(C)$ is the first polar. A united point of $T\diamondsuit T^{-1}$ is one of the two points of the intersection of a bitangent with the curve. We have $s = d(d-1)-2, r = d-2, \nu = 2$. Applying the Cayley-Brill formula, we find that the number b of bitangents is expressed by the following formula

$$2b = 2(d(d-1)-2)(d-3)+(d-1)(d-2)(d(d-1)-6) = d(d-2)(d^2-9). \tag{5.33}$$

As in the case of bitangents to the plane quartic, there exists a plane curve of degree $(d-2)(d^2-9)$ (a *bitangential curve* which cuts out on C the set of tangency points of bitangents (see [492], pp. 342-357).

There are many other applications of the Cayley-Brill formula to enumerative geometry. Many of them go back to Cayley and can be found in Baker's book. Modern proofs of some of these formulas are available in the literature and we omit them.

Recall that a *k-secant line* of an irreducible space curve $C \subset \mathbb{P}^3$ of degree d is a line ℓ such that a general plane contaning ℓ intersects C at $d-k$ points outside ℓ. Equivalently, the projection from ℓ defines a finite map $C \to \mathbb{P}^1$ of degree $d-k$.

The proof of the following formula can be found in [267], Chapter 2, §5.

Proposition 5.5.6 *Let C be a general space curve of genus g and degree d. Then the number of 4-secant lines of C is given by the following formula:*

$$q = \frac{1}{12}(d-2)(d-3)^2(d-4) - \tfrac{1}{2}g(d^2-7d+13-g). \tag{5.34}$$

There is a precise meaning of generality of a curve. We refer to loc. cit. or [358] for the explanation.

The set of trisecant lines is infinite and parameterized by a curve of degree

$$t = (d-2)\frac{(d-1)(d-3)-3g}{3}. \tag{5.35}$$

(see [358]).

5.5.2 Scorza correspondence

Let C be a nonsingular projective curve of genus $g > 0$ and ϑ be a non-effective theta-characteristic on C.

Let

$$d_1 : C \times C \to \mathrm{Jac}(C),\ (x, y) \mapsto [x - y] \tag{5.36}$$

be the difference map. Let $\Theta = W^0_{g-1} - \vartheta$ be the symmetric theta divisor corresponding to ϑ. Define

$$R_\vartheta = d_1^{-1}(\Theta).$$

Set-theoretically,

$$(R_\vartheta)_{\mathrm{red}} = \{(x, y) \in C \times C : h^0(x + \vartheta - y) > 0\}.$$

Lemma 5.5.7 *R_ϑ is a symmetric correspondence of type (g, g), with valence equal to -1 and without united points.*

Proof Since Θ is a symmetric theta divisor, the divisor $d_1^{-1}(\Theta)$ is invariant with respect to the switch of the factors of $X \times X$. This shows that R_ϑ is symmetric.

Fix a point x_0 and consider the map $i : C \to \mathrm{Jac}(C)$ defined by $i(x) = [x - x_0]$. It is known (see [43], Chapter 11, Corollary (2.2)) that

$$\Theta \cdot \iota_*(C) = (C \times \{t_0\}) \cdot d_1^*(\Theta) = g.$$

This shows that R_ϑ is of type (g, g). Also it shows that $R_\vartheta(x_0) - x_0 + \vartheta \in W_{g-1}$. For any point $x \in C$, we have $h^0(\vartheta + x) = 1$ because ϑ is non-effective. Thus $R_\vartheta(x)$ is the unique effective divisor linearly equivalent to $x + \vartheta$. By definition, the valence of R_ϑ is equal to -1. Applying the Cayley-Brill formula we obtain that R_ϑ has no united points. □

Definition 5.5.8 *The correspondence R_ϑ is called the* Scorza correspondence.

Example 5.5.9 Assume $g = 1$ and fix a point on C equipping C with a structure of an elliptic curve. Then ϑ is a nontrivial 2-torsion point. The Scorza correspondence R_ϑ is the graph of the translation automorphism defined by ϑ.

In general, R_ϑ could be neither reduced nor irreducible correspondence. However, for general curve X of genus g everything is as expected.

Proposition 5.5.10 *Assume C is general in the sense that* $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$. *Then R_ϑ is reduced and irreducible.*

Proof The assumption that $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$ implies that any correspondence on $C \times C$ has valence. This implies that the Scorza correspondence is an irreducible curve and is reduced. In fact, it is easy to see that the valence of the sum of two correspondences is equal to the sum of valences. Since R_ϑ has no united points, it follows from the Cayley-Brill formula that the valence of each part must be negative. Since the valence of R_ϑ is equal to -1, we get a contradiction. □

It follows from (5.32) that the divisor class of R_ϑ is equal to

$$R_\vartheta \sim p_1^*(\vartheta) + p_2^*(\vartheta) + \Delta. \tag{5.37}$$

Since $K_{C\times C} = p_1^*(K_C) + p_2^*(K_C)$, applying the adjunction formula and using that $\Delta \cap R = \emptyset$ and the fact that $p_1^*(\vartheta) = p_2^*(\vartheta)$, we easily find

$$\omega_{R_\vartheta} = 3p_1^*\omega_C. \tag{5.38}$$

In particular, the arithmetic genus of R_ϑ is given by

$$p_a(R_\vartheta) = 3g(g-1) + 1. \tag{5.39}$$

Note that the curve R_ϑ is very special, for example, it admits a fixed-point free involution defined by the switching the factors of $X \times X$.

Proposition 5.5.11 *Assume that C is not hyperelliptic. Let R be a symmetric correspondence on $C \times C$ of type (g, g), without united points and some valence. Then there exists a unique non-effective theta characteristic ϑ on C such that $R = R_\vartheta$.*

Proof It follows from the Cayley-Brill formula that the valence ν of R is equal to -1. Thus the divisor class of $R(x) - x$ does not depend on x. Since R has no united points, the divisor class $D = R(x) - x$ is not effective, i.e. $h^0(R(x) - x) = 0$. Consider the difference map $d_1 : C \times C \to \mathrm{Jac}(C)$. For any $(x, y) \in R$, the divisor $R(x) - y \sim D + x - y$ is effective and of degree $g - 1$. Thus $d_1(R) + D \subset W_{g-1}^0$. Let $\sigma : X \times X \to X \times X$ be the switch of the factors. Then

$$\phi(R) = d_1(\sigma(R)) = [-1](d_1(R)) \subset [-1](W_{g-1}^0 - D) \subset W_{g-1}^0 + D',$$

where $D' = K_C - D$. Since $R \cap \Delta = \emptyset$ and C is not hyperelliptic, the equality $d_1(x, y) = d_1(x', y')$ implies $(x, y) = (x', y')$. Thus the difference map d_1 is injective on R. This gives

$$R = d_1^{-1}(W_{g-1}^0 - D) = d_1^{-1}(W_{g-1}^0 - D').$$

Restricting to $\{x\} \times C$ we see that the divisor classes D and D' are equal. Hence D is a theta characteristic ϑ. By assumption, $h^0(R(x) - x) = h^0(\vartheta) = 0$, hence ϑ is non-effective. The uniqueness of ϑ follows from formula (5.37). □

Let $x, y \in R_\vartheta$. Then the sum of two positive divisors $(R_\vartheta(x)-y)+(R_\vartheta(y)-x)$ is linearly equivalent to $x + \vartheta - y + y + \vartheta - x = 2\vartheta = K_C$. This defines a map

$$\gamma : R_\vartheta \to |K_C|, \ (x, y) \mapsto (R_\vartheta(x) - y) + (R_\vartheta(y) - x). \tag{5.40}$$

Recall from [267], p. 360, that the theta divisor Θ defines the *Gauss map*

$$\mathcal{G} : \Theta^0 \to |K_C|,$$

where Θ^0 is the open subset of nonsingular points of Θ. It assigns to a point z the tangent space $T_z(\Theta)$ considered as a hyperplane in

$$T_z(\mathrm{Jac}(C)) \cong H^1(C, \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C(K_C))^\vee.$$

More geometrically, $\mathcal{G}$ assigns to $D - \vartheta$ the linear span of the divisor D in the canonical space $|K_C|^\vee$ (see [10], p. 246). Since the intersection of hyperplane $\gamma(x, y)$ with the canonical curve C contains the divisors $R(x) - y$ (and $R(y) - x$), and they do not move, we see that

$$\gamma = \mathcal{G} \circ d_1.$$

Lemma 5.5.12

$$\gamma^*(\mathcal{O}_{|K_C|}(1)) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta) \cong p_1^*(K_C).$$

Proof The Gauss map $\mathcal{G}$ is given by the normal line bundle $\mathcal{O}_\Theta(\Theta)$. Thus the map γ is given by the line bundle

$$d_1^*(\mathcal{O}_\Theta(\Theta)) = \mathcal{O}_{R_\vartheta}(d_1^*(\Theta)) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta).$$

It remains for us to apply formula (5.37). □

The Gauss map is a finite map of degree $\binom{2g-2}{g-1}$. It factors through the map $\Theta^0 \to \Theta^0/(\iota)$, where ι is the negation involution on $\mathrm{Jac}(C)$. The map γ also factors through the involution of $X \times X$. Thus the degree of the map $R_\vartheta \to \gamma(R_\vartheta)$ is equal to $2d(\vartheta)$, where $d(\vartheta)$ is some numerical invariant of the theta characteristic ϑ. We call it the *Scorza invariant*. Let

$$\Gamma(\vartheta) := \gamma(R_\vartheta).$$

We considered it as a curve embedded in $|K_C|$. Applying Lemma 5.5.12, we obtain the following.

Corollary 5.5.13

$$\deg \Gamma(\vartheta) = \frac{g(g-1)}{d(\vartheta)}.$$

Remark 5.5.14 Let C be a canonical curve of genus g and R_ϑ be a Scorza correspondence on C. For any $x, y \in C$ consider the degree $2g$ divisor $D(x,y) = R_\vartheta(x) + R_\vartheta(y) \in |K_C + x + y|$. Since $|2K_C - (K_C + x + y)| = |K_C - x - y|$, we obtain that the linear system of quadrics through $D(x,y)$ is of dimension $\frac{1}{2}g(g+1) - 2g = \dim |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| - 2g + 1$. This shows that the set $D(x,y)$ imposes one less condition on quadrics passing through this set. For example, when $g = 3$ we get that $D(x,y)$ is on a conic. If $g = 4$ it is the base set of a net of quadrics. We refer to [176] and [209] for projective geometry of sets imposing one less condition on quadrics (called *self-associated sets*).

5.5.3 Scorza quartic hypersurfaces

The following construction due to G. Scorza needs some generality assumption on C.

Definition 5.5.15 *A pair (C, ϑ) is called* Scorza general *if the following properties are satisfied*

(i) R_ϑ *is a connected nonsingular curve;*
(ii) $d(\vartheta) = 1$*;*
(iii) $\Gamma(\vartheta)$ *is not contained in a quadric.*

We will see in the next chapter that a general canonical curve of genus 3 is Scorza general. For higher genus this was proven in [557].

We continue to assume that C is non-hyperelliptic. Consider the canonical embedding $C \hookrightarrow |K_C|^\vee \cong \mathbb{P}^{g-1}$ and identify C with its image (the canonical model of C). For any $x \in C$, the divisor $R_\vartheta(x)$ consists of g points y_i. If all of them are distinct we have g hyperplanes $\gamma(x, y_i) = \langle R_\vartheta(x) - y_i \rangle$, or, g points on the curve $\Gamma(\vartheta)$. More generally, we have a map $C \to C^{(g)}$ defined by the projection $p_1 : R_\vartheta \to C$. The composition of this map with the map $\gamma^{(g)} : C^{(g)} \to \Gamma(\vartheta)^{(g)}$ is a regular map $\phi : C \to \Gamma(\vartheta)^{(g)}$. Let $H \cap C = x_1 + \cdots + x_{2g-2}$ be a hyperplane section of C. Adding up the images of the points x_i under the map ϕ we obtain $g(2g-2)$ points on $\Gamma(\vartheta)$.

Proposition 5.5.16 *Let $D = x_1 + \cdots + x_{2g-2}$ be a canonical divisor on C. Assume (C, ϑ) is Scorza general. Then the divisors*

$$\phi(D) = \sum_{i=1}^{2g-2} \phi(x_i), \quad D \in |K_C|,$$

span a linear system of divisors on $\Gamma(\vartheta)$ *which are cut out by quadrics.*

Proof First note that the degree of the divisor is equal to $2\deg\Gamma(\vartheta)$. Let $(x,y) \in R_\vartheta$ and $D_{x,y} = \gamma(x,y) = (R_\vartheta(x) - y) + (R_\vartheta(y) - x) \in |K_C|$. For any $x_i \in R_\vartheta(x) - y$, the divisor $\gamma(x,x_i)$ contains y. Similarly, for any $x_j \in R_\vartheta(y) - x$, the divisor $\gamma(y,x_j)$ contains x. This means that $\phi(D_{x,y})$ is cut out by the quadric $Q_{x,y}$ equal to the sum of two hyperplanes $\check{H}_x, \check{H}_y$ corresponding to the points $x, y \in C \subset |K_C|^\vee$ via the duality. The image of $|K_C|$ in $\Gamma(\vartheta)^{(g(2g-2))}$ spans a linear system L (since any map of a rational variety to $\mathrm{Jac}(\Gamma(\vartheta))$ is constant). Since $\Gamma(\vartheta)$ is not contained in a quadric, it generates $|K_C|$. This shows that all divisors in L are cut out by quadrics. The quadrics $Q_{x,y}$ span the space of quadrics in $|K_C|$ since otherwise there exists a quadric in $|K_C|^\vee$ apolar to all quadrics $Q_{x,y}$. This would imply that for a fixed $x \in C$, the divisor $R_\vartheta(x)$ lies in a hyperplane, the polar hyperplane of the quadric with respect to the point x. However, because ϑ is non-effective, $\langle R_\vartheta(x)\rangle$ spans $\mathbb{P}^{g-1}$. Thus $\dim L \geq g(g+1)/2$, and, since no quadrics contains $\Gamma(\vartheta)$, L coincides with the linear system of divisors on $\Gamma(\vartheta)$ cut out by quadrics. □

Let $E = H^0(C,\omega_C)^\vee$. We can identify the space of quadrics in $|E|$ with $\mathbb{P}(S^2(E))$. Using the previous Proposition, we obtain a map $|E^\vee| \to |S^2(E)|$. The restriction of this map to the curve $\Gamma(\vartheta)$ is given by the linear system $|\mathcal{O}_{\Gamma(\vartheta)}(2)|$. This shows that the map is given by quadratic polynomials, so defines a linear map

$$\alpha : S^2(E^\vee) \to S^2(E).$$

The proof of the Proposition implies that this map is bijective.

Theorem 5.5.17 *Assume* (C,ϑ) *is Scorza general. Then there exists a unique quartic hypersurface* $V(f)$ *in* $|E| = \mathbb{P}^{g-1}$ *such that the inverse linear map* α^{-1} *is equal to the polarization map* $\psi \mapsto D_\psi(f)$.

Proof Consider $\alpha^{-1} : S^2(E) \to S^2(E^\vee)$ as a tensor $U \in S^2(E^\vee) \otimes S^2(E^\vee) \subset (E^\vee)^{\otimes 4}$ viewed as a 4-multilinear map $E^4 \to \mathbb{C}$. It is enough to show that U is totally symmetric. Then α^{-1} is defined by the apolarity map associated to a quartic hypersurface. Fix a reduced divisor $R_\vartheta(x) = x_1 + \cdots + x_g$. Let H_i be the hyperplane in $|E|$ spanned by $R_\vartheta(x) - x_i$. Choose a basis $(t_1,\ldots,t_g)$ in $E^\vee$ such that $H_i = V(t_i)$. It follows from the proof of Proposition 5.5.16 that the quadratic map $\mathbb{P}(E^\vee) \to \mathbb{P}(S^2(E))$ assigns to the hyperplane H_i the quadric Q_{x,x_i} equal to the union of two hyperplanes associated

to x and x_i via the duality. The corresponding linear map α satisfies

$$\alpha(t_j^2) = \xi_j(\sum_{i=1}^{g} b_i\xi_i),\ j = 1, \dots, g, \tag{5.41}$$

where $(\xi_1, \dots, \xi_g)$ is the dual basis to $(t_1, \dots, t_g)$, and $(b_1, \dots, b_g)$ are the coordinates of the point x. This implies that

$$U(\xi_j, \sum_{i=1}^{g} b_i\xi_i, \xi_k, \xi_m) = \begin{cases} 1 & \text{if } j = k = m, \\ 0 & \text{otherwise} \end{cases} = U(\xi_k, \sum_{i=1}^{g} b_i\xi_i, \xi_j, \xi_m).$$

This shows that U is symmetric in the first and the third arguments when the second argument belongs to the curve $\Gamma(\vartheta)$. Since the curve $\Gamma(\vartheta)$ spans $\mathbb{P}(E^\vee)$, this is always true. It remains for us to use that U is symmetric in the first and the second arguments, as well as in the third and the fourth arguments. □

Definition 5.5.18 *Let (C, ϑ) be Scorza general pair consisting of a canonical curve of genus g and a non-effective theta characteristic ϑ. Then the quartic hypersurface $V(f)$ is called the* Scorza quartic hypersurface *associated to (C, ϑ).*

We will study the Scorza quartic plane curves in the case $g = 3$. Very little is known about Scorza hypersurfaces for general canonical curves of genus > 3. We do not even know whether they are nonsingular. However, it follows from the construction that the hypersurface is given by a nondegenerate homogeneous form.

The Scorza correspondence has been recently extended to pairs (C, θ), where C is a curve of genus $g > 1$ and θ is an effective even theta characteristic [317], [270].

5.5.4 Contact hyperplanes of canonical curves

Let C be a nonsingular curve of genus $g > 0$. Fixing a point c_0 on C allows one to define an isomorphism of algebraic varieties $\text{Pic}^d(C) \to \text{Jac}(C)$, $[D] \mapsto [D - dc_0]$. Composing this map with the map $u_d : C^{(d)} \to \text{Pic}^d(C)$ we obtain a map

$$u_d(c_0) : C^{(d)} \to \text{Jac}(C). \tag{5.42}$$

If no confusion arises, we drop c_0 from this notation. For $d = 1$, this map defines an embedding

$$u_1 : C \hookrightarrow \text{Jac}(C).$$

For the simplicity of the notation, we will identify C with its image. For any $c \in C$ the tangent space of C at a point c is a 1-dimensional subspace of the tangent space of $\mathrm{Jac}(C)$ at c. Using a translation automorphism, we can identify this space with the tangent space $T_0\mathrm{Jac}(C)$ at the zero point. Under the Abel-Jacobi map, the space of holomorphic 1-forms on $\mathrm{Jac}(C)$ is identified with the space of holomorphic forms on C. Thus we can identify $T_0\mathrm{Jac}(C)$ with the space $H^0(C, K_C)^\vee$. As a result, we obtain the *canonical map* of C

$$\varphi : C \to \mathbb{P}(H^0(C, K_C)^\vee) = |K_C|^\vee \cong \mathbb{P}^{g-1}.$$

If C is not hyperelliptic, the canonical map is an embedding.

We continue to identify $H^0(C, K_C)^\vee$ with $T_0\mathrm{Jac}(C)$. A symmetric odd theta divisor $\Theta = W^0_{g-1} - \vartheta$ contains the origin of $\mathrm{Jac}(C)$. If $h^0(\vartheta) = 1$, the origin is a nonsingular point on Θ, and hence Θ defines a hyperplane in $T_0(\mathrm{Jac}(C))$, the tangent hyperplane $T_0\Theta$. Passing to the projectivization we have a hyperplane in $|K_C|^\vee$.

Proposition 5.5.19 *The hyperplane in $|K_C|^\vee$ defined by Θ is a contact hyperplane to the image $\varphi(C)$ under the canonical map.*

Proof Consider the difference map (5.36) $d_1 : C \times C \to \mathrm{Jac}(C)$. In the case when Θ is an even divisor, we proved in (5.37) that

$$d_1^*(\Theta) \sim p_1^*(\theta) + p_2^*(\theta) + \Delta. \tag{5.43}$$

Since two theta divisors are algebraically equivalent the same is true for an odd theta divisor. The only difference is that $d_1^*(\Theta)$ contains the diagonal Δ as the preimage of 0. It follows from the definition of the map $u_1(c_0)$ that

$$u_1(c_0)(C) \cap \Theta = d_1^{-1}(\Theta) \cap p_1^{-1}(c_0) = c_0 + D_\vartheta,$$

where D_ϑ is the unique effective divisor linearly equivalent to ϑ. Let $\mathcal{G} : \Theta \to \mathbb{P}(T_0\mathrm{Jac}(C))$ be the Gauss map defined by translation of the tangent space at a nonsingular point of Θ to the origin. It follows from the proof of Torelli Theorem [10] that the Gauss map ramifies at any point where Θ meets $u_1(C)$. So, the image of the Gauss map intersects the canonical image with multiplicity ≥ 2 at each point. This proves the assertion. □

More explicitly, the equation of the contact hyperplane corresponding to Θ is given by the linear term of the Taylor expansion of the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right]$ corresponding to Θ. Note that the linear term is a linear function on $H^0(C, K_C)^\vee$, hence can be identified with a holomorphic differential

$$h_\Theta = \sum_{i=1}^{g} \frac{\partial\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](z,\tau)}{\partial z_i}(0)\omega_i,$$

where $(z_1, \dots, z_g)$ are coordinates in $H^0(C, K_C)^\vee$ defined by a normalized basis $\omega_1, \dots, \omega_g$ of $H^0(C, K_C)$. A nonzero section of $\mathcal{O}_{\mathrm{Jac}(C)}(\Theta)$ can be viewed as a holomorphic differential of order $\frac{1}{2}$. To make this more precise, i.e. describe how to get a square root of a holomorphic 1-form, we use the following result (see [217], Proposition 2.2).

Proposition 5.5.20 *Let Θ be a symmetric odd theta divisor defined by the theta function $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$. Then, for all $x, y \in C$,*

$$\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-y))^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y))E(x,y)^2,$$

where $E(x, y)$ is a certain section of $\mathcal{O}_{C\times C}(\Delta)$ (the prime-form*).*

An attentive reader should notice that the equality is not well-defined in many ways. First, the vector $\varphi(x)$ is defined only up to proportionality and the value of a section of a line bundle is also defined only up to proportionality. To make sense of this equality we pass to the universal cover of $\mathrm{Jac}(C)$ identified with $H^0(C, K_C)^\vee$ and to the universal cover U of $C \times C$ and extend the difference map and the map φ to the map of universal covers. Then the prime-form is defined by a certain holomorphic function on U and everything makes sense. As the equality of the corresponding line bundles, the assertion trivially follows from (5.43).

Let

$$\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y) = \frac{\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-y))}{E(x,y)}.$$

Since $E(x, y) = -E(y, x)$ and $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x, y) = \mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](y, x)$ for any $x, y \in C \times C \setminus \Delta$. It satisfies

$$\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y)^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y)). \tag{5.44}$$

Note that $E(x, y)$ satisfies $E(x, y) = -E(y, x)$, since $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x, y) = \mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](y, x)$ for any $x, y \in C \times C \setminus \Delta$.

Now let us fix a point $y = c_0$, so we can define the *root function* on C. It is a rational function on the universal cover of C defined by $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x, c_0)$.

Thus every contact hyperplane of the canonical curve defines a root function.

Suppose we have two odd theta functions $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right], \theta\left[\begin{smallmatrix}\epsilon'\\ \eta'\end{smallmatrix}\right]$. Then the ratio of the corresponding root functions is equal to $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-c_0))/\theta\left[\begin{smallmatrix}\epsilon'\\ \eta'\end{smallmatrix}\right](d_1(x-c_0))$ and its square is a rational function on C, defined uniquely up to a constant factor depending on the choice of c_0. Its divisor is equal to the difference $2\vartheta - 2\vartheta'$. Thus we can view the ratio as a section of $K_X^{\frac{1}{2}}$ with divisor $\theta - \theta'$. This section is not defined on C but on the double cover of C corresponding to the 2-torsion

point $\vartheta - \vartheta'$. If we have two pairs $\vartheta_1, \vartheta', \vartheta_2, \vartheta'_2$ of odd theta characteristics satisfying $\vartheta_1 - \vartheta' = \vartheta_2 - \vartheta'_2 = \epsilon$, i.e. forming a syzygetic tetrad, the product of the two ratios is a rational function on C with divisor $\vartheta_1 + \vartheta'_2 - \vartheta'_1 - \vartheta_2$. Following Riemann [471] and Weber [595], we denote this function by $(\vartheta_1\vartheta'_1/\vartheta_2\vartheta'_2)^{1/2}$. By Riemann-Roch, $h^0(\vartheta_1 + \vartheta'_2) = h^0(K_C + \epsilon) = g - 1$, hence any g pairs $(\vartheta_1, \vartheta'_1), \ldots, (\vartheta_g, \vartheta'_g)$ of odd theta characteristics in a Steiner complex define g linearly independent functions $(\vartheta_1\vartheta'_1/\vartheta_g\vartheta'_g)^{1/2}, \ldots, (\vartheta_{g-1}\vartheta'_{g-1}/\vartheta_g\vartheta'_g)^{1/2}$. After scaling, and getting rid of squares by using (5.44), we obtain a polynomial in $h_{\Theta_1}(\varphi(x)), \ldots, h_{\Theta_g}(\varphi(x))$ vanishing on the canonical image of C.

Example 5.5.21 Let $g = 3$. We take three pairs of odd theta functions and get the equation

$$\sqrt{\vartheta_1\vartheta'_1} + \sqrt{\vartheta_2\vartheta'_2} + \sqrt{\vartheta_3\vartheta'_3} = 0. \tag{5.45}$$

After getting rid of square roots, we obtain a quartic equation of C

$$(lm + pq - rs)^2 - 4lmpq = 0, \tag{5.46}$$

where l, m, p, q, rs are the linear functions in z_1, z_2, z_3 defining the linear terms of the Taylor expansion at 0 of the odd theta functions corresponding to three pairs in a Steiner complex. The number of possible ways to write the equation of a plane quartic in this form is equal to $63 \cdot 20 = 1260$.

Remark 5.5.22 For any nonzero 2-torsion point, the linear system $|K_C + \epsilon|$ maps C to $\mathbb{P}^{g-2}$, the map is called the *Prym canonical map*. We have seen that the root functions $(\vartheta_1\vartheta'_1/\vartheta_2\vartheta'_2)^{1/2}$ belong to $H^0(C, K_C + \epsilon)$ and can be used to define the Prym canonical map. For $g = 3$, the map is a degree 4 cover of $\mathbb{P}^1$ and we express the quartic equation of C as a degree 4 cover of $\mathbb{P}^1$.

Exercises

5.1 Let C be an irreducible plane curve of degree d with a $(d-2)$-multiple point. Show that its normalization is a hyperelliptic curve of genus $g = d - 2$. Conversely, show that any hyperelliptic curve of genus g admits such a plane model.

5.2 Show that a nonsingular curve of genus 2 has a vanishing theta characteristic but a nonsingular curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve.

5.3 Show that a nonsingular non-hyperelliptic curve of genus 4 has a vanishing theta characteristic if and only if its canonical model lies on a quadratic cone.

5.4 Find the number of vanishing theta characteristics on a hyperelliptic curve of genus g.

5.5 Show that a canonical curve of genus 5 has 10 vanishing even theta characteristics if and only if it is isomorphic to the intersection of three simultaneously diagonalized quadrics in $\mathbb{P}^4$.

5.6 Compute the number of syzygetic tetrads contained in a Steiner complex.

5.7 Show that the composition of two correspondences (defined as the composition of the multi-valued maps defined by the correspondences) with valences ν and ν' is a correspondence with valence $-\nu\nu'$.

5.8 Let $f : X \to \mathbb{P}^1$ be a non-constant rational function on a nonsingular projective curve X. Consider the fibered product $X \times_{\mathbb{P}^1} X$ as a correspondence on $X \times X$. Show that it has valence and compute the valence. Show that the Cayley-Brill formula is equivalent to the Hurwitz formula.

5.9 Suppose that a nonsingular projective curve X admits a non-constant map to a curve of genus > 0. Show that there is a correspondence on X without valence.

5.10 Show that any correspondence on a nonsingular plane cubic has valence unless the cubic is harmonic or equianharmonic.

5.11 Describe all symmetric correspondences of type $(4, 4)$ with valence 1 on a canonical curve of genus 4.

5.12 Let R_ϑ be the Scorza correspondence on a curve C. Prove that a point $(x, y) \in R_\vartheta$ is singular if and only if x and y are ramification points of the projections $R_\vartheta \to C$.

Historical Notes

It is too large a task to discuss the history of theta functions. We mention only that the connection between odd theta functions with characteristics and bitangents to a quartic curves goes back to Riemann [471], [595]. There are numerous expositions of the theory of theta functions and Jacobian varieties (e.g. [10], [114], [406]). The theory of fundamental sets of theta characteristics goes back to A. Göpel and J. Rosenhein. A good exposition can be found in Krazer's book [348]. As an abstract symplectic geometry over the field of two elements it is presented in Coble's book [121], which we followed. Some additional material can be found in [116] (see also a modern exposition in [484]).

The theory of correspondences on an algebraic curve originates from *Charles' Principle of Correspondence* [94] which is the special case of the Cayley-Brill formula in the case $g = 0$. However, the formula was known and used earlier by E. de Jonquières [159], and later but before Chasles, by L. Cremona in [141]. We refer to C. Segre [521] for a careful early history of this discovery and the polemic between Chasles and de Jonquières on the priority of this discovery.

We have already encountered with the application of Chasles' Principles to Poncelet polygons in Chapter 2. This application was first found by A. Cayley [85]. He was also the first to extend Chasles' Principle to higher genus [85] although with incomplete proof. The first proof of the Cayley-Brill formula was given by A. Brill [55]. The notion of valence (die Werthigeit) was

introduced by Brill. The fact that only correspondences with valence exist on a general curve was first pointed out by A. Hurwitz [309]. He also showed the existence of correspondences without valence. A good reference to many problems solved by the theory of correspondences is Baker's book [21], vol. 6. We refer to [535] for a fuller history of the theory of correspondences.

The number of bitangents to a plane curve was first computed by J. Plücker [448], [449]. The equations of bitangential curves were given by A. Cayley [79], G. Salmon [492] and O. Dersch [165].

The study of correspondences of type (g, g) with valence -1 was initiated by G. Scorza [511], [512]. His construction of a quartic hypersurface associated to a non-effective theta characteristic on a canonical curve of genus g was given in [513]. A modern exposition of Scorza' theory was first given in [177].

6
Plane Quartics

6.1 Bitangents

6.1.1 28 bitangents

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded in $\mathbb{P}^2$ by its canonical linear system $|K_C|$. It has no vanishing theta characteristics, so the only effective theta characteristics are odd ones. The number of them is $28 = 2^2(2^3 - 1)$. Thus C has exactly 28 contact lines, which in this case coincide with bitangents. Each bitangent is tangent to C at two points that may coincide. In the latter case the bitangent is called a *inflection bitangent*.

We can apply the results from Section 5.4 to the case $g = 3$. Let $V = \mathrm{Pic}(C)[2] \cong \mathbb{F}_2^6$ with the symplectic form ω defined by the Weil pairing. The elements of $Q(V)_-$, i.e. quadratic forms of odd type on V, will be identified with bitangents.

The union of four bitangents forming a syzygetic tetrad cuts out in C an effective divisor of degree 8 equal to the intersection of C with some conic $V(q)$. There are $t_3 = 315$ syzygetic tetrads which are in a bijective correspondence with the set of isotropic planes in $\mathrm{Pic}(C)[2]$.

Since a syzygetic tetrad of bitangents and the conic $V(q)$ cuts out the same divisor, we obtain the following.

Proposition 6.1.1 *A choice of a syzygetic tetrad of bitangents $V(l_i), i = 1, \dots, 4$, puts the equation of C in the form*

$$C = V(l_1 l_2 l_3 l_4 + q^2). \tag{6.1}$$

Conversely, each such equation defines a syzygetic tetrad of bitangents. There are 315 ways to write f in this form.

There are 63 Steiner complexes of bitangents. Each complex consists of six

pairs of bitangents ℓ_i, ℓ_i' such that the divisor class of $\ell_i \cap C - \ell_i' \cap C$ is a fixed nonzero 2-torsion divisor class.

Proposition 6.1.2 *Let $(l, m), (p, q), (r, s)$ be three pairs of linear forms defining three pairs of bitangents from a Steiner complex. Then, after scaling the forms, one can write the equation of C in the form*

$$4lmpq - (lm + pq - rs)^2 = 0, \tag{6.2}$$

which is equivalent to the equation

$$\sqrt{lm} + \sqrt{pq} + \sqrt{rs} = 0 \tag{6.3}$$

after getting rid of square roots. Conversely, an equation of this form is defined by three pairs of bitangents from a Steiner complex. The number of ways in which the equation can be written in this form is equal to $1260 = \binom{6}{3} \cdot 63$.

Proof By (6.1), we can write

$$C = V(lmpq - a^2) = V(lmrs - b^2)$$

for some quadratic forms a, b. After subtracting the equations, we get

$$lm(pq - rs) = (a + b)(a - b).$$

If l divides $a + b$ and m divides $a - b$, then the quadric $V(a)$ passes through the point $l \cap m$. But this is impossible since no two bitangents intersect at a point on the quartic. Thus, we obtain that lm divides either $a + b$ or $a - b$. Without loss of generality, we get $lm = a + b, pq - rs = a - b$, and hence $a = \frac{1}{2}(lm + pq - rs)$. Therefore, we can define the quartic by the equation $4lmpq - (lm + pq - rs)^2 = 0$. Conversely, Equation (6.2) defines a syzygetic tetrad $V(l), V(m), V(p), V(q)$. By the symmetry of Equation (6.3), we obtain two other syzygetic tetrads $V(l), V(m), V(r), V(s)$ and $V(p), V(q), V(r), V(s)$. Obviously, the pairs $(l, m), (p, q), (r, s)$ define the same 2-torsion divisor class, so they belong to a Steiner hexad. □

In the previous Chapter we found this equation by using theta functions (see (5.45)).

Remark 6.1.3 Consider the 4-dimensional algebraic torus

$$T = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in (\mathbb{C}^*)^6 : z_1 z_2 = z_3 z_4 = z_5 z_6\} \cong (\mathbb{C}^*)^4.$$

It acts on 6-tuples of linear forms $(l_1, \ldots, l_6) \in (\mathbb{C}^3)^6 \cong \mathbb{C}^{18}$ by scalar multiplication. The group $G = \mathbb{F}_2^3 \rtimes \mathfrak{S}_3$ of order 48 acts on the same space by permuting two forms in each pair $(l_i, l_{i+1}), i = 1, 3, 5$, and permuting the three pairs. This action commutes with the action of T and defines a linear action of

the group $T \times G$ on $\mathbb{P}^{17} = \mathbb{C}^{18} \setminus \{0\}/\mathbb{C}^*$. The GIT-quotient $X = \mathbb{P}^{17}/(T \times G)$ is a projective variety of dimension 14. A rational map $X \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|$ which assigns to a general orbit of $T \times G$ the quartic curve $V(\sqrt{l_l l_2} + \sqrt{l_3 l_4} + \sqrt{l_5 l_6})$ is a $\mathrm{SL}(3)$-equivariant and of degree $48 \cdot 1260$. I do not know whether $X/\mathrm{SL}(3)$ is a rational variety; the orbit space $|\mathcal{O}_{\mathbb{P}^2}(4)|/\mathrm{SL}(3)$ is known to be a rational variety [330], [46].

We know that two Steiner complexes have either four or six common bitangents, depending on whether they are syzygetic or not. Each isotropic plane in $\mathrm{Pic}(C)[2]$ defines three Steiner complexes with common four bitangents. Two azygetic Steiner complexes have 6 common bitangents. The number of azygetic triads is equal to 336.

The projection from the intersection point of two bitangents defines a g^1_4 with two members of the form $2p + 2q$. It is possible that more than two bitangents are concurrent. However, we can prove the following.

Proposition 6.1.4 *No three bitangents forming an azygetic triad can intersect at one point.*

Proof Let $\vartheta_1, \vartheta_2, \vartheta_3$ be the corresponding odd theta characteristics. The 2-torsion divisor classes $\epsilon_{ij} = \vartheta_i - \vartheta_j$ form a non-isotropic plane. Let ϵ be a nonzero point in the orthogonal complement. Then $q_{\eta_i}(\epsilon) + q_{\eta_j}(\epsilon) + \langle \eta_{ij}, \epsilon \rangle = 0$ implies that q_{η_i} take the same value at ϵ. We can always choose ϵ such that this value is equal to 0. Thus the three bitangents belong to the same Steiner complex $\Sigma(\epsilon)$. Obviously, no two differ by ϵ, hence we can form 3 pairs from them. These pairs can be used to define Equation (6.2) of C. It follows from this equation that the intersection point of the three bitangents lies on C. But this is impossible because C is nonsingular. □

Remark 6.1.5 A natural question is whether the set of bitangents determines the quartic, i.e. whether two quartics with the same set of bitangents coincide. Surprizingly it has not been answered by the ancients. Only recently it was proven that the answer is yes: [62] (for a general curve), [356] (for any nonsingular curve).

6.1.2 Aronhold sets

We know that in the case $g = 3$ a normal fundamental set of eight theta characteristics contains seven odd theta characteristics. The corresponding unordered set of seven bitangents is called an *Aronhold set*. It follows from (5.30) that the number of Aronhold sets is equal to $\#\mathrm{Sp}(6, \mathbb{F}_2)/7! = 288$.

A choice of an ordered Aronhold set defines a unique fundamental set that

contains it. The eighth theta characteristic is equal to the sum of the characteristics from the Aronhold set. Thus an Aronhold set can be defined as an azygetic set of seven bitangents.

A choice of an ordered Aronhold set allows one to index all 2-torsion divisor classes (resp. odd theta characteristics) by subsets of even cardinality (resp. of cardinality 2) of $\{1,\dots,8\}$, up to complementary set. Thus we have 63 2-torsion classes $\epsilon_{ab}, \epsilon_{abcd}$ and 28 bitangents ℓ_{ij} corresponding to 28 odd theta characteristics ϑ_{ij}. The bitangents from the Aronhold set correspond to the subsets $(18, 28, \dots, 78)$.

We also know that $\vartheta_A - \vartheta_B = \epsilon_{A+B}$. This implies, for example, that four bitangents $\ell_A, \ell_B, \ell_C, \ell_D$ form a syzygetic tetrad if and only if $A + B + C + D = 0$.

Following Cayley, we denote a pair of numbers from the set $\{1,\dots,8\}$ by a vertical line $|$. If two pairs have a common number we make them intersect. For example, we have the following.

- Pairs of bitangents: 210 of type $||$ and 168 of type $\vee$.
- Triads of bitangents:
  1. (syzygetic) 420 of type $\sqcup$, 840 azygetic of type $|||$;
  2. (azygetic) 56 of type $\triangle$, 1680 of type $\vee\,|$, and 280 of type $\diagdown\!\!\diagup$.
- Tetrads of bitangents:
  1. (syzygetic) 105 azygetic of types $||||$, 210 of type $\square$;
  2. (asyzygetic) 560 of types $|\,\triangle$, 280 of type $\diagdown\!\!\!\angle$, 1680 of type $|\diagdown\!\!\diagup$, 2520 of type $\vee\vee$;
  3. (non syzygetic but containing a syzygetic triad) 2520 of type $||\,\vee$, 5040 of type $|\,\sqcup$, 3360 of type $\sqcup\!\!\sqcap$, 840 of type $\triangle\!\!\diagup$, 3360 of type $\diagdown\!\!\wedge$.

There are two types of Aronhold sets: $\ast$, $\diagdown\!\!\!\angle\,\triangle$. They are represented by the sets $(12, 13, 14, 15, 16, 17, 18)$ and $(12, 13, 23, 45, 46, 47, 48)$. The number of the former type is 8, the number of the latter type is 280. Note that the different types correspond to orbits of the subgroup of $\mathrm{Sp}(6, \mathbb{F}_2)$ isomorphic to the permutation group $\mathfrak{S}_8$. For example, we have two orbits of $\mathfrak{S}_8$ on the set of Aronhold sets consisting of 8 and 280 elements.

Lemma 6.1.6 *Three odd theta characteristics $\vartheta_1, \vartheta_2, \vartheta_3$ in a Steiner complex $\Sigma(\epsilon)$, no two of which differ by ϵ, are azygetic.*

Proof Let $\vartheta_i' = \vartheta_i + \epsilon$, $i = 1, 2, 3$. Then $\{\vartheta_1, \vartheta_1', \vartheta_2, \vartheta_2'\}$ and $\{\vartheta_1, \vartheta_1', \vartheta_3, \vartheta_3'\}$ are syzygetic and have two common theta characteristics. By Proposition 5.4.13, the corresponding isotropic planes do not span an isotropic 3-space. Thus $\langle \vartheta_1 - \vartheta_2, \vartheta_3 - \vartheta_1 \rangle = 1$, hence $\vartheta_1, \vartheta_2, \vartheta_3$ is an azygetic triad. □

The previous Lemma suggests a way to construct an Aronhold set from a Steiner set $\Sigma(\epsilon)$. Choose another Steiner set $\Sigma(\eta)$ azygetic to the first one. They intersect at six odd theta characteristics $\vartheta_1, \ldots, \vartheta_6$, no two of which differ by ϵ. Consider the set $\{\vartheta_1, \ldots, \vartheta_5, \vartheta_6 + \epsilon, \vartheta_6 + \eta\}$. We claim that this is an Aronhold set. By the previous Lemma all triads $\vartheta_i, \vartheta_j, \vartheta_k, i, j, k \leq 5$ are azygetic. Any triad $\vartheta_i, \vartheta_6 + \epsilon, \vartheta_6 + \eta, i \leq 5$, is azygetic too. In fact $\vartheta_i((\vartheta_6 + \epsilon) + (\vartheta_6 + \eta)) = \vartheta_i(\epsilon + \eta) \neq 0$ since $\vartheta_i \notin \Sigma(\epsilon + \eta)$. So the assertion follows from Lemma 5.4.2. We leave it to the reader to check that remaining triads $\{\vartheta_i, \vartheta_j, \vartheta_6 + \epsilon\}, \{\vartheta_i, \vartheta_j, \vartheta_6 + \eta\}, i \leq 5$, are azygetic.

Proposition 6.1.7 *Any six lines in an Aronhold set are contained in a unique Steiner complex.*

We use that the symplectic group $\mathrm{Sp}(6, \mathbb{F}_2)$ acts transitively on the set of Aronhold sets. So it is enough to check the assertion for one Aronhold set. Let it correspond to the index set $(12, 13, 14, 15, 16, 17, 18)$. It is enough to check that the first six are contained in a unique Steiner complex. By Proposition 5.4.7, it is enough to exhibit a 2-torsion divisor class ϵ_{ij} such that $\vartheta_{1k}(\epsilon_{ij}) = 0$ for $k = 2, 3, 4, 5, 6, 7$, and show its uniqueness. By Proposition 5.4.19, ϵ_{18} does the job.

Recall that a Steiner subset of theta characteristics on a genus 3 curve consists of 12 elements. A subset of six elements will be called a *hexad*.

Corollary 6.1.8 *Any Steiner complex contains 2^6 azygetic hexads. Half of them are contained in another Steiner complex, necessarily azygetic to the first one. Any other hexad can be extended to a unique Aronhold set.*

Proof Let $\Sigma(\epsilon)$ be a Steiner complex consisting of six pairs of odd theta characteristics. Consider it as G-set, where $G = (\mathbb{Z}/2\mathbb{Z})^6$ whose elements, identified with subsets I of $[1, 6]$, act by switching elements in i-th pairs, $i \in I$. It is clear that G acts simply transitively on the set of azygetic sextuples in $\Sigma(\epsilon)$. For any azygetic complex $\Sigma(\eta)$, the intersection $\Sigma(\epsilon) \cap \Sigma(\eta)$ is an azygetic hexad. Note that two syzygetic complexes have only four bitangents in common. The number of such hexads is equal to $2^6 - 2^5 = 2^5$. Thus the set of azygetic hexads contained in a unique Steiner complex is equal to $2^5 \cdot 63$. But this number is equal to the number $7 \cdot 288$ of subsets of cardinality 6 of Aronhold sets. By the previous Proposition, all such sets are contained in a unique Steiner complex. □

Let $(\vartheta_{18}, \ldots, \vartheta_{78})$ be an Aronhold set. By Proposition 6.1.7, the hexad ϑ_{28}, $\ldots, \vartheta_{78}$ is contained in a unique Steiner complex $\Sigma(\epsilon)$. Let $\vartheta'_{28} = \vartheta_{28} + \epsilon$. By Proposition 5.4.19, the only 2-torsion point ϵ_{ij} at which all quadrics

$\vartheta_{28}, \ldots, \vartheta_{78}$ vanish is the point ϵ_{18}. Thus $\vartheta'_{28} = \vartheta_{28} + \epsilon_{18} = \vartheta_{12}$. This shows that the bitangent defined by ϑ'_{28} coincides with ϑ_{12}. Similarly, we see that the bitangents corresponding to $\vartheta_{i8} + \epsilon, i = 3, \ldots, 7$, coincide with the bitangents ϑ_{1i}.

6.1.3 Riemann's equations for bitangents

Here we show how to write equations of all bitangents knowing the equations of an Aronhold set of bitangents.

Let $\ell_1 = V(l_1), \ldots, \ell_7 = V(l_7)$ be an Aronhold set of bitangents of C. By Proposition 6.1.4, any three lines are not concurrent. We may assume that

$$\ell_1 = V(t_0), \ell_2 = V(t_1), \ell_3 = V(t_2), \ell_4 = V(t_0 + t_1 + t_2)$$

and the remaining ones are $\ell_{4+i} = V(a_{0i}t_0 + a_{1i}t_1 + a_{2i}t_2), i = 1, 2, 3$.

Theorem 6.1.9 *There exist linear forms u_0, u_1, u_2 such that, after rescaling the linear forms,*

$$C = V(\sqrt{t_0u_0} + \sqrt{t_1u_1} + \sqrt{t_2u_2}).$$

The forms u_i can be found from equations

$$\begin{aligned}
u_0 + u_1 + u_2 + t_0 + t_1 + t_2 &= 0,\\
\frac{u_0}{a_{01}} + \frac{u_1}{a_{11}} + \frac{u_2}{a_{21}} + k_1a_{01}t_0 + k_1a_{11}t_1 + k_1a_{21}t_2 &= 0,\\
\frac{u_0}{a_{02}} + \frac{u_1}{a_{12}} + \frac{u_2}{a_{22}} + k_2a_{02}t_0 + k_2a_{12}t_1 + k_2a_{22}t_2 &= 0,\\
\frac{u_0}{a_{03}} + \frac{u_1}{a_{13}} + \frac{u_2}{a_{23}} + k_3a_{03}t_0 + k_3a_{13}t_1 + k_3a_{23}t_2 &= 0,
\end{aligned}$$

where k_1, k_2, k_3 can be found from solving first linear equations:

$$\begin{pmatrix} \frac{1}{a_{01}} & \frac{1}{a_{02}} & \frac{1}{a_{03}} \\ \frac{1}{a_{11}} & \frac{1}{a_{12}} & \frac{1}{a_{13}} \\ \frac{1}{a_{21}} & \frac{1}{a_{22}} & \frac{1}{a_{23}} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

and then solving the equations

$$\begin{pmatrix} \lambda_0a_{01} & \lambda_1a_{11} & \lambda_2a_{21} \\ \lambda_0a_{02} & \lambda_1a_{12} & \lambda_2a_{22} \\ \lambda_0a_{03} & \lambda_1a_{13} & \lambda_2a_{23} \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The equations of the remaining 21 bitangents are:

(1) $u_0 = 0,\ u_1 = 0,\ u_2 = 0,$

(2) $t_0 + t_1 + u_2 = 0,\ t_0 + t_2 + u_1 = 0,\ t_1 + t_2 + u_0 = 0,$
(3) $\frac{u_0}{a_{01}} + k_i(a_{1i}t_1 + a_{2i}t_2) = 0,\ i = 1, 2, 3,$
(4) $\frac{u_1}{a_{11}} + k_i(a_{0i}t_0 + a_{2i}t_2) = 0,\ i = 1, 2, 3,$
(5) $\frac{u_2}{a_{21}} + k_i(a_{0i}t_0 + a_{1i}t_1) = 0,\ i = 1, 2, 3,$
(6) $\frac{t_0}{1-k_i a_{1i}a_{2i}} + \frac{t_1}{1-k_i a_{0i}a_{2i}} + \frac{t_2}{1-k_i a_{0i}a_{1i}} = 0,\ i = 1, 2, 3,$
(7) $\frac{t_0}{a_{0i}(1-k_i a_{1i}a_{2i})} + \frac{t_1}{a_{1i}(1-k_i a_{0i}a_{2i})} + \frac{t_2}{a_{2i}(1-k_i a_{0i}a_{1i})} = 0,\ i = 1, 2, 3.$

Proof By Proposition 6.1.7, six bitangents in our set of seven bitangents $\ell_1, \dots, \ell_7$ are contained in a unique Steiner complex. Throwing away ℓ_1, ℓ_2, ℓ_3, we find three Steiner complexes partitioned in pairs

$$\begin{aligned} &(\ell_2, \xi_3),\ (\ell_3, \xi_2),\ (\ell_4, \xi_{41}),\ \dots,\ (\ell_7, \xi_{71}), \\ &(\ell_3, \xi_1),\ (\ell_1, \xi_3),\ (\ell_4, \xi_{42}),\ \dots,\ (\ell_7, \xi_{72}), \\ &(\ell_1, \xi_2),\ (\ell_2, \xi_1),\ (\ell_4, \xi_{43}),\ \dots,\ (\ell_7, \xi_{73}). \end{aligned} \tag{6.4}$$

Since two Steiner complexes cannot contain more than six common bitangents, the bitangents $\xi_i = V(u_{i-1})$ and $\xi_{ij} = V(l_{ij})$ are all different and differ from $\ell_1, \dots, \ell_7$. We continue to identify bitangents with odd theta characteristics, and the corresponding odd quadratic forms.

Now we have

$$\ell_2 - \xi_3 = \ell_3 - \xi_2,\ \ell_3 - \xi_1 = \ell_1 - \xi_3,\ \ell_1 - \xi_2 = \ell_2 - \xi_1.$$

This implies that $\ell_1 - \xi_1 = \ell_2 - \xi_2 = \ell_3 - \xi_3$, i.e. the pairs $(\ell_1, \xi_1), (\ell_2, \xi_2)$, and (ℓ_3, ξ_3) belong to the same Steiner complex Σ. One easily checks that

$$\langle \ell_1 - \xi_1, \ell_1 - \xi_2 \rangle = \langle \ell_2 - \xi_2, \ell_2 - \xi_3 \rangle = \langle \ell_3 - \xi_3, \ell_3 - \xi_1 \rangle = 0,$$

and hence Σ is syzygetic to the three complexes (6.4) and therefore it does not contain $\ell_i, i \geq 4$.

By Proposition 6.1.2 and its proof, we can write, after rescaling u_0, u_1, u_2,

$$C = V(4t_0t_1u_0u_1 - q_3^2) = V(4t_0t_2u_0u_2 - q_2^2) = V(4t_1t_2u_1u_2 - q_1^2), \tag{6.5}$$

where

$$\begin{aligned} q_1 &= -t_0u_0 + t_1u_1 + t_2u_2, \\ q_2 &= t_0u_0 - t_1u_1 + t_2u_2, \\ q_3 &= t_0u_0 + t_1u_1 - t_2u_2. \end{aligned} \tag{6.6}$$

Next, we use the first Steiner complex from (6.4) to do the same by using the first three pairs. We obtain

$$C = V(4t_1u_2l_4l_{41} - q^2).$$

As in the proof of Proposition (6.1.2), we find that

$$q_1 - q = 2\lambda_1 t_1 u_2,\ q_1 + q = \frac{2(t_2u_2 - l_4l_{41})}{\lambda_1}.$$

Hence

$$q_1 = \lambda_1 t_1 u_2 + \frac{t_2u_1 - l_4l_{41}}{\lambda_1} = -t_0u_0 + t_1u_1 + t_2u_3,$$

and we obtain

$$\begin{aligned} l_4l_{41} &= t_2u_1 - \lambda_1(-t_0u_0 + t_1u_1 + t_2u_3) + \lambda_1^2 t_1u_2, \\ l_4l_{42} &= t_1u_0 - \lambda_2(t_0u_0 - t_1u_1 + t_2u_3) + \lambda_2^2 t_2u_0, \\ l_4l_{43} &= t_0u_2 - \lambda_3(t_0u_0 + t_1u_1 - t_2u_3) + \lambda_3^2 t_0u_1. \end{aligned} \tag{6.7}$$

The last two equations give

$$l_4\big(\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3}\big) = t_0\big(-2u_0 + \lambda_3u_1 + \frac{u_2}{\lambda_3}\big) + u_0\big(\lambda_2t_2 + \frac{t_1}{\lambda_3}\big). \tag{6.8}$$

The lines ℓ_4, ℓ_1, and ξ_1 belong to the third Steiner complex (6.4), and by Lemma 6.1.6 form an azygetic triad. By Proposition 6.1.4, they cannot be concurrent. This implies that the line $V(\lambda_2t_2 + \frac{t_1}{\lambda_3})$ passes through the intersection point of the lines ξ_1 and ℓ_4. This gives a linear dependence between the linear functions $l_4 = a_0t_0 + a_1t_1 + a_2t_2$, $l_1 = t_0$ and $\lambda_2t_2 + \frac{t_1}{\lambda_3}$ (we can assume that $a_0 = a_1 = a_2 = 1$ but will do it later). This can happen only if

$$\lambda_2 = c_1a_2, \quad \frac{1}{\lambda_3} = c_1a_1,$$

for some constant c_1. Now $\lambda_2t_2 + \frac{1}{\lambda_3}t_1 = c_1(a_2t_2 + a_1t_1) = c_1(l_4 - a_0t_0)$, and we can rewrite (6.8) in the form

$$c_1l_4\big(\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} - c_1u_0\big) = t_0\big(-c_1(2 + a_0c_1)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}\big).$$

This implies that

$$\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = c_1u_0 + \frac{k_1}{c_1}t_0, \tag{6.9}$$

$$k_1l_4 = -c_1(2 + c_1a_0)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}, \tag{6.10}$$

for some constant k_1. Similarly, we get

$$k_2l_4 = -c_2(2 + c_2a_1)u_1 + \frac{u_0}{a_0} + \frac{u_2}{a_2},$$

$$k_3l_4 = -c_3(2 + c_3a_2)u_2 + \frac{u_1}{a_0} + \frac{u_2}{a_1}.$$

It is easy to see that this implies that

$$k_1 = k_2 = k_3 = k,\ c_1 = -a_0,\ c_2 = -a_1, c_3 = -a_2.$$

Equations (6.9) and (6.10) become

$$\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = -a_0 u_0 - \frac{k}{a_0} t_0, \tag{6.11}$$

$$k l_4 = \frac{u_0}{a_0} + \frac{u_1}{a_1} + \frac{u_2}{a_2}. \tag{6.12}$$

At this point, we can scale the coordinates to assume

$$a_1 = a_2 = a_2 = 1 = -k = 1,$$

and obtain our first equation

$$t_0 + t_1 + t_2 + u_0 + u_1 + u_2 = 0.$$

Replacing l_{41} with l_{51}, l_{61}, l_{71} and repeating the argument, we obtain the remaining three equations relating u_0, u_1, u_2 with t_0, t_1, t_2.

Let us find the constants k_1, k_2, k_3 for ℓ_5, ℓ_6, ℓ_7. We have found four linear equations relating six linear functions $t_0, t_1, t_2, u_0, u_1, u_2$. Since three of them form a basis in the space of linear functions, there must be one relation. We may assume that the first equation is a linear combination of the last three with some coefficients $\lambda_1, \lambda_2, \lambda_3$. This leads to the system of linear equations from the statement of the Theorem.

Finally, we have to find the equations of the 21 bitangents. The equations (6.5) show that the lines ξ_1, ξ_2, ξ_3 are bitangents. Equation (6.11) and similar equations

$$\frac{l_{43}}{\lambda_3} + \frac{l_{41}}{\lambda_1} = -u_1 + t_1,$$

$$\frac{l_{41}}{\lambda_1} + \frac{l_{42}}{\lambda_2} = -u_2 + t_2,$$

after adding up, give

$$\frac{l_{41}}{\lambda_1} + \frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = t_0 + t_1 + t_2,$$

and then

$$\frac{l_{41}}{\lambda_1} = u_0 + t_1 + t_2,$$

$$\frac{l_{42}}{\lambda_1} = u_1 + t_0 + t_2,$$

$$\frac{l_{43}}{\lambda_1} = u_2 + t_0 + t_1.$$

This gives us three equations of type (2). Similarly, we get the expressions for l_{5i}, l_{6i}, l_{7i} which are the nine equations of types (3), (4), and (5).

Let us use the Aronhold set $(\ell_1, \dots, \ell_7)$ to index bitangents by subsets (ij) of $\{1, \dots, 8\}$. As we explained at the end of the previous section, we have

$$\xi_1 = \vartheta_{23}, \quad \xi_2 = \vartheta_{13}, \quad \xi_3 = \vartheta_{12},$$

$$\xi_{4k} = \vartheta_{k4}, \quad \xi_{5k} = \vartheta_{k5}, \quad \xi_{6k} = \vartheta_{k6}, \quad \xi_{7k} = \vartheta_{k7}, \quad k = 1, 2, 3.$$

The remaining bitangents are $\vartheta_{56}, \vartheta_{57}, \vartheta_{67}, \vartheta_{45}, \vartheta_{46}, \vartheta_{47}$. The first three look like $\vartheta_{23}, \vartheta_{13}, \vartheta_{12}$, they are of type $\triangle$. The second three look like $\vartheta_{5k}, \vartheta_{6k}, \vartheta_{7k}$, they are of type $\curlyvee$. To find the equations of triples of bitangents of type $\triangle$, we interchange the roles of the lines ℓ_1, ℓ_2, ℓ_3 with the lines ℓ_5, ℓ_6, ℓ_7. Our lines will be the new lines analogous to the lines ξ_1, ξ_2, ξ_3. Solving the system, we find their equations. To find the equations of the triple of bitangents of type $\curlyvee$, we delete ℓ_4 from the original Aronhold set, and consider the Steiner complex containing the remaining lines as we did in (6.4). The lines making the pairs with ℓ_5, ℓ_6, ℓ_7 will be our lines. We find their equations in the same manner as we found the equations for $\xi_{5k}, \xi_{6k}, \xi_{7k}$. □

Remark 6.1.10 The proof of the Theorem implies the following result, which can be found in [271]. Let (ℓ_1, ξ_1), be three pairs of bitangents from the same Steiner complex. Let (ℓ_4, ξ_4) be a fourth pair of bitangents from the Steiner complex given by pairs $(\ell_1, \xi_2), (\ell_2, \xi_1)$ as in (6.4) (where $\xi_4 = \xi_{43}$). Choose some linear forms l_i, m_i representing ℓ_i, ξ_i. Then the equation of C can be given by

$$\big((l_4l_2l_3)(l_4m_2m_3)l_1m_1 + (l_1l_4l_3)(m_1l_4m_3)l_2m_2 - (l_1l_2l_4)(m_1m_2l_4)l_3m_3\big)^2$$

$$-4(l_4l_2l_3)(l_4m_2m_3)(l_1l_4l_3)(m_1l_4m_3)l_1m_1l_2m_2 = 0,$$

where the brackets denote the determinants of the matrix formed by the coefficients of the linear forms. In fact, this is Equation (6.5), where the determinants take care of scaling of the forms u_0, u_1, u_2 (use that, $V(l_4)$ can be taken to be $V(l_1 + l_2 + l_3)$ and we must keep the relation $l_1 + l_2 + l_3 + u_1 + u_2 + u_3 = 0$).

One can also find in loc. cit. paper of J. Guàrdia the expressions for l_i, m_i in terms of the period matrix of C.

Remark 6.1.11 We will see later in Subsection 6.3.3 that any seven lines in a general linear position can be realized as an Aronhold set for a plane quartic curve. Another way to see it can be found in [596], p. 447.

6.2 Determinant equations of a plane quartic

6.2.1 Quadratic determinantal representations

Suppose an aCM invertible sheaf $\mathcal{F}$ on a nonsingular plane curve C of degree $d = 2k$ defines a resolution

$$0 \to \underline{U}^{\vee}(-2) \xrightarrow{\phi} \underline{V} \to \mathcal{F} \to 0, \tag{6.13}$$

where $\dim U = \dim V = k$. Taking the cohomology, we obtain

$$H^0(C, \mathcal{F}) \cong V, \quad H^0(C, \mathcal{F}(-1)) \cong U, \tag{6.14}$$

$$H^0(C, \mathcal{F}(-1)) = H^1(C, \mathcal{F}) = 0, \tag{6.15}$$

The map ϕ is defined by a linear map $S^2 \to U \otimes V$. In coordinates, it is defined by a $k \times k$-matrix $A = (a_{ij})$ with quadratic forms as its entries. The transpose matrix defines a resolution

$$0 \to \underline{V}^{\vee}(-2) \xrightarrow{\phi} \underline{U} \to \mathcal{G} \to 0, \tag{6.16}$$

where

$$\mathcal{G} \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2})(-2) \cong \mathcal{F}^{\vee}(d-2).$$

If we set $\mathcal{L} = \mathcal{F}(1-k), \mathcal{M} = \mathcal{G}(1-k)$, then $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C$ and conditions (6.15) can be rephrased in terms of $\mathcal{L}$. They are

$$H^0(C, \mathcal{L}(k-2)) = H^1(C, \mathcal{L}(k-1)) = 0. \tag{6.17}$$

Twisting the exact sequence (6.13) by $\mathcal{O}_{\mathbb{P}^2}(1-k)$, and applying Riemann-Roch, we obtain, after easy calculation,

$$\deg \mathcal{L} = g - 1 + \chi(\mathcal{L}) = g - 1 + k(\chi(\mathcal{O}_{\mathbb{P}^2}(1-k)) - \chi(\mathcal{O}_{\mathbb{P}^2}(-1-k)) = 0.$$

Conversely, given an aCM invertible sheaf $\mathcal{L}$ satisfying (6.17), then $\mathcal{F} = \mathcal{L}(k-1)$ admits a resolution of the form (4.20). Taking cohomology, one can easily show that $a_1 = \ldots = a_k = 0, b_1 = \ldots = b_k = 2$. This gives the following.

Theorem 6.2.1 *The equivalence classes of quadratic determinantal representations of a nonsingular plane curve C of degree $d = 2k$ are in a bijective correspondence with invertible sheaves $\mathcal{L}$ on C of degree 0 satisfying*

$$H^0(C, \mathcal{L}(k-2)) = H^1(C, \mathcal{L}(k-1)) = 0.$$

Changing $\mathcal{L}$ to $\mathcal{L}^{-1}$ corresponds to the transpose of the matrix defining the determinantal representation.

As we know, resolutions (6.13) and (6.16) define the two maps

$$\mathfrak{l} : C \to \mathbb{P}(U), \quad \mathfrak{r} : C \to \mathbb{P}(V),$$

given by the left and right kernels of the matrix A, considered as a bilinear form on $U \otimes V$. These maps are given by the linear systems $|(k-1)h + a|$, $|(k-1)h - a|$, where $\mathcal{L} \cong \mathcal{O}_C(a)$ and $\mathcal{O}_C(1) \cong \mathcal{O}_C(h)$.

Consider the map

$$(\mathfrak{l}, \mathfrak{r}) : C \to \mathbb{P}(U) \times \mathbb{P}(V)$$

with the image S. We identify the product of the projective spaces with its image in $\mathbb{P}(U \otimes V)$ under the Segre embedding. Consider the restriction map

$$\nu : H^0(\mathbb{P}(U \otimes V), \mathcal{O}_{\mathbb{P}(U\otimes V)}(1)) = U \otimes V \to H^0(S, \mathcal{O}_S(1))$$

$$= H^0(C, \mathcal{L}(k-1) \otimes \mathcal{M}(k-1)) = H^0(C, \mathcal{O}_C(2k-2)), \tag{6.18}$$

Passing to the projectivizations, and composing it with the Segre map $\mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V)$, it corresponds to the multiplication map

$$\mu : |(k-1)h + a| \times |(k-1)h - a| \to |\mathcal{O}_{\mathbb{P}^2}(2k-2)|, \ (D_1, D_2) \mapsto \langle D_1, D_2 \rangle,$$

where $\langle D_1, D_2 \rangle$ is the unique curve of degree $2k-2$ that cuts out the divisor $D_1 + D_2$ on C. Composing the linear map (6.18) with the linear map $\phi : S^2E \to U \otimes V$, we get a linear map

$$\nu' : S^2(E) \xrightarrow{\phi} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = S^{2k-2}(E^\vee). \tag{6.19}$$

A similar proof as used in the case of linear determinantal representations shows that this map coincides with the apolarity map corresponding to C.

For any $x \in C$, consider the tensor $\mathfrak{l}(x) \otimes \mathfrak{r}(x)$ as a hyperplane in $|U \otimes V|$. It intersects $|U| \times |V|$ at the subvariety of points whose image under the map μ vanishes at x. Choose a basis $(s_1, \dots, s_k)$ in U and a basis $(s'_1, \dots, s'_k)$ in V. The map ϕ is given by $\phi(x) = \sum a_{ij} s_i \otimes s_j$. It follows from the above that the matrix $(\nu(s_i \otimes s_j))$ and the matrix $\mathrm{adj}((a_{ij}))$ coincide when restricted at C (up to a multiplicative factor). Since its entries are polynomials of degree less than $\deg C$, we see that they coincide for all x. This shows that the map ν can be written by the formula

$$(\sum u_i s_i, \sum v_j s'_j) \mapsto -\det \begin{pmatrix} a_{11}(t) & \dots & a_{1k}(t) & v_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1}(t) & \dots & a_{kk}(t) & v_k \\ u_1 & \dots & u_k & 0 \end{pmatrix}. \tag{6.20}$$

Under the composition of the map, the zero set of the bordered determinant

is a curve of degree $2k-2$. Consider the discriminant hypersurface $\mathsf{D}_{d-2}(2)$ of curves of degree $d-2 = 2k-2$. The preimage of $\mathsf{D}_{d-2}(2)$ under the map (6.18) is a hypersurface X in $\mathbb{P}(U)\times\mathbb{P}(V) \cong \mathbb{P}^{k-1}\times\mathbb{P}^{k-1}$ given by a bihomogeneous equation of bidegree $(3(d-3)^2, 3(d-3)^2)$. Here we use that $\deg \mathsf{D}_d(2) = 3(d-1)^2$.

Now it is time to specialize to the case $d=4$. In this case, the map $|\nu|$ is the map

$$|\nu| : |K_C + a| \times |K_C - a| \to |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

In coordinates, the map ν is given by

$$(u_1 s_1 + u_2 s_2, v_1 s_1' + v_2 s_2') \mapsto -u_0 v_0 a_{11} + u_0 v_1 a_{12} - u_1 v_0 a_{21} - u_1 v_1 a_{22}. \tag{6.21}$$

The map ϕ is given by

$$\phi(x) = \sum a_{ij}(x) s_i^* \otimes s_j'^*,$$

where $(s_1^*, s_2^*), (s_1'^*, s_2'^*)$ are the dual bases in $U^\vee$ and $V^\vee$. One can also explicitly see the kernel maps $\mathfrak{l}$ and $\mathfrak{r}$:

$$\mathfrak{l}(x) = [-a_{21}(x), a_{11}(x)] = [-a_{22}(x), a_{12}(x)], \tag{6.22}$$

$$\mathfrak{r}(x) = [-a_{12}(x), a_{11}(x)] = [-a_{22}(x), a_{21}(x)]. \tag{6.23}$$

The intersection of the conics $V(a_{21}(t)) \cap V(a_{21}(t))$ lies on C, so $\mathfrak{l}$ is given by a pencil of conics with four base points $x_1, \dots, x_4$ on C. The map $\mathfrak{r}$ is given by another pencil of conics whose base points $y_1, \dots, y_4$ together with the base points $x_1, \dots, x_4$ are cut out by a conic.

The hypersurface $X \subset \mathbb{P}(U)\times\mathbb{P}(V)$ is of type $(3,3)$. It is a curve of arithmetic genus 4. Its image under the Segre map is a canonical curve equal to the intersection of a nonsingular quadric and a cubic surface. The cubic surface is the preimage of the determinantal cubic. It is a cubic symmetroid. We will discuss such cubics in Chapter 8. As we explained in Subsection 4.2.6, a cubic symmetroid surface admits a unique double cover ramified along the nodes. The restriction of this cover to X is an irreducible unramified cover $r : \tilde{X} \to X$. Let τ be the nontrivial 2-torsion divisor class on X corresponding to this cover (it is characterized by the property that $r^*(\tau) = 0$). The linear system $|K_X + \tau|$ maps X to $\mathbb{P}^2$. The image is a *Wirtinger plane sextic* with double points at the vertices of a complete quadrilateral. Conversely, we will explain in Chapter 9 that a cubic symmetroid with 4 nodes is isomorphic to the image of the plane under the linear system of cubics passing through the six vertices of a complete quadrilateral. This shows that any Wirtinger sextic is isomorphic to the intersection of a quadric and a cubic symmetroid. In this way

we see that any general curve of genus 4 is isomorphic to the curve X arising from a quadratic determinantal representation of a nonsingular plane quartic. We refer for this and for more of the geometry of Wirtinger sextics to [70].

The map (6.19) is just the apolarity map $\mathrm{ap}_2 : S^2(E) \to S^2(E^\vee)$ defined by the quartic C. It is bijective if the quartic C is nondegenerate. Under the composition $|E| \to |S^2(E)| \to |S^2(E^\vee)|$, the preimage of the discriminant cubic hypersurface is the Hessian sextic of C.

Consider the hypersurface W of type $(1, 1, 2)$ in $|U| \times |V| \times |E|$ defined by the section of $\mathcal{O}_{\mathbb{P}(U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(E)}(2)$ corresponding to the tensor defining the linear map $\phi : S^2E \to U \otimes V$. It is immediate that

$$W = \{(D_1, D_2, x) \in |K_C + a| \times |K_C - a| \times \mathbb{P}^2 : x \in \langle D_1, D_2\rangle\}. \quad (6.24)$$

In coordinates, the equation of W is given by the bordered determinant (6.20).

Consider the projections

$$\mathrm{pr}_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathrm{pr}_2 : W \to \mathbb{P}^1. \quad (6.25)$$

The fibres of pr_1 are isomorphic (under pr_2) to conics.The discriminant curve is the curve X. The fibre of pr_2 over a point $x \in \mathbb{P}^2$ is isomorphic, under pr_1, to a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1)$. In the Segre embedding, it is a conic. The discriminant curve is the curve C. Thus W has two structures of a conic bundle. The two discriminant curves, X and C, come with the natural double cover parameterizing irreducible components of fibres. In the first case, it corresponds to the 2-torsion divisor class τ and X is a nontrivial unramified double cover. In the second case, the cover splits (since the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ come with an order).

Remark 6.2.2 Recall that, for any unramified double cover of nonsingular curves $\pi : \tilde{S} \to S$, the Prym variety $\mathrm{Prym}(\tilde{S}/S)$ is defined to be the connected component of the identity of $\mathrm{Jac}(\tilde{S})/\ \pi^*\mathrm{Jac}(S)$.

$$\mathrm{Prym}(\tilde{X}/X) \cong \mathrm{Jac}(C).$$

This is a special case of the *trigonal construction* applied to trigonal curves (like ours X) discovered by S. Recillas [455] (see a survey of R. Donagi [189] about this and more general constructions of this sort). Note that, in general, the curve X could be singular even when C is not. However, the Prym variety is still defined.

Let $\mathcal{R}_g$ be the coarse moduli space of isomorphism classes of pairs $(S, \tilde{S})$, where S is a nonsingular curve of genus g and $\tilde{S} \to S$ is its unramified double cover. There is a *Prym map*

$$\mathfrak{p}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}, \quad (S, \tilde{S}) \mapsto \mathrm{Prym}(\tilde{S}/S),$$

where $\mathcal{A}_{g-1}$ is the coarse moduli space of principally polarized abelian varieties of dimension $g-1$. In our case $g=4$, the quadratic determinantal constructions allows us to describe the fiber over the Jacobian variety of a nonsingular canonical curve of genus 3. It is isomorphic to the *Kummer variety* $\mathrm{Kum}(C) = \mathrm{Jac}(C)/(\iota)$, where ι is the negation involution $a \mapsto -a$.

The map $\mathfrak{p}_g$ is known to be generically injective for $g \geq 7$ [227], a finite map of degree 27 for $g=6$ [192], and dominant for $g \leq 5$ with fibres of dimension $3g-3-\frac{1}{2}g(g-1)$. We refer to [190] for the description of fibres.

The varieties $\mathcal{R}_g$ are known to be rational ([186] for $g=2$, [186], [330] for $g=3$, [70] for $g=4$) and unirational for $g=5$ [316], [589], $g=6$ [191], [587] and $g=7$ [589]). It is known to be of general type for $g>13$ and $g \neq 15$ [216].

6.2.2 Symmetric quadratic determinants

By Theorem 6.2.1, the equivalence classes of symmetric quadratic determinantal representations of a nonsingular plane curve C of degree $d=2k$ correspond bijectively to nontrivial 2-torsion divisors $\epsilon \in \mathrm{Jac}(C)$ such that

$$H^0(C, \mathcal{O}_C(\epsilon)(k-2)) = H^1(C, \mathcal{O}_C(\epsilon)(k-2)) = 0.$$

Each such ϵ defines a quadratic determinantal representation

$$C := \det \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} = 0,$$

where $a_{ij} = a_{ji}$ are homogeneous forms of degree 2. It comes with the maps,

$$\phi : |E| \to |S^2(U^\vee)|, \quad x \mapsto (a_{ij}(x)),$$

$$\mathfrak{l} : C \to \mathbb{P}(U) \cong \mathbb{P}^{k-1}, \quad x \mapsto |N(A(x))|.$$

It is given by the linear system $|(k-1)h+\epsilon|$. The map (6.18) becomes the restriction map of quadrics in $\mathbb{P}(U)$ to the image S of C under the map $\mathfrak{l}$

$$\nu : S^2(U) \to H^0(C, \mathcal{O}_C(2k-2)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2k-2)).$$

The map μ is the composition of the map $U \times U \to |S^2(U)|$ given by the complete linear system of quadrics in $|U|$ and the map ν. It factors through the symmetric square of $|U|$, and defines a map

$$|U|^{(2)} \to |\mathcal{O}_{\mathbb{P}^2}(2k-2)|. \tag{6.26}$$

Recall that $|U|^{(2)}$ is isomorphic to the secant variety of $\mathsf{v}_2(|U|)$ in $|S^2(U)|$. The preimage $X(\epsilon)$ of the determinantal hypersurface $\mathsf{D}_{2k-2}(2)$ of curves of degree $2k-2$ in $|S^2(U)|$ is a hypersurface of degree $3(d-3)^2$. Its intersection with $|U| \times |U|$, embedded by Segre, is a hypersurface of bidegree $(3(d-3)^2, 3(d-3)^2)$. It is invariant with respect to the switch involution of $|U| \times |U|$ and descends to a hypersurface in the quotient. Its preimage under the Veronese map is a hypersurface $B(\epsilon)$ of degree $6(d-3)^2$ in $\mathbb{P}(U)$.

In coordinates, the multiplication map is given by the bordered determinant (6.20). Since A is symmetric, we have $D(A;u,v) = D(A;v,u)$, and the bordered determinantal identity (4.11) gives

$$D(A;u,v)^2 - D(A;u,u)D(A;v,v) = |A|P(t;u,v),$$

where $P(t;u,v)$ is of degree $2k-4$ in (t_0,t_1,t_2) and of bidegree $(2,2)$ in u,v. The curves $V(D(A;u,u))$ define a quadratic family of contact curves of degree $2k-2$. So, we have $2^{2g}-1$ of such families, where g is the genus of C.

Now let us specialize to the case $k=2$. The determinantal equation of C corresponding to ϵ must be given by a symmetric quadratic determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2. \tag{6.27}$$

Thus we obtain the following.

Theorem 6.2.3 *An equation of a nonsingular plane quartic can be written in the form*

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = 0,$$

where a_1, a_2, a_3 are homogeneous forms of degree 2. The set of equivalence classes of such representations is in a bijective correspondence with the set of 63 nontrivial 2-torsion divisor classes in $\mathrm{Pic}(C)$.

The bordered determinant

$$D(A;u,u) = \begin{vmatrix} a_{11} & a_{12} & u_0 \\ a_{21} & a_{22} & u_1 \\ u_0 & u_1 & 0 \end{vmatrix} = -(a_{11}u_0^2 + 2a_{12}u_0u_1 + a_{22}u_1^2)$$

defines a family of *contact conics* of C. Each conic from the family touches C along a divisor from $|K_C + \epsilon|$.

Also identity (4.12) between the bordered determinants becomes in our case

$$\det \begin{pmatrix} D(A;u,u) & D(A;u,v) \\ D(A;u,v) & D(A;v,v) \end{pmatrix} = |A|P(u,v), \tag{6.28}$$

where $P(u, v)$ is a bihomogeneous polynomial in u, v of bidegree $(2, 2)$. Note that $P(u, v)$ is symmetric in u, v and $P(u, u) = 0$. This shows that $P(u, v)$ can be written in the form

$$P(u, v) = (u_0v_1 - u_1v_0)(\alpha u_0v_0 + \beta(u_0v_1 + u_1v_0) + \gamma u_1v_1),$$

where α, β, γ are some constants.

The variety $X(\epsilon)$ in $|U| \times |U| \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bidegree $(3, 3)$. The difference from the general case of quadratic determinantal representations of C is that the curve $X(\epsilon)$ is defined by a symmetric bihomogeneous form. The symmetric product $|U|^{(2)}$ is isomorphic to $|S^2(U)| \cong \mathbb{P}^2$. The image of $X(\epsilon)$ in the plane is a curve $F(\epsilon)$ of degree 3. In intersects the Veronese curve $|E| \hookrightarrow |S^2(U)|$ at 6 points. They are the images of the hypersurface $B(\epsilon) \subset |U|$ under the Veronese map $|E| \hookrightarrow |S^2(U)|$. So, we see another special property of $X(\epsilon)$. If it is nonsingular, it is a canonical *bielliptic curve* of genus 4. One can easily compute the number of moduli of such curves. It is equal to six instead of nine for a general curve of genus 4. This agrees with our construction since we have 6 moduli for pairs (C, ϵ).

It follows from the definition that the curve $F(\epsilon)$ parameterizes unordered pairs D_1, D_2 of divisors $D \in |K_C + \epsilon|$ such that the conic $\langle D_1, D_2\rangle$ is equal to the union of two lines.

Let $\Pi(\epsilon)$ be the plane in $|\mathcal{O}_{\mathbb{P}^2}(2)|$ equal to the image of the map (6.26). It is a net of conics in $|E| = \mathbb{P}^2$. It is spanned by the contact conics to C. We can take for the basis of the net the conics

$$V(a_{11}) = \langle 2D_1\rangle,\ V(a_{12}) = \langle D_1, D_2\rangle,\ V(a_{22}) = \langle 2D_2\rangle,$$

where D_1, D_2 span $|K_C + \epsilon|$. In particular, we see that $\Pi(\epsilon)$ is base-point-free. Its discriminant curve is equal to the curve $F(\epsilon)$.

Proposition 6.2.4 *The cubic curve $F(\epsilon)$ is nonsingular if and only if the linear system $|K_C + \epsilon|$ does not contain a divisor of the form $2a + 2b$.*

Proof Let $\mathsf{D} = \mathsf{D}_2(2) \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the discriminant cubic. The plane section $\Pi(\epsilon) \cap \mathsf{D}_2(2)$ is singular if and only if $\Pi(\epsilon)$ contains a singular point of D represented by a double line, or if it is tangent to D at a nonsingular point. We proved in Example 1.2.3 that the tangent hypersurface of D at a nonsingular point represented by a reducible conic Q is equal to the space of conics passing through the singular point q of Q. If L is contained in the tangent hyperplane, then all conics from $\Pi(\epsilon)$ pass through q. But, as we saw earlier, the net of conics $\Pi(\epsilon)$ is base-point-free. This shows that $\Pi(\epsilon)$ intersects D transversally at each nonsingular point.

In particular, $F(\epsilon)$ is singular if and only if $\Pi(\epsilon)$ contains a double line.

Assume that this happens. Then we get two divisors $D_1, D_2 \in |K_C + \epsilon|$ such that $D_1 + D_2 = 2A$, where $A = a_1 + a_2 + a_3 + a_4$ is cut out by a line ℓ. Let $D_1 = p_1 + p_2 + p_3 + p_4$, $D_2 = q_1 + q_2 + q_3 + q_4$. Then the equality of divisors (not the divisor classes)

$$p_1 + p_2 + p_3 + p_4 + q_1 + q_2 + q_3 + q_4 = 2(a_1 + a_2 + a_3 + a_4)$$

implies that either D_1 and D_2 share a point x, or $D_1 = 2p_1 + 2p_2$, $D_2 = 2q_1 + 2q_2$. The first case is impossible, since $|K_C + \epsilon - x|$ is of dimension 0. The second case happens if and only if $|K_C + \epsilon|$ contains a divisor $D_1 = 2a + 2b$. The converse is also true. For each such divisor the line $\overline{ab}$ defines a residual pair of points c, d such that $D_2 = 2c + 2d \in |K_C + \epsilon|$ and $\varphi(D_1, D_2)$ is a double line. □

Remark 6.2.5 By analyzing possible covers of a plane cubic unramified outside the singular locus, one can check that $F(\epsilon)$ is either nonsingular or a nodal cubic, maybe reducible.

From now on we assume that $F(\epsilon)$ is a nonsingular cubic. Since it parameterizes singular conics in the net $\Pi(\epsilon)$, it comes with a natural nontrivial 2-torsion point η. Recall that the corresponding unramified double cover of $F(\epsilon)$ is naturally isomorphic to the Cayleyan curve in the dual plane $\Pi(\epsilon)^\vee$ which parameterizes irreducible components of singular conics in the net.

Theorem 6.2.6 *Let $\Sigma(\epsilon) = \{(\ell_1, \ell_1'), \dots, (\ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents associated to the 2-torsion divisor class ϵ. Each pair, considered as a divisor $D_i = \ell_i + \ell_i' \in |K_C + \epsilon| = |U|$ is mapped under the Veronese map $|U| \to |S^2(U)|$ to a point in $F(\epsilon)$. It belongs to the set $B(\epsilon)$ of six ramification points of the cover $X(\epsilon) \to F(\epsilon)$. The 12 bitangents, considered as points in the dual plane $|S^2(U^\vee)|$, lie on the cubic curve $\tilde{F}(\epsilon)$.*

Proof Let $(\vartheta_i, \vartheta_i')$ be a pair of odd theta characteristics corresponding to a pair (ℓ_i, ℓ_i') of bitangents from $\Sigma(\epsilon)$. They define a divisor $D = \vartheta_i + \vartheta_i' \in |K_C + \epsilon|$ such that D is the divisor of contact points of a reducible contact conic, i.e. the union of two bitangents. This shows that $\vartheta_i, \vartheta_i' \in \tilde{F}(\epsilon)$. The point $(D, D) \in |K_C + \epsilon| \times |K_C + \epsilon|$ belongs to the diagonal in $|U| \times |U|$. These are the ramification points of the cover $X(\epsilon) \to F(\epsilon)$. They can be identified with the branch points of the cover $X(\epsilon) \to F(\epsilon)$. □

So, we have a configuration of 63 cubic curves $\tilde{F}(\epsilon)$ in the plane $|S^2(U^\vee)|$ (beware that this plane is different from the plane $|E|$ containing C). Each contains 12 bitangents from a Steiner complex. Let S_1, S_2, S_3 be a syzygetic (resp. azygetic) triad of Steiner complexes. They define three cubic curves $\tilde{F}(\epsilon), \tilde{F}(\eta), \tilde{F}(\eta + \epsilon)$ which have four (resp. six) points in common.

Let us see what happens in the symmetric case with the two-way conic bundle $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ from (6.24) which we discussed in the previous Subsection. First, its intersection with the product of the diagonal Δ of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{P}^2$ defines the universal family $\mathsf{U}(\epsilon)$ of the contact conics. It is isomorphic to a surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 2)$. The projection to $\mathbb{P}^2$ is a double cover branched along the quartic C. As we will see later, $\mathsf{U}(\epsilon)$ is isomorphic to a *del Pezzo surface of degree 2*. Its isomorphism class does not depend on ϵ. The projection $\mathsf{U}(\epsilon) \to \mathbb{P}^1$ is a conic bundle. It has six singular fibres that lie over six points at which the diagonal intersects the curve $X(\epsilon)$, i.e. the ramification points of the cover $X(\epsilon) \to F(\epsilon)$. The six branch points lie on a conic, the image of the diagonal Δ in $\mathbb{P}^2$. We will see later that a del Pezzo surface of degree 2 has 126 conic bundle structures; they divided into 63 pairs which correspond to nonzero 2-torsion divisor classes on C.

The threefold W is invariant with respect to the involution of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ which switches the first two factors. The quotient $W^s = W/(\iota)$ is a hypersurface of bidegree $(2, 2)$ in $(\mathbb{P}^1 \times \mathbb{P}^1)/(\iota) \times \mathbb{P}^2 \cong \mathbb{P}^2 \times \mathbb{P}^2$. The projection to the first factor is a conic bundle with the discriminant curve $B(\epsilon)$. The projection to the second factor is no longer a conic bundle. It is isomorphic to the pullback of the universal family of lines $X(\epsilon) \to \mathbb{P}^2$ under the map of the base $\mathbb{P}^2 \to \mathbb{P}^2$ given by the net of conics $\Pi(\epsilon)$.

Remark 6.2.7 One can easily describe the Prym map $\mathfrak{p}_3 : \mathcal{R}_3 \to \mathcal{A}_2$ restricted to the open subset of canonical curves of genus 3. A pair (C, η) defines an elliptic curve $F(\epsilon)$ and six branch points of the cover $X(\epsilon) \to F(\epsilon)$. The six points lie on the Veronese conic $|E| \hookrightarrow |S^2(U)|$. The cover $\tilde{C} \to C$ defined by ϵ is a curve of genus 5. The Prym variety $\mathrm{Prym}(\tilde{C}/C)$ is a principally polarized abelian variety of dimension 2. One can show that it is isomorphic to the Jacobian variety of the hyperelliptic curve of genus 2 which is isomorphic to the branch cover of the Veronese conic with the branch locus $B(\epsilon)$ (see [356], [357]). Other description of the Prym map $\mathfrak{p}_3$ can be found in [588].

6.3 Even theta characteristics

6.3.1 Contact cubics

Recall that each even theta characteristic ϑ on a nonsingular quartic C defines a 3-dimensional family of contact cubics. The universal family of contact cubics is a hypersurface $\mathsf{W}_\vartheta \subset |E| \times \mathbb{P}(U) \cong \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2, 3)$. If we choose coordinates (t_0, t_1, t_2) in $|E|$ and coordinates u_0, u_1, u_2, u_3 in $\mathbb{P}(U)$, then the

equation of the contact family is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_0 \\ a_{21} & a_{22} & a_{23} & a_{24} & u_1 \\ a_{31} & a_{32} & a_{33} & a_{34} & u_2 \\ a_{41} & a_{42} & a_{43} & a_{44} & u_3 \\ u_0 & u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0, \tag{6.29}$$

where (a_{ij}) is the symmetric matrix defining the net N_ϑ of quadrics defined by ϑ. The first projection $\mathsf{W}_\vartheta \to |E|$ is a quadric bundle with discriminant curve equal to C. Its fiber over a point $x \notin C$ is the dual of the quadric $Q_x = \phi(x)$. Its fiber over a point $x \in C$ is the double plane corresponding to the vertex of the quadric cone $\phi(x)$. Scheme-theoretically, the discriminant hypersurface of the quadric bundle is the curve C taken with multiplicity 3.

The second projection $\mathsf{W}_\vartheta \to \mathbb{P}^3$ is a fibration with fibres equal to contact cubics. Its discriminant surface D_ϑ is the preimage of the discriminant hypersurface $D_3(2)$ of plane cubic curves in $|\mathcal{O}_{\mathbb{P}^2}(3)|$ under the map $\mathbb{P}^3 \to |\mathcal{O}_{\mathbb{P}^2}(3)|$ given by quadrics. This implies that D_ϑ is of degree 24 and its equation is of the form $F_8^3 + G_{12}^2 = 0$, where F_8 and G_{12} are homogeneous forms in $u_0, \dots, u_3$ of the degrees indicated by the subscript.

Proposition 6.3.1 *The discriminant surface D_ϑ of the contact family of cubics is reducible and non-reduced. It consists of the union of 8 planes and a surface of degree 8 taken with multiplicity 2.*

Proof Let N_θ be the net of quadrics in $\mathbb{P}^3$ defined by ϑ. □

Let F_ξ be a contact nodal cubic represented by a general point ξ in one of the eight plane components. It is tangent to C at 6 nonsingular points. On the other hand, if F_ξ is a general point of the other component of D_ϑ, then it is a nodal cubic with a node at C.

We can see other singular contact cubics too. For example, 56 planes through three base points of the pencil N_ϑ correspond to the union of three asyzygetic bitangents. Another singular contact cubic is a *biscribed triangle*. It is the union of three lines such that C is tangent to the sides and also passes through the three vertices of the triangle. It is proved in [401] that the number of biscribed triangles in each of 36 families of contact cubics is equal to 8.

Remark 6.3.2 Note that each cubic curve F in the family of contact cubics comes with a distinguished 2-torsion point defined by the divisor class $\eta = d - 2h$, where $C \cap F = 2d$, and h is the intersection of F with a line. One can show that the 2-torsion point is nontrivial. The locus of zeros of the invariant surface $V(G_{12})$ of degree 12 parameterizes harmonic contact cubics F together with

a nontrivial 2-torsion divisor class η. The group μ_4 of complex multiplications of $\mathrm{Jac}(F)$ acts on the set of 2-torsion divisor classes with two fixed points. If η is invariant with respect to μ_4,

6.3.2 Cayley octads

Let N_ϑ be the net of quadrics defined by the pair (C, ϑ) and Q_1, Q_2, Q_3 be its basis. The base locus of N_ϑ is the complete intersection of these quadrics. One expects that it consists of eight distinct points. Let us see that this is indeed true.

Proposition 6.3.3 *The set of base points of the net of quadrics N_ϑ consists of eight distinct points, no three of which are collinear, no four are coplanar.*

Proof If we have fewer than eight base points, then all nonsingular quadrics share the same tangent line at a base point. This implies that N_ϑ contains a quadric Q with a singular point at a base point. The computation of the tangent space of the discriminant hypersurface given in (1.45) shows that Q is a singular point of the discriminant curve C, a contradiction.

Suppose three points are on a line ℓ. This includes the case when two points coincide. This implies that ℓ is contained in all quadrics from $\mathcal{N}$. Take a point $x \in \ell$. For any quadric $Q \in \mathsf{N}_\vartheta$, the tangent plane of Q at x contains the line ℓ. Thus the tangent planes form a pencil of planes through ℓ. Since N_ϑ is a net, there must be a quadric which is singular at x. Thus each point of ℓ is a singular point of some quadric from N_ϑ. However, the set of singular points of quadrics from N_ϑ is equal to the nonsingular sextic S, the image of C under the map given by the linear system $|\vartheta(1)|$. This shows that no three points are collinear.

Suppose that four points lie in a plane Π. The restriction of N_ϑ to Π defines a linear system of conics through four points no three of which are collinear. It is of dimension 1. Thus, there exists a quadric in N_ϑ which contains Π. However, since C is nonsingular, all quadrics in N_ϑ are of corank ≤ 1. □

Definition 6.3.4 *A set of eight distinct points in $\mathbb{P}^3$ which is a complete intersection of three quadrics is called a* Cayley octad.

From now on we assume that a Cayley octad satisfies the properties from Proposition 6.3.3.

Let S be the sextic model of C defined by the linear system $|K_C + \vartheta|$.

Theorem 6.3.5 *Let $q_1, \ldots, q_8$ be a Cayley octad. Each line $\overline{q_i q_j}$ intersects the sextic curve S at two points $\varphi(p_i), \varphi(p_j)$. The line $\overline{p_i p_j}$ is a bitangent of C.*

Proof The quadrics containing the line $\ell_{ij} = \overline{q_i q_j}$ form a pencil $\mathcal{P}$ in N_ϑ. Its base locus consists of the line ℓ_{ij} and a rational normal cubic curve R which intersects the line at two points (they could be equal). Note that the locus of singular quadrics in the net of quadrics containing R is a conic. Thus the pencil $\mathcal{P}$ contains two (or one) singular quadric with singular points at the intersection of R and ℓ_{ij}. In the net N_ϑ this pencil intersects the discriminant curve C at two points. Suppose one of these two points is an ordinary cusp. It is easy to check that the multiplicity of a zero of the discriminant polynomial of the pencil of quadrics is equal to the corank of the corresponding quadric. Since our pencil does not contain reducible quadrics, we see that this case does not occur. Hence the pencil $\mathcal{P}$ in N_ϑ is a bitangent.

□

We can also see all even theta characteristics.

Theorem 6.3.6 *Let $q_1, \ldots, q_8$ be the Cayley octad associated to an even theta characteristic ϑ. Let ϑ_{ij} be the odd theta characteristic corresponding to the lines $\overline{q_i q_j}$. Then any even theta characteristic different from ϑ can be represented by the divisor class*

$$\vartheta_{i,jkl} = \vartheta_{ij} + \vartheta_{ik} + \vartheta_{il} - K_C$$

for some distinct i, j, k, l.

Proof Suppose that $\vartheta_{i,jkl}$ is an odd theta characteristic ϑ_{mn}. Consider the plane π which contains the points q_i, q_j, q_k. It intersects S at six points corresponding to the theta characteristics $\vartheta_{ij}, \vartheta_{ik}, \vartheta_{jk}$. Since the planes cut out divisors from $|K_C + \vartheta|$, we obtain

$$\vartheta_{ij} + \vartheta_{ik} + \vartheta_{jk} \sim K_C + \vartheta.$$

This implies that

$$\vartheta_{jk} + \vartheta_{il} + \vartheta_{mn} \sim K_C + \vartheta.$$

Hence the lines $\overline{q_j q_k}$ and $\overline{q_i q_l}$ lie in a plane π'. The intersection point of the lines $\overline{q_j q_k}$ and $\overline{q_i q_l}$ is a base point of two pencils in $\mathcal{N}$ and hence is a base point of $\mathcal{N}$. However, it does not belong to the Cayley octad. This contradiction proves the assertion. □

Remark 6.3.7 Note that

$$\vartheta_{i,jkl} = \vartheta_{j,ikl} = \vartheta_{k,ijl} = \vartheta_{l,ijk}.$$

Thus $\vartheta_{i,jkl}$ depends only on the choice of a subset of four elements in $\{1, \ldots, 8\}$.

Also it is easy to check that the complementary set defines the same theta characteristic. This gives $35 = \binom{8}{4}/2$ different even theta characteristics. Together with $\vartheta = \vartheta_\emptyset$ we obtain 36 even theta characteristics. Observe now that the notation ϑ_{ij} for odd thetas and $\vartheta_{i,jkl}, \vartheta_\emptyset$ agrees with the notation we used for odd even theta characteristics on curves of genus 3. For example, any set $\vartheta_{18}, \ldots, \vartheta_{78}$ defines an Aronhold set. Or, a syzygetic tetrad corresponds to four chords forming a spatial quadrangle, for example $\overline{p_1p_3}, \overline{p_2p_4}, \overline{p_2p_3}, \overline{p_1p_4}$.

Here is another application of Cayley octads.

Proposition 6.3.8 *There are 1008 azygetic hexads of bitangents of C such that their 12 contact points lie on a cubic.*

Proof Let ℓ_1, ℓ_2, ℓ_3 be an azygetic triad of bitangents. The corresponding odd theta characteristics add up to $K_C + \vartheta$, where ϑ is an even theta characteristic. Let O be the Cayley octad corresponding to the net of quadrics for which C is the Hessian curve and let $S \subset \mathbb{P}^3 = |K_C + \vartheta|^\vee$ be the corresponding sextic model of C. We know that the restriction map

$$|\mathcal{O}_{\mathbb{P}^3}(2)| \to |\mathcal{O}_S(2)| = |\mathcal{O}_C(3K_C)| = |\mathcal{O}_{\mathbb{P}^2}(3)|$$

is a bijection. We also know that the double planes in $|\mathcal{O}_{\mathbb{P}^3}(2)|$ are mapped to contact cubics corresponding to ϑ. The cubic curve $\ell_1 + \ell_2 + \ell_3$ is one of them. Using the interpretation of bitangents as chords of the Cayley octad given in Theorem 6.3.5, we see that the union of the three chords corresponding to ℓ_1, ℓ_2, ℓ_3 cut out on S six coplanar points.This means that the three chords span a plane in $\mathbb{P}^3$. Obviously, the chords must be of the form $\overline{q_iq_j}, \overline{q_iq_k}, \overline{q_jq_k}$, where $1 \le i < j < k \le 8$. The number of such triples is $\binom{8}{3} = 56$. Fixing such a triple of chords, we can find $\binom{5}{3} = 10$ triples disjoint from the fixed one. The sum of the six corresponding odd theta characteristics is equal to $3K$ and hence the contact points are on a cubic. We can also see it by using the determinantal identity (4.11). Other types of azygetic hexads can be found by using the previous Remark. □

Altogether we find (see [492]) the following possible types of such hexads.

- 280 of type $(12, 23, 31, 45, 56, 64)$;
- 168 of type $(12, 34, 35, 36, 37, 38)$;
- 560 of type $(12, 13, 14, 56, 57, 58)$.

Recall that the three types correspond to three orbits of the permutation group $\mathfrak{S}_8$ on the set of azygetic hexads whose contact points are on a cubic. Note that not every azygetic hexad has this property. For example, a subset of an Aronhold set does not have this property.

For completeness sake, let us give the number of not azygetic hexads whose contact points are on a cubic.The number of them is equal 5040. Here is the list.

- 840 of type $(12, 23, 13, 14, 45, 15)$;
- 1680 of type $(12, 23, 34, 45, 56, 16)$;
- 2520 of type $(12, 34, 35, 36, 67, 68)$.

6.3.3 Seven points in the plane

Let $\mathcal{P} = \{p_1, \dots, p_7\}$ be a set of seven distinct points in $\mathbb{P}^2$. We assume that the points satisfy the following condition:

$(*)$ no three of the points are collinear and no six lie on a conic.

Consider the linear system L of cubic curves through these points. The conditions on the points imply that L is of dimension 2 and each member of L is an irreducible cubic. A subpencil in L has two base points outside the base locus of L. The line spanned by these points (or the common tangent if these points coincide) is a point in the dual plane $\mathbb{P}(E)$. This allows us to identify the net L with the plane $\mathbb{P}^2$ where the seven points lie. Nets of curves with this special property are *Laguerre nets* which we will discuss later in Example 7.3.12.

Proposition 6.3.9 *The rational map $L \dashrightarrow L^\vee$ given by the linear system L is of degree 2. It extends to a regular degree 2 finite map $\pi : X \to L^\vee \cong \mathbb{P}^2$, where X is the blow-up of the set $\mathcal{P}$. The branch curve of ϕ is a nonsingular plane quartic C in $L^\vee$. The ramification curve R is the proper transform of a curve $B \subset L$ of degree 6 with double points at each p_i. Conversely, given a nonsingular plane quartic C, the double cover of $\mathbb{P}^2$ ramified over C is a nonsingular surface isomorphic to the blow-up of 7 points $p_1, \dots, p_7$ in the plane satisfying the condition above.*

We postpone the proof of this Proposition until Chapter 8. The surface X is a del Pezzo surface of degree 2 .

Following our previous notation, we denote the plane $L^\vee$ by $|E|$ for some vector space E of dimension 3. Thus L can be identified with $\mathbb{P}(E)$. Let $\sigma : X \to \mathbb{P}^2$ be the blowing up map. The curves $E_i = \sigma^{-1}(p_i)$ are exceptional curves of the first kind, (-1)-curves for short. We will often identify L with its proper transform in S equal to

$$|-K_X| = |3h - E_1 - \cdots - E_7|,$$

where $h = c_1(\sigma^*\mathcal{O}_{\mathbb{P}^2}(1))$ is the divisor class of the preimage of a line in $\mathbb{P}^2$.

The preimage of a line $\ell \subset |E|$ in $\mathbb{P}(E) = L$ is a nonsingular member of L if and only if ℓ intersects transversally C. In this case, it is a double cover of ℓ branched over $\ell \cap C$. The preimage of a tangent line is a singular member, the singular points lie over the contact points. Thus, the preimage of a general tangent line is an irreducible cubic curve with a singular point at $\sigma(R)$. The preimage of a bitangent is a member of $|-K_X|$ with two singular points (they may coincide if the bitangent is an inflection bitangent). It is easy to see that its image in the plane is either an irreducible cubic F_i with a double point at p_i or the union of a line $\overline{p_ip_j}$ and the conic K_{ij} passing through the point $p_k, k \neq i, j$. In this way we can account for all 28 = 7+21 bitangents. If we denote the bitangents corresponding to F_i by ℓ_{i8} and the bitangents corresponding to $\overline{p_ip_j} + K_{ij}$ by ℓ_{ij}, we can accommodate the notation of bitangents by subsets of cardinality 2 of $[1, 8]$. We will see below that this notation agrees with the previous notation. In particular, the bitangents corresponding to the curves F_i's form an Aronhold set.

Let $\ell' \in |h|$. Its image $\pi(\ell')$ in $|L|^\vee = |E|$ is a plane cubic G. The preimage of G in X is the union of ℓ' and a curve ℓ'' in the linear system $3(3h - \sum E_i) - h| = |8h - 3\sum E_i|$. The curves ℓ' and ℓ'' intersect at 6 points. Since the cubic G splits in the cover π, it must touch the branch curve C at each intersection point with it. Thus it is a contact cubic and hence the divisor $D = \phi(\ell' \cap \ell'')$ belongs to $|K_C + \vartheta|$ for some even theta characteristic ϑ. This shows that ℓ' cuts out in R the divisor from the linear system $|K_R + \vartheta|$. In other words, the inverse of the isomorphism $\pi|R : R \to C$ is given by a 2-dimensional linear system contained in $|K_C + \vartheta|$. The image B of R in the plane $|L|$ is a projection of a sextic model of C in $\mathbb{P}^3$ defined by the linear system $|K_C + \vartheta|$.

We can easily locate an Aronhold set defined by θ. The full preimage of a curve F_i on X cuts out on R the divisor $2a_i+2b_i$, where $a_i, b_i \in E_i$ correspond to the branches of F_i at p_i. Thus the full preimage of the divisor $F_i - F_j$ cuts out on E_i the divisor

$$(2a_i + 2b_i) - (a_i + b_i) + (a_j + b_j) - 2(a_j + b_j) = a_i + b_i - a_j - b_j.$$

But $F_i - F_j$ is the divisor of a rational function on $\mathbb{P}^2$. This shows that the images of the F_i's in the plane $|E|$ are seven bitangents defined by odd theta characteristics ϑ_i such that $\vartheta_i - \vartheta_j$ is the same 2-torsion divisor class. Thus, the seven bitangents form an Aronhold set.

Let us record what we have found so far.

Proposition 6.3.10 *A choice of seven unordered points $p_1, \dots, p_7$ in the plane $\mathbb{P}(E)$ satisfying condition * defines a nonsingular plane quartic C in*

the dual plane $|E|$ and an even theta characteristic ϑ on C. The linear system of cubic curves through the seven points maps each its member with a double point at p_i to a bitangent ϑ_i of C. The seven bitangents $\vartheta_1, \dots, \vartheta_7$ form an Aronhold set of bitangents.

Let us now see the reverse construction of a set of 7 points defined by a pair (C, ϑ) as above.

Let N_ϑ be the linear system of quadrics in $|K_C + \vartheta| \cong \mathbb{P}^3$ defined by an even theta characteristic ϑ on C. Let $X \to \mathbb{P}^3$ be the blow-up of the Cayley octad $\mathsf{O} = \{q_1, \dots, q_8\}$ of its base points. The linear system N_ϑ defines an elliptic fibration $f : X \to \mathsf{N}_\vartheta^\vee$. If we identify N_ϑ with $|E|$ by using the determinantal representation $\phi : |E| \to |\mathcal{O}_{\mathbb{P}^3}(2)|$, then N_ϑ can be identified with $\mathbb{P}(E)$. The images of fibres of f in $\mathbb{P}^3$ are quartic curves passing through O. The projection map from $\mathbb{P}^3$ from $q_8 \in \mathsf{O}$ is defined by a 2-dimensional linear subsystem H of $|K_C + \vartheta|$. The projections of quartic curves are cubic curves passing through the set $\mathcal{P} = \{p_1, \dots, p_7\}$, where p_i is the projection of q_i. In this way we get a set of seven points that defines (C, ϑ). The Aronhold set of bitangents ϑ_{i8} obtained from the Cayley octad corresponds to the Aronhold set $\vartheta_1, \dots, \vartheta_7$ defined by the cubic curves F_i. They are the projections of the rational curve of degree R_i which together with the line $\overline{q_j q_8}$ form the base locus of the pencil of quadrics with singular points on the line $\overline{q_j q_8}$.

So, we have proved the converse.

Proposition 6.3.11 *A nonsingular plane quartic curve $C \subset |E|$ together with an even theta characteristic defines a unique Cayley octad $\mathsf{O} \subset |K_C + \vartheta| = \mathbb{P}^3$ such that the linear system of quadrics through O is the linear system of quadrics associated to (C, ϑ). The projection of O from one of its points to $\mathbb{P}^2$ plus a choice of an isomorphism $\mathbb{P}^2 \cong |E^\vee|$ defines a net of cubics through seven points $p_1, \dots, p_7$. The blow-up of the seven points is a del Pezzo surface and its anticanonical linear system defines a degree 2 finite map $X \to |E|$ branched over C. The ramification curve R of the map is the projection of the image of C under the linear system $|K_C + \vartheta|$.*

Note that in this way we account for all $288 = 8 \times 36$ Aronhold sets of seven bitangents. They are defined by a choice of an even theta characteristic and a choice of a point in the corresponding Cayley octad. We also obtain the following.

Corollary 6.3.12 *The moduli space U_2^7 of projective equivalence classes of unordered seven points in the plane is birationally isomorphic to the moduli space $\mathcal{M}_3^{\mathrm{ar}}$ of curves of genus 3 together with an Aronhold set of bitangents. It is (birationally) a $8:1$-cover of the moduli space $\mathcal{M}_3^{\mathrm{ev}}$ of curves of genus*

3 together with an even theta characteristic. The latter space is birationally isomorphic to the moduli space of projective equivalence classes of Cayley octads.

Remark 6.3.13 Both of the moduli spaces $\mathcal{M}_3^{\text{ar}}$ and $\mathcal{M}_3^{\text{ev}}$ are known to be rational varieties. The rationality of U_2^7 was proven by P. Katsylo in [329]. The rationality of $\mathcal{M}_3^{\text{ev}}$ P. Katsylo [331]. Also known, and much easier to prove, that the moduli space $\mathcal{M}^{\text{odd}}$ of curves of genus 3 with an odd theta characteristic is rational [23].

Remark 6.3.14 The elliptic fibration $f : X \to \mathbb{P}(E)$ defined by the linear system N_ϑ has 8 sections corresponding to the exceptional divisors over the points q_j. Its discriminant locus consists of lines in $|E|$ tangent to C, that is, the dual curve $C^\vee$ of C. If we fix one section, say the exceptional divisor over q_8, then all nonsingular fibres acquire a group structure. The closure of the locus of nontrivial 2-torsion points is a smooth surface W in X. Its image in $\mathbb{P}^3$ is a surface of degree 6 with triple points at $q_1, \dots, q_8$, called the *Cayley dianode surface*. It is a determinantal surface equal to the Jacobian surface of the linear system of quartic surfaces with double points at $q_1, \dots, q_7$. The linear system of quartics defines a map $X \to \mathbb{P}^6$ whose image is the cone over a Veronese surface in a hyperplane. The map is a double cover onto the image. The exceptional divisor over q_8 is mapped to the vertex of the cone. The surface W is the ramification locus of this map. Its image in $\mathbb{P}^6$ is the complete intersection of the cone and a cubic hypersurface. It is a surface of degree 12 with 28 nodes, the images of the lines $\overline{q_i q_j}$. The surface W is a minimal surface of general type with $p_g = 3$ and $K_W^2 = 3$. It is birationally isomorphic to the quotient of a symmetric theta divisor in $\mathrm{Jac}(C)$ modulo the involution $x \mapsto -x$. All of this is discussed in [121] and [176].

There is another similar elliptic fibration over $\mathbb{P}(E)$. Consider the universal family of the net L:

$$\mathsf{U} = \{(x, F) \in |E| \times L : x \in F\}.$$

The fiber of the first projection $\pi_1 : \mathsf{U} \to X$ over a point $x \in X$ can be identified, via the second projection, with the linear subsystem $L(x) \subset L$ of curves passing through the point x. If $x \notin \mathcal{P}$, $L(x)$ is a pencil, otherwise it is the whole L. The second projection $\pi_2 : \mathsf{U} \to L$ is an elliptic fibration, its fiber over the point $\{F\}$ is isomorphic to F. It has seven regular sections

$$s_i : L \to \mathsf{U}, \quad F \mapsto (p_i, F).$$

There is another natural rational section $s_8 : L \to \mathsf{U}$ defined as follows. We know from 3.3.2 that any g_2^1 on a nonsingular cubic curve F is obtained by

projection from the coresidual point $p \in F$ to a line. Take a curve $F \in L$ and restrict L to F. This defines a g^1_2 on F, and hence defines the coresidual point $\mathfrak{c}_F$. The section s_8 maps F to $\mathfrak{c}_8$. Although the images S_i of the first sections are disjoint in U, the image S_8 of s_8 intersects each $S_j, j \neq 8$, at the point (p_j, F_j) (in this case the g^1_2 on F_i has a base point p_i, which has to be considered as the coresidual point of F_j). The universal family U is singular because the net N_ϑ has base points. The singular points are the intersection points of the sections S_j and $S_8, j \neq 8$. The variety X is a small resolution of the singular points. The exceptional curves are the proper transforms of lines $\overline{q_j q_8}$.

6.3.4 The Clebsch covariant quartic

Here we shall specialize the Scorza construction in the case of plane quartic curves. Consider the following symmetric correspondence on $\mathbb{P}^2$

$$R = \{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2 : \text{rank} P_{yx}(C) = 1\}.$$

We know that a cubic curve has a polar quadric of rank 1 if and only if it lies in the closure of the projective equivalence class of the Fermat cubic. Equivalently, a cubic curve $G = V(g)$ has this property if and only if the Aronhold invariant S vanishes on g. We write in this case $\mathsf{S}(G) = 0$.

Consider the projection of R to one of the factors. It is equal to

$$\mathfrak{C}(C) := \{x \in \mathbb{P}^2 : \mathsf{S}(P_x(C)) = 0\}.$$

By symmetry of polars, if $x \in \mathfrak{C}(C)$, then $R(x) \subset \mathfrak{C}(C)$. Thus $S = \mathfrak{C}(C)$ comes with a symmetric correspondence

$$R_C := \{(x, y) \in S \times S : \text{rank} P_{xy}(C) = 1\}.$$

Since the Aronhold invarariant S is of degree 4 in coefficients of a ternary quartic, we obtain that $\mathfrak{C}(C)$ is a quartic curve or the whole $\mathbb{P}^2$. The case when $\mathfrak{C}(C) = \mathbb{P}^2$ happens, for example, when C is a Fermat quartic. For any point $x \in \mathbb{P}^2$ and any vertex y of the polar triangle of the Fermat cubic $P_x(C)$, we obtain $P_{yx}(C) = \mathbb{P}^2$.

The assignment $C \to \mathfrak{C}(C)$ lifts to a covariant

$$\mathfrak{C} : S^4(E^\vee) \to S^4(E^\vee)$$

which we call the *Scorza covariant* of quartics. We use the same notation for the associated rational map

$$\mathfrak{C} : |\mathcal{O}_{\mathbb{P}^2}(4)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|.$$

Example 6.3.15 Assume that the equation of C is given in the form

$$at_0^4 + bt_1^4 + ct_2^4 + 6ft_1^2t_2^2 + 6gt_0^2t_2^2 + 6ht_0^2t_1^2 = 0.$$

Then the explicit formula for the Aronhold invariant S (see [492], p. 270) gives

$$\mathfrak{C}(C) := a't_0^4 + b't_1^4 + c't_2^4 + 6f't_1^2t_2^2 + 6g't_0^2t_2^2 + 6h't_0^2t_1^2 = 0,$$

where

$$\begin{aligned} a' &= 6e^2h^2, \quad b' = 6h^2f^2, \quad c' = 6f^2g^2,\\ d' &= bcgh - f(bg^2 + ch^2) - ghf^2,\\ e' &= acfh - g(ch^2 + af^2) - fhg^2,\\ h' &= abfg - h(af^2 + bg^2) - fgh^2. \end{aligned}$$

For a general f the formula for $\mathfrak{C}$ is too long.

Consider the pencil of quartics defined by the equation

$$t_0^4 + t_1^4 + t_2^4 + 6\alpha(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2) = 0, \quad \alpha \neq 0. \tag{6.30}$$

Then $\mathfrak{C}(C)$ is given by the equation

$$t_0^4 + t_1^4 + t_2^4 + 6\beta(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2) = 0,$$

where

$$6\beta\alpha^2 = 1 - 2\alpha - \alpha^2.$$

We find that $\mathfrak{C}(C) = C$ if and only if α satisfies the equation

$$6\alpha^3 + \alpha^2 + 2\alpha - 1 = 0.$$

One of the solutions is $\alpha = 1/3$; it gives a double conic. Two other solutions are $\alpha = \frac{1}{4}(-1 \pm \sqrt{-7})$. They give two curves isomorphic to the *Klein curve* $V(t_0^3t_1 + t_1^3t_2 + t_2^3t_0)$ with 168 automorphisms. We will discuss this curve later in the Chapter.

We will be interested in the open subset of $|\mathcal{O}_{\mathbb{P}^2}(4)|$ where the map $\mathfrak{C}$ is defined and its values belong to the subset of nonsingular quartics.

Proposition 6.3.16 *Suppose $\mathfrak{C}(C)$ is a nonsingular quartic. Then C is either nondegenerate, or it has a unique irreducible apolar conic.*

Proof Suppose C does not satisfy the assumption. Then C admits either a pencil of apolar conics or one reducible apolar conic. In any case there is a reducible conic, hence there exist two points x, y such that $P_{xy}(C) = \mathbb{P}^2$. This implies that $P_x(C)$ is a cone with triple point y. It follows from the explicit formula for the Aronhold invariant S that the curve $P_x(C)$ is a singular point

in the closure of the variety of Fermat cubics. Thus the image of the polar map $x \mapsto P_x(C)$ passes through the singular point. The preimage of this point under the polar map is a singular point of C. □

Theorem 6.3.17 *Let $C = V(f)$ be a general plane quartic. Then $S = \mathfrak{C}(C)$ is a nonsingular curve and there exists an even theta characteristic ϑ on S such that R_C coincides with the Scorza correspondence R_ϑ on S. Every nonsingular S together with an even theta characteristic is obtained in this way.*

Proof To show that $\mathfrak{C}(C)$ is nonsingular for a general quartic, it suffices to give one example when it happens. The Klein curve from Example 6.3.15 will do.

Let S be a nonsingular quartic and R_ϑ be the Scorza correspondence on S defined by a theta characteristic ϑ. It defines the Scorza quartic C. It follows immediately from (5.41) in the proof of Theorem 5.5.17 that for any point $(x, y) \in R_\vartheta$ the second polar $P_{x,y}(C)$ is a double line (in notation in the proof of the loc. cit. Theorem, $(x, y) = (x, x_i)$ and $V(t_i^2)$ is the double line). This shows that $P_x(C)$ is a Fermat cubic, and hence $\mathfrak{C}(C) = S$. Thus, we obtain that the Clebsch covariant $\mathfrak{C}$ is a dominant map whose image contains nonsingular quartics. Moreover, it inverts the Scorza rational map which assigns to (S, ϑ) the Scorza quartic. Thus a general quartic curve C is realized as the Scorza quartic for some (S, ϑ), the correspondence R_C coincides with R_ϑ and $S = \mathfrak{C}(C)$. □

Suppose C is plane quartic with nonsingular $S = \mathfrak{C}(C)$. Suppose $R_C = R_\theta$ for some even theta characteristic on S. Let C' be the Scorza quartic assigned to (S, ϑ). Then, for any $x \in S$, $P_x(C) = P_x(C')$. Since S spans $\mathbb{P}^2$, this implies that $C = C'$. The generality condition in order that $R_C = R_\vartheta$ happens can be made more precise.

Proposition 6.3.18 *Suppose $S = \mathfrak{C}(C)$ satisfies the following conditions*

- *S is nonsingular;*
- *the Hessian of C is irreducible;*
- *S does not admit nonconstant maps to curves of genus 1 or 2.*

Then $R_C = R_\theta$ for some even theta characteristic θ and C is the Scorza quartic associated to (S, θ).

Proof It suffices to show that R_C is a Scorza correspondence on S. Obviously, R_C is symmetric. As we saw in the proof of Proposition 6.3.16, the first condition shows that no polar $P_x(C), x \in S$, is the union of three concurrent lines. The second condition implies that the Steinerian of C is irreducible and

hence does not contain S. This shows that, for any general point $x \in S$, the first polar $P_x(C)$ is projectively equivalent to a Fermat cubic. This implies that R_C is of type $(3,3)$. Since C is nonsingular, $P_{x^2}(C)$ is never a double line or $\mathbb{P}^2$. Thus R_C has no united points.

By Proposition 5.5.11, it remains for us to show that R_C has valence -1. Take a general point $x \in S$. The divisor $R_C(x)$ consists of the three vertices of its unique polar triangle. For any $y \in R_C(x)$, the side $\lambda = V(l)$ opposite to y is defined by $P_y(P_x(C)) = P_x(P_y(C)) = V(l^2)$. It is a common side of the polar triangles of $P_x(C)$ and $P_y(C)$. We have $\ell \cap S = y_1 + y_2 + x_1 + x_2$, where $R_C(x) = \{y, y_1, y_2\}$ and $R_C(y) = \{x, x_1, x_2\}$. This gives

$$y_1 + y_2 + x_1 + x_2 = (R_C(x) - x) + (R_C(y) - y) \in |K_S|.$$

Consider the map $\alpha : S \to \mathrm{Pic}^2(S)$ given by $x \to [R(x) - x]$. Assume R_C has no valence, i.e. the map α is not constant. If we replace in the previous formula y with y_1 or y_2, we obtain that $\alpha(y) = \alpha(y_1) = \alpha(y_2) = K_S - \alpha(x)$. Thus $\alpha : S \to \alpha(S) = S'$ is a map of degree ≥ 3. It defines a finite map of degree ≥ 3 from S to the normalization $\tilde{S}'$ of S'. Since a rational curve does not admit non-constant maps to an abelian variety, we obtain that $\tilde{S}'$ is of positive genus. By assumption, this is impossible. Hence R_C has valence $v = -1$.

□

Let $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{snd}}$ be the open subset of plane quartics C such that $\mathfrak{C}(C)$ is a nonsingular quartic and the correspondence R_C is a Scorza correspondence R_ϑ. The Clebsch covariant defines a regular map

$$\tilde{\mathfrak{C}} : |\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{snd}} \to \mathcal{TC}_4^{\mathrm{ev}}, \quad C \mapsto (\mathfrak{C}(C), R_C). \tag{6.31}$$

By Proposition 5.2.3 the variety $\mathcal{TC}_4^{\mathrm{ev}}$ is an irreducible cover of degree 36 of the variety $|\mathcal{O}_{\mathbb{P}^2}(4)|$ of nonsingular quartics. By Proposition 5.2.3 the variety $\mathcal{TC}_4^{\mathrm{ev}}$ is an irreducible cover of degree 36 of the variety $|\mathcal{O}_{\mathbb{P}^2}(4)|$ of nonsingular quartics. The Scorza map defines a rational section of $\tilde{\mathfrak{C}}$. Since both the source and the target of the map are irreducible varieties of the same dimension, this implies that (6.31) is a birational isomorphism.

Passing to the quotients by $\mathrm{PGL}(3)$, we obtain the following.

Theorem 6.3.19 *Let $\mathcal{M}_3^{\mathrm{ev}}$ be the moduli space of curves of genus 3 together with an even theta characteristic. The birational map $S : |\mathcal{O}_{\mathbb{P}^2}(4)| \to \mathcal{TC}_4$ has the inverse defined by assigning to a pair (C, ϑ) the Scorza quartic. It induces a birational isomorphism*

$$\mathcal{M}_3 \cong \mathcal{M}_3^{\mathrm{ev}}.$$

The composition of this map with the forgetting map $\mathcal{M}_3^{\text{ev}} \to \mathcal{M}_3$ is a rational self-map of $\mathcal{M}_3$ of degree 36.

Remark 6.3.20 The Corollary generalizes to genus 3 the fact that the map from the space of plane cubics $|\mathcal{O}_{\mathbb{P}^2}(3)|$ to itself defined by the Hessian is a birational map to the cover $|\mathcal{O}_{\mathbb{P}^2}(3)|^{\text{ev}}$, formed by pairs (X, ϵ), where ϵ is a nontrivial 2-torsion divisor class (an even characteristic in this case). Note that the Hessian covariant is defined similarly to the Clebsch invariant. We compose the polarization map $V \times S^3(E^\vee) \to S^2(E^\vee)$ with the discriminant invariant $S^2(E^\vee) \to \mathbb{C}$.

6.3.5 Clebsch and Lüroth quartics

Since five general points in the dual plane lie on a singular quartic (a double conic), a general quartic does not admit a polar pentagon, although the count of constants suggests that this is possible. This remarkable fact was first discovered by J. Lüroth in 1868. Suppose a quartic C admits a polar pentagon $\{[l_1], \dots, [l_5]\}$ (or the *polar pentalateral* $V(l_1), \dots, V(l_5)$). Let $Q = V(q)$ be a conic in $\mathbb{P}(E)$ passing through the points $[l_1], \dots, [l_5]$. Then $q \in \text{AP}_2(f)$. The space $\text{AP}_2(f) \neq \{0\}$ if and only if $\det \text{Cat}_2(f) = 0$. Thus the set of quartics admitting a polar pentagon is the locus of zeros of the catalecticant invariant on the space $\mathbb{P}(S^4(E^\vee))$. It is a polynomial of degree 6 in the coefficients of a ternary form of degree 4.

Definition 6.3.21 *A plane quartic admitting a polar pentagon is called a* Clebsch quartic.

Lemma 6.3.22 *Let $C = V(f)$ be a Clebsch quartic. The following properties are equivalent.*

- *(i) C admits polar pentagon $\{[l_1], \dots, [l_5]\}$ such that $l_1^2, \dots, l_5^2 \in S^2(E^\vee)$ are linearly independent;*
- *(ii) $\dim \text{AP}_2(f) = 1$;*
- *(iii) for any polar pentagon $\{[l_1], \dots, [l_5]\}$ of C, $l_1^2, \dots, l_5^2$ are linearly independent;*
- *(iv) for any polar pentagon $\{[l_1], \dots, [l_5]\}$ of C, no four of the points $[l_i]$ are collinear.*

Proof (i) $\Rightarrow$ (ii) For any $\psi \in \text{AP}_2(f)$, we have

$$0 = D_\psi(f) = \sum D_\psi(l_i^2) l_i^2.$$

Since l_i^2 are linearly independent, this implies $D_\psi(l_i^2) = 0, i = 1, \dots, 5$. This

means that $V(\psi)$ is a conic passing through the points $[l_1], \ldots, [l_5]$. Five points in the plane determine unique conic unless four of the points are collinear. It is easy to see that in this case the quadratic forms $l_1^2, \ldots, l_5^2$ are linearly dependent. Thus $\dim \mathrm{AP}_2(f) = 1$.

(ii) $\Rightarrow$ (ii) Suppose $\{[l_1], \ldots, [l_5]\}$ is a polar pentagon of C with linearly dependent $l_1^2, \ldots, l_5^2$. Then there exist two linearly independent functions ψ_1, ψ_2 in $S^2(E^\vee)^\vee = S^2(E)$ vanishing at $l_1^2, \ldots, l_5^2$. They are apolar to f, contradicting the assumption.

(iii) $\Rightarrow$ (iv) Suppose $\{[l_1], \ldots, [l_4]\}$ are collinear. Then, we can choose coordinates to write $l_1 = t_0, l_1 = t_1, l_3 = at_0 + bt_1, l_4 = ct_0 + dt_1$. Taking squares, we see that the five l_i^2 are linear combinations of four forms $t_0^2, t_1^2, t_0t_1, l_5$. This contradicts the assumption.

(iv) $\Rightarrow$ (i) Let $\{[l_1], \ldots, l_5]\}$ be a polar pentagon with no four collinear points. It is easy to see that it implies that we can choose four of the points such that no three among them are collinear. Now change coordinates to assume that the corresponding quadratic forms are $t_0^2, t_1^2, t_2^2, a(t_0 + t_1 + t_2)^2$. Suppose $l_1^2, \ldots, l_5^2$ are linearly dependent. Then we can write

$$l_5^2 = \alpha_1 t_0^2 + \alpha_2 t_1^2 + \alpha_3 t_2^2 + \alpha_4 (t_0 + t_1 + t_2)^2.$$

If two of the coefficients α_i are not zero, then the quadratic form in the right-hand side is of rank ≥ 2. The quadratic form in the left-hand side is of rank 1. Thus, three of the coefficients are zero, but the two of the points $[l_i]$ coincide. This contradiction proves the implication. □

Definition 6.3.23 *A Clebsch quartic is called* weakly nondegenerate *if it satisfies one of the equivalent conditions from the previous Lemma. It is called nondegenerate if the unique polar conic is irreducible.*

This terminology is somewhat confusing since a quartic was earlier called nondegenerate if it does not admit an apolar conic. I hope the reader can live with this.

It follows immediately from the definition that each polar pentalateral of a nondegenerate Clebsch quartic consists of five sides, no three of which pass through a point (a *complete pentalateral*). Considered as a polygon in the dual plane, this means that no three vertices are collinear. On the other hand, the polar pentalateral of a weakly nondegenerate Clebsch quartic may contain one or two triple points.

Let $C = V(\sum l_i^4)$ be a Clebsch quartic. If x lies in the intersection of two

sides $V(l_i)$ and $V(l_j)$ of the polar pentalateral, then

$$P_x(C) = V(\sum_{k\neq i,j} l_k(x) l_k^3),$$

hence it lies in the closure of the locus of Fermat cubics. This means that the point x belongs to the quartic $\mathfrak{C}(C)$. When C is a general Clebsch quartic, $\mathfrak{C}(C)$ passes through each of 10 vertices of the polar complete pentalateral. In other words, $\mathfrak{C}(C)$ is a *Darboux plane curve* of degree 4 in sense of the definition below.

Let $\ell_1, \ldots, \ell_N$ be a set of N distinct lines in the planes, the union of which is called a *N-lateral*, or an *arrangement of lines*. A point of intersection x_{ij} of two of the lines ℓ_i and ℓ_j is called a *vertex* of the N-lateral. The number of lines intersecting at a vertex is called the *multiplicity* of the vertex. An N-lateral with all vertices of multiplicity 2 is called a *complete N-lateral* (or a *general arrangement*). Considered as a divisor in the plane, it is a normal crossing divisor. The dual configuration of an N-lateral (the dual arrangement) consists of a set of N points corresponding to the lines and a set of lines corresponding to points. The number of points lying on a line is equal to the multiplicity of the line considered as a vertex in the original N-lateral.

Let $\mathcal{J}$ be the ideal sheaf of functions vanishing at each vertex x_{ij} with multiplicity $\geq \nu_{ij} - 1$, where ν_{ij} is the multiplicity of x_{ij}. A nonzero section of $\mathcal{J}(k)$ defines a plane curve of degree k that has singularities at each x_{ij} of multiplicity $\geq \nu_{ij} - 1$.

Lemma 6.3.24 *Let $\mathcal{A} = \{\ell_1, \ldots, \ell_N\}$ be an N-lateral. Then*

$$h^0(\mathbb{P}^2, \mathcal{J}(N-1)) = N.$$

Proof Let ℓ be a general line in the plane. It defines an exact sequence

$$0 \to \mathcal{J}(N-2) \to \mathcal{J}(N-1) \to \mathcal{J}(N-1) \otimes \mathcal{O}_\ell \to 0.$$

Since the divisor of zeros of a section of $\mathcal{J}(N-2)$ contains the divisor $\ell_i \cap (\sum_{j\neq i} \ell_j)$ of degree $N-1$, it must be the whole ℓ_i. Thus $h^0(\mathcal{J}(N-2)) = 0$. Since $\mathcal{J}(N-1) \otimes \mathcal{O}_\ell \cong \mathcal{O}_{\mathbb{P}^1}(N-1)$, we have $h^0(\mathcal{J}(N-1) \otimes \mathcal{O}_\ell) = N$. This shows that $h^0(\mathcal{J}(N-1)) \leq N$. On the other hand, we can find N linear independent sections by taking the products f_j of linear forms defining $\ell_i, j \neq i$. This proves the equality. □

Definition 6.3.25 *A* Darboux curve *of degree $N-1$ is a plane curve defined by a nonzero section of the sheaf $\mathcal{J}(N-1)$ for some N-lateral of lines in the plane. A Darboux curve of degree 4 is called a* Lüroth quartic curve.

Obviously, any conic (even a singular one) is a Darboux curve. The same is true for cubic curves. The first case where a Darboux curve must be a special curve is the case $N = 5$.

It follows from the proof of Lemma 6.3.24 that a Darboux curve can be given by an equation

$$\sum_{i=1}^{N} \prod_{j \neq i} l_j = \prod_{i=1}^{N} l_i (\sum_{i=1}^{N} \frac{1}{l_i}) = 0 \tag{6.32}$$

where $\ell_i = V(l_i)$.

From now on, we will be dealing with the case $N = 5$, i.e with Lüroth quartics. The details for the next computation can be found in the original paper by Lüroth [372], p. 46.

Lemma 6.3.26 *Let $C = V(\sum l_i^2)$ be a Clebsch quartic in $\mathbb{P}^2 = |E|$. Choose a volume form on E to identify $l_i \wedge l_j \wedge l_k$ with a number $|l_i l_j l_k|$. Then*

$$\mathfrak{C}(C) = V(\sum_{s=1}^{5} k_s \prod_{i \neq s} l_i),$$

where

$$k_s = \prod_{i<j<k, r \notin \{i,j,k\}} |l_i l_j l_k|.$$

Proof This follows from the known symbolic expression of the Aronhold invariant

$$\mathsf{S} = (abc)(abd)(acd)(bcd).$$

If we polarize $4D_a(f) = \sum l_i(a) l_i^3$, we obtain a tensor equal to the tensor $\sum l_i(a) l_i \otimes l_i \otimes l_i \in (E^\vee)^{\otimes 3}$. The value of S is equal to the sum of the determinants $l_i(a) l_j(a) l_k(a) |l_i l_j l_k|$. When $[a]$ runs $\mathbb{P}^2$, we get the formula from the assertion of the Lemma. □

Looking at the coefficients $k_1, \ldots, k_5$, we observe that

- $k_1, \ldots, k_5 \neq 0$ if and only if C is nondegenerate;
- two of the coefficients $k_1, \ldots, k_5$ are equal to zero if and only if C is weakly degenerate and the polar pentalateral of C has one triple point;
- three of the coefficients $k_1, \ldots, k_5$ are equal to zero if and only if C is weakly nondegenerate and the polar pentalateral of C has two triple points;
- $\mathfrak{C}(C) = \mathbb{P}^2$ if the polar pentalateral has a point of multiplicity 4.

It follows from this observation, that a Lüroth quartic of the form $\mathfrak{C}(C)$ is always reducible if C admits a degenerate polar pentalateral. Since $\mathfrak{C}(C)$ does not depend on a choice of a polar pentalateral, we also see that all polar pentalaterals of a weakly nondegenerate Clebsch quartic are complete pentalaterals (in the limit they become generalized polar 5-hedra).

Thus we see that, for any Clebsch quartic C, the quartic $\mathfrak{C}(C)$ is a Lüroth quartic. One can prove that any Lüroth quartic is obtained in this way from a unique Clebsch quartic (see [177]).

Let $C = V(f)$ be a nondegenerate Clebsch quartic. Consider the map

$$c : \mathrm{VSP}(f, 5)^o \to |\mathcal{O}_{\mathbb{P}^2}(2)| \tag{6.33}$$

defined by assigning to $\{\ell_1, \dots, \ell_5\} \in VSP(f, 5)^o$ the unique conic passing through these points in the dual plane. This conic is nonsingular and is apolar to C. The fibres of this map are polar pentagons of f inscribed in the apolar conic. We know that the closure of the set of Clebsch quartics is defined by one polynomial in coefficients of quartic, the catalecticant invariant. Thus the varierty of Clebsch quartics is of dimension 13.

Let $\mathcal{E}^5$ be the variety of 5-tuples of distinct nonzero linear forms on E. Consider the map $\mathcal{E}^5 \to |\mathcal{O}_{\mathbb{P}^2}(4)|$ defined by $(l_1, \dots, l_5) \mapsto V(l_1^4 + \cdots + l_5^4)$. The image of this map is the hypersurface of Clebsch quartics. A general fiber must be of dimension $15 - 13 = 2$. However, scaling the l_i by the same factor, defines the same quartic. Thus the dimension of the space of all polar pentagons of a general Clebsch quartic is equal to 1. Over an open subset of the hypersurface of Clebsch quartics, the fibres of c are irreducible curves.

Proposition 6.3.27 *Let $C = V(f)$ be a nondegenerate Clebsch quartic and Q be its apolar conic. Consider any polar pentagon of C as a positive divisor of degree 5 on Q. Then $\mathrm{VSP}(f, 5)^o$ is an open non-empty subset of a g^1_5 on Q.*

Proof Consider the correspondence

$$X = \{(x, \{\ell_1, \dots, \ell_5\}) \in Q \times \mathrm{VSP}(f, 5)^o : x = [l_i] \text{ for some } i = 1, \dots, 5\}.$$

Let us look at the fibres of the projection to Q. Suppose we have two polar pentagons of f with the same side $[l]$. We can write

$$f - l^4 = l_1^4 + \cdots + l_4^4,$$

$$f - \lambda l^4 = m_1^4 + \cdots + m_4^4.$$

For any $\psi \in S^2(E)$ such that $\psi(l_i) = 0, i = 1, \dots, 4$, we get $D_\psi(f) = 12\psi(l)l^2$. Similarly, for any $\psi' \in S^2(E)$ such that $\psi'(m_i) = 0, i = 1, \dots, 4$,

we get $D_{\psi'}(f) = 12\lambda\psi'(l)l^2$. This implies that $V(\psi(l)\psi' - \psi'(l)\psi)$ is an apolar conic to C. Since C is a general Clebsch quartic, there is only one apolar conic. The set of $V(\psi)$'s is a pencil with base points $V(l_i)$, the set of $V(\psi')$ is a pencil with base points $V(l_i)$. This gives a contradiction unless the two pencils coincide. But then their base points coincide and the two pentagons are equal. This shows that the projection to Q is a one-to-one map. In particular, X is an irreducible curve.

Now it is easy to finish the proof. The set of degree 5 positive divisors on $Q \cong \mathbb{P}^1$ is the projective space $|\mathcal{O}_{\mathbb{P}^1}(5)|$. The closure $\mathcal{P}$ of our curve of polar pentagons lies in this space. All divisors containing one fixed point in their support form a hyperplane. Thus the polar pentagons containing one common side $[l]$ correspond to a hyperplane section of $\mathcal{P}$. Since we know that there is only one such pentagon and we take $[l]$ in an open Zariski subset of Q, we see that the curve is of degree 1, i.e. a line. So our curve is contained in a 1-dimensional linear system of divisors of degree 5. □

Remark 6.3.28 The previous Proposition shows why Lüroth quartics are special among Darboux curves. By Lemma 6.3.24, the variety of pairs consisting of an N-lateral and a curve of degree $N-1$ circumscribing it is of dimension $3N-1$. This shows that the dimension of the variety of Darboux curves of degree $N-1$ is equal to $3N-1-k$, where k is the dimension of the variety of N-laterals inscribed in a general Darboux curve. We can construct a Darboux curve by considering an analog of a Clebsch curve, namely a curve C admitting a polar N-gon. Counting constants shows that the expected dimension of the locus of such curves is equal to $3N-1-m$, where m is the dimension of the variety of polar N-gons of C. Clearly every such C defines a Darboux curve as the locus of $x \in \mathbb{P}^2$ such that $P_x(C)$ admits a polar $(N-2)$-gon. The equation of a general Darboux curve shows that it is obtained in this way from a generalized Clebsch curve. In the case $N=5$, we have $k=m=1$. However, already for $N=6$, the variety of Darboux quintics is known to be of dimension 17, i.e. $k=0$ [24]. This shows that there are only finitely many N-laterals that a general Darboux curve of degree 5 could circumscribe.

Suppose C is an irreducible Lüroth quartic. Then it comes from a Clebsch quartic C' if and only if it circumscribes a complete pentalateral and C' is a nondegenerate Clebsch quartic. For example, an irreducible singular Lüroth quartic circumscribing a pentalateral with a triple point does not belong to the image of the Clebsch covariant. In any case, a Darboux curve of degree $N-1$ given by Equation (6.32), in particular, a Lüroth quartic, admits a natural

symmetric linear determinantal representation:[1]

$$\det \begin{pmatrix} l_1+l_2 & l_1 & l_1 & \dots & l_1 \\ l_1 & l_1+l_3 & l_1 & \dots & l_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_1 & \dots & \dots & l_1 & l_1+l_N \end{pmatrix} = 0. \tag{6.34}$$

It is clear that, if $l_1(x) = l_2(x) = l_3(x) = 0$, the corank of the matrix at the point x is greater than 1. Thus, if the N-lateral is not a complete N-lateral, the theta characteristic defining the determinantal representation is not an invertible one. However, everything goes well if we assume that the Lüroth quartic comes from a nondegenerate Clebsch quartic. Before we state and prove the next Theorem, we have to recall some facts about cubic surfaces which we will prove and discuss later in Chapter 9. A cubic surface K always admits a polar pentahedron, maybe a generalized one. Suppose that K is general enough so that it admits a polar pentahedron $V(L_1), \dots, V(L_5)$ such that no four of the forms L_i are linearly dependent. In this case K is called a *Sylvester nondegenerate cubic* and the polar pentahedron is unique. If we write $K = V(L_1^3 + \cdots + L_5^3)$, then the Hessian surface of K can be written by the equation

$$\sum_{i=1}^{5} \prod_{i \neq j} L_i(z) = 0. \tag{6.35}$$

Obviously, a general plane section of the Hessian surface is isomorphic to a Lüroth quartic.

Theorem 6.3.29 *Let $\mathcal{N}$ be a net of quadrics in $\mathbb{P}^3$. The following properties are equivalent.*

(i) There exists a basis (Q_1, Q_2, Q_3) of $\mathcal{N}$ such that the quadrics Q_i can be written in the form

$$Q_j = V(\sum_{i=1}^{5} a_{ij} L_i^2), \quad j = 1, 2, 3, \tag{6.36}$$

where L_i are linear forms with any four of them being linearly independent.

(ii) There exists a Sylvester nondegenerate cubic surface K in $\mathbb{P}^3$ such that $\mathcal{N}$ is equal to a net of polar quadrics of K.

[1] This was communicated to me by B. van Geemen, but also can be found in Room's book [476], p. 178.

(iii) The discriminant curve C of $\mathcal{N}$ is a Lüroth quartic circumscribing a complete pentalateral $\{V(l_1), \dots, V(l_5)\}$ and $\mathcal{N}$ corresponds to the symmetric determinantal representation (6.34) of C.

Proof (i) $\Rightarrow$ (ii) Consider the Sylvester nondegenerate cubic surface K given by the *Sylverster equation*

$$K = V(L_1^3 + \cdots + L_5^3).$$

For any point $x = [v] \in \mathbb{P}^3$, the polar quadric $P_x(K)$ is given by the equation $V(\sum L_i(v)L_i^2)$. Let $A = (a_{ij})$ be the 5×3 matrix defining the equations of the three quadrics. Let

$$L_i = \sum_{j=0}^{3} b_{ij} z_j, \quad i = 1, \dots, 5,$$

and let $B = (b_{ij})$ be the 5×4-matrix of the coefficients. By assumption, rank$A = 4$. Thus we can find a 4×3-matrix $C = (c_{ij})$ such that $B \cdot C = A$. If we take the points x_1, x_2, x_3 with coordinate vectors v_1, v_2, v_3 equal to the columns of the matrix C, then we obtain that $L_i(v_j) = a_{ij}$. This shows that $Q_i = P_{x_i}(K), i = 1, 2, 3$.

(ii) $\Rightarrow$ (i) Suppose we can find three non-collinear points $x_i = [v_i]$ and a Sylvester nondegenerate cubic surface K such that $Q_i = P_{x_i}(K), i = 1, 2, 3$. Writing K as a sum of 5 cubes of linear forms L_i, we obtain (i).

(i) $\Rightarrow$ (iii) Consider the five linear forms $l_i = a_{i1}t_0 + a_{i2}t_1 + a_{i3}t_2$. Our net of quadrics can be written in the form

$$Q(t_0, t_1, t_2) = V(\sum_{i=1}^{5} l_i(t_0, t_1, t_2) L_i(z_0, z_1, z_2, z_3)^2).$$

By scaling coordinates t_i and z_j, we may assume that the forms l_i and L_j satisfy

$$\begin{aligned} l_1 + l_2 + l_3 + l_4 + l_5 &= 0, \\ L_1 + L_2 + L_3 + L_4 + L_5 &= 0. \end{aligned} \tag{6.37}$$

The quadric $Q(a)$ is singular at a point x if and only if

$$\text{rank} \begin{pmatrix} l_1(a)L_1(x) & \dots & l_5(a)L_5(x) \\ 1 & \dots & 1 \end{pmatrix} = 1.$$

This is equivalent to that

$$l_1(a)L_1(x) = \dots = L_5(a)l_5(x). \tag{6.38}$$

Taking into account (6.37), we obtain

$$\sum_{i=1}^{5} \frac{1}{l_i(a)} = 0,$$

or

$$\sum_{i=1}^{5} \prod_{i \neq j} l_i(a) = 0. \tag{6.39}$$

This shows that the discriminant curve is a Lüroth quartic given by the determinantal Equation (6.34).

(iii) $\Rightarrow$ (ii) Computing the determinant, we find the equation of C in the form (6.39). Then we linearly embed $\mathbb{C}^3$ in $\mathbb{C}^4$ and find five linear forms L_i such that restriction of L_i to the image is equal to l_i. Since no four of the l_i are linearly dependent, no four of the L_i are linearly dependent. Thus $K = V(\sum L_i^3)$ is a Sylvester nondegenerate cubic surface. This can be chosen in such a way that $\sum L_i = 0$ generates the space of linear relations between the forms. By definition, the image of C in $\mathbb{P}^3$ given by the forms l_i is the discriminant curve of the net of polars of K. □

Definition 6.3.30 *The even theta characteristic on a Lüroth curve defined by the determinantal representation* (5.5.11) *is called a* pentalateral *theta characteristic.*

By changing the pentalateral inscribed in a weakly nondegenerate Lüroth quartic C, we map $\mathbb{P}^1$ to the variety of nets of quadrics in $\mathbb{P}^3$ with the same discriminant curve C. Its image in the moduli space of nets of quadrics modulo projective transformations of $\mathbb{P}^3$ is irreducible. Since there are only finitely many projective equivalence classes of nets with the same discriminant curve, we obtain that the pentalateral theta characteristic does not depend on the choice of the pentalateral.

Suppose C is a nondegenerate Lüroth quartic equal to $\mathfrak{C}(C')$ for some Clebsch quartic C'. It is natural to guess that the determinantal representation of C given by determinant (6.34) corresponds to the pentalateral theta characteristic defined by the Scorza correspondence $R_{C'}$ on C. The guess is correct. We refer for the proof to [177], Theorem 7.4.1.

Remark 6.3.31 Since the locus of Clebsch quartics is a hypersurface (of degree 6) in the space of all quartics, the locus of Lüroth quartics is also a hypersurface. Its degree is equal to 54 ([393]). Modern proofs of this fact can be found in [359], [570], and in [422]. We also refer to a beautiful paper of H. Bateman which discusses many aspects of the theory of Lüroth quartics, some

of this was revised in [422] and [423]. For example, in the second paper, G. Ottaviani and E. Sernesi study the locus of singular Lüroth quartics and prove that it consists of two irreducible components. One of them is contained in the image of the Clebsch covariant. The other component is equal to the locus of Lüroth quartics circumscribing a pentalateral with a double point.

Note that the degree of the locus of three quadrics (Q_1, Q_2, Q_3) with discriminant curve isomorphic to a Lüroth quartic is equal to $4 \cdot 54 = 216$. It consists of one component of degree 6, the zero set of the Toeplitz invariant, and the other component of degree 210. The component of degree 6 corresponds to a choice of a pentagonal theta characteristic, the other component corresponds to other 35 theta characteristics, for which the monodromy is irreducible.

6.3.6 A Fano model of VSP$(f, 6)$

Recall that a nondegenerate ternary quartic $f \in S^4(E^\vee)$ is one of the special cases from Theorem 1.3.19 where Corollary 1.4.13 applies. So, the variety $\mathrm{VSP}(f,6)^o$ embeds in the Grassmann variety $G(3, \mathrm{AP}_3(f)^\vee) \cong G(3,7)$. The image is contained in the subvariety $G(3, \mathrm{AP}_3(f))_\sigma$ of isotropic subspaces of the skew-symmetric linear map $\sigma : \Lambda^2 E \to \bigwedge^2 \mathrm{AP}_3(f)$. Choosing a basis in E and identifying $\Lambda^2 E$ with $E^\vee$, we can view this map as a skew-symmetric 7×7-matrix M whose entries are linear functions on E. Let $L \subset \mathrm{AP}_3(f)^\vee$ be an isotropic subspace of σ. In appropriate coordinates (t_0, t_1, t_2), we can write M in the block-form

$$M = \begin{pmatrix} B & A \\ -{}^t A & 0 \end{pmatrix},$$

where B is a square skew-symmetric 4×4 matrix and A is a 4×3 matrix. The maximal minors of the matrix A generate an ideal in $\mathbb{C}[t_0, t_1, t_2]$ defining a closed 0-dimensional subscheme Z of length 6. This defines the map

$$G(3, \mathrm{AP}_3(f))_\sigma \to \mathrm{VSP}(f,6)$$

which is the inverse of the map $\mathrm{VSP}(f,6)^o \to G(3, \mathrm{AP}_3(f))_\sigma$ (see [454]).

The following Theorem is originally due to S. Mukai [400] and was reproved by a different method by K. Ranestad and F.-O. Schreyer [454], [502].

Theorem 6.3.32 *Let $f \in S^4(E^\vee)$ be a nondegenerate quartic form in three variables. Then the map $VSP(f,6)^o \to G(3, \mathrm{AP}_3(f)^\vee)_\sigma$ extends to an isomorphism*

$$\mu : \mathrm{VSP}(f,6) \to G(3, \mathrm{AP}_3(f)^\vee)_\sigma.$$

If f is a general quartic, the variety $G(3, \mathrm{AP}_3(f)^\vee)_\sigma$ is a smooth threefold.

Its canonical class is equal to $-H$, where H is a hyperplane section in the Plücker embedding of the Grassmannian.

Recall that a *Fano variety* of dimension n is a projective variety X with ample $-K_X$. If X is smooth, and $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $-K_X = mH$, where H is an ample generator of the Picard group, then X is said to be of *index* m. The *degree* of X is the self-intersection number H^n . The number $g = \frac{1}{2}K_X^n + 1$ is called the *genus*.

In fact, in [399] S. Mukai announced a more precise result. The variety $\mathrm{VSP}(f, 6)$ is a Gorenstein Fano variety if f is not a Lüroth quartic and it is smooth, if $V(f)$ is nondegenerate and does not admit complete quadrangles as its a polar 6-side (a *complete quadrangle* is the union of six lines joining two out of four general points in the plane).

Remark 6.3.33 A Fano variety V_{22} Also he shows that through each point on V_{22} passes 6 conics taken with multiplicities. In the dual plane they correspond to a generalized polar hexagon of f (see [400], [401]).

By the same method, Ranestad and Schreyer extended the previous result to all exceptional cases listed in Subsection 1.4.3, where $n = 2$. We have

Theorem 6.3.34 *Let f be a general ternary form of degree $2k$. Then*

- $k = 1$*:* $\mathrm{VSP}(f, 3) \cong G(2, 5)_\sigma$ *is isomorphic to a Fano variety of degree 5 and index 2;*
- $k = 2$*:* $\mathrm{VSP}(f, 6) \cong G(3, 7)_\sigma$ *is isomorphic to the Fano variety V_{22} of degree 22 and index 1;*
- $k = 3$*:* $\mathrm{VSP}(f, 10) \cong G(4, 9)_\sigma$ *is isomorphic to a K3 surface of degree* 38 *in* $\mathbb{P}^{20}$*;*
- $k = 4$*:* $\mathrm{VSP}(f, 15) \cong G(5, 11)_\sigma$ *is a set of 16 points.*

In the two remaining cases $(n, k) = (1, k)$ and $(n, k) = (3, 2)$, the variety $\mathrm{VSP}(f, k+1)$ is isomorphic to $\mathbb{P}^1$ (see 1.5.1) in the first case and, in the second case, the birational type of the variety $\mathrm{VSP}(f, 10)$ is unknown at present.

Let $C = V(f)$ be a nonsingular plane quartic and θ is an even theta characteristic on C. Let N_ϑ be the corresponding net of quadrics in $\mathbb{P}(H^0(C, \theta(1)))$. Let $\mathsf{N}_\vartheta^\perp$ be the apolar linear system of quadrics in the dual projective space $\check{\mathbb{P}}^3$. Its dimension is equal to 6. We say that a rational normal cubic R in $\check{\mathbb{P}}^3$ is associated to N_ϑ if the net of quadrics $|\mathcal{J}_R(2)|$ vanishing on R is contained in $\mathsf{N}_\vartheta^\perp$. In [502] F.-O. Schreyer constructs a linear map $\alpha : \bigwedge^2 \mathsf{N}_\vartheta^\perp \to \mathsf{N}_\vartheta$ and shows that the nets of quadrics defining the associated rational normal curves is parameterized by the subvariety $G(3, \mathsf{N}_\vartheta^\perp)_\alpha$ of isotropic subspaces of α. This

reminds us of the construction of $G(3, \mathrm{AP}_3(f)^\vee)_\sigma$. In fact, consider the transpose map ${}^t\alpha : \mathsf{N}_\vartheta^\vee \to \bigwedge^2(E^\perp)^\vee$ and pass to the third symmetric power to get a linear map

$$S^3(\mathsf{N}_\vartheta^\vee) \to S^3 \bigwedge^2 (\mathsf{N}_\vartheta^\perp)^\vee \to \bigwedge^6 (\mathsf{N}_\vartheta^\perp)^\vee \to \mathsf{N}_\vartheta^\perp.$$

Its kernel can be identified with $\mathrm{AP}_3(g)$, where $V(g)$ is the quartic such $V(f)$ is its Scorza quartic. This gives another proof of the Scorza birational isomorphism between $\mathcal{M}_3$ and $\mathcal{M}_3^{\mathrm{ev}}$. A similar construction was announced earlier by S. Mukai [399].

6.4 Invariant theory of plane quartics

Let $\mathsf{I}(d)$ denote the space of $\mathrm{SL}(3)$-invariants of degree d in the linear action of $\mathrm{SL}(3)$ on the space of quartic ternary forms. We have already encountered an invariant F_6 of degree 6, the catalecticant invariant. It vanishes on the space of Clebsch quartics. Another familiar invariant is the discriminant invariant F_{27} of degree 27. There is also an invariant F_3 of degree 3 with symbolic expression $(abc)^4$. We will explain its geometric meaning a little later.

Let us introduce the generating function

$$P(T) = \sum_{d=0}^{\infty} \dim_{\mathbb{C}} \mathsf{I}(d) T^d.$$

It has been computed by T. Shioda [533], and the answer is

$$P(T) = \frac{N(T)}{\prod_{i=1}^{6}(1-T^{3i})(1-T^{27})}, \tag{6.40}$$

where

$$N(T) = 1 + T^9 + T^{12} + T^{15} + 2T^{18} + 3T^{21} + 2T^{24} + 3T^{27} + 4T^{30} + 3T^{33} + 4T^{36}$$
$$+4T^{39} + 3T^{42} + 4T^{45} + 3T^{48} + 2T^{51} + 3T^{54} + 2T^{57} + T^{60} + T^{63} + T^{66} + T^{75}.$$

It was proven by J. Dixmier [168] that the algebra of invariants is finite over the free subalgebra generated by seven invariants of degrees $3, 6, 9, 12, 15, 18, 27$. This was conjectured by Shioda. He also conjectured that one needs six more invariants of degrees $9, 12, 15, 18, 21, 21$ to generate the whole algebra of invariants. This is still open. We know some of the covariants of plane quartics. These are the Hessian He of order 6 and degree 3, and the Clebsch covariant $\mathfrak{C}_4$ of order 4 and degree 3. Recall that it assigns to a general quartic the closure of the locus of points whose polar is equianharmonic cubic. There is a

similar covariant $\mathfrak{C}_6$ of degree 4 and order 6 that assigns to a general quartic curve the closure of the locus of points whose polars are harmonic cubics. The Steinerian covariant of degree 12 and order 12 is a linear combination of $\mathfrak{C}_4^3$ and $\mathfrak{C}_6^2$.

The dual analogs of the covariants $\mathfrak{C}_4$ and $\mathfrak{C}_6$ are the *harmonic* contravariant Φ_6 of class 6 and degree 3 and the *equianharmonic* contravariant Φ_4 of class 4 and degree 2. The first (resp. the second) assigns to a general quartic the closure of the locus of lines which intersect the quartic at a harmonic (resp. equianharmonic) set of four points.

The invariant A_3 vanishes on the set of curves C such that the quartic envelope $\Phi_4(C)$ is apolar to C. One can generate a new invariant by using the polarity pairing between covariants and contravariants of the same order. The obtained invariant, if not zero, is of degree equal to the sum of degrees of the covariant and the contravariant. For example, $(\Phi_4(C), \mathfrak{C}_4(C))$ or $(\Phi_6(C), \mathrm{He}(C))$ give invariants of degree 6. It follows from (6.40) that all invariants of degree 6 are linear combinations of A_3^2 and A_6. However, $(\Phi_6(C), \mathfrak{C}_6(C))$ is a new invariant of degree 9. Taking here the Hessian covariant instead of $\mathfrak{C}_6(C)$, one obtains an invariant of degree 6.

There is another contravariant Ω of class 4 but of degree 5. It vanishes on the set of lines ℓ such that the unique anti-polar conic of ℓ contains ℓ (see [177], p. 274). The contravariant $A_3\Phi_4$ is of the same degree and order, but the two contravariants are different.

We can also generate new covariants and contravariants by taking the polar pairing at already known covariants and contravariants. For example, one gets a covariant conic σ of degree 5 by operating $\Phi_4(C)$ on $\mathrm{He}(C)$. Or we may operate C on $\Phi_6(C)$ to get a contravariant conic of degree 4.

Applying known invariants to covariants or contravariants gets a new invariant. However, they are of large degrees. For example, taking the discriminant of the Hessian, we get an invariant of degree 215. However, it is reducible, and contains a component of degree 48 representing an invariant that vanishes on the set of quartics which admit a polar conic of rank 1 [563]. There are other known geometrically meaningful invariants of large degree. For example, the Lüroth invariant of degree 54 vanishing on the locus of Lüroth quartics and the *Salmon invariant* of degree 60 vanishing on the locus of quartics with an inflection bitangent (see [122]).

The GIT-quotient of $|\mathcal{O}_{\mathbb{P}^2}(4)|$ by $\mathrm{SL}(3)$ and other compactifications of the moduli space of plane quartic curves were studied recently from different aspects. Unfortunately, it is too large a topic to discuss it here. We refer to [13], [15],[284], [346], [367], [368].

6.5 Automorphisms of plane quartic curves

6.5.1 Automorphisms of finite order

Since an automorphism of a nonsingular plane quartic curve C leaves the canonical class K_C invariant, it is defined by a projective transformation. We first describe all possible cyclic groups of automorphisms of C.

Lemma 6.5.1 *Let σ be an automorphism of order $n > 1$ of a nonsingular plane quartic $C = V(f)$. Then one can choose coordinates in such a way that a generator of the cyclic group (σ) is represented by the diagonal matrix*

$$\mathrm{diag}[1, \zeta_n^a, \zeta_n^b], \quad 0 \le a < b < n,$$

where ζ_n is a primitive n-th root of unity, and f is given in the following list.

(i) $(n = 2), (a, b) = (0, 1)$,

$$t_2^4 + t_2^2 g_2(t_0, t_1) + g_4(t_0, t_1);$$

(ii) $(n = 3), (a, b) = (0, 1)$,

$$t_2^3 g_1(t_0, t_1) + g_4(t_0, t_1);$$

(iii) $(n = 3), (a, b) = (1, 2)$,

$$t_0^4 + \alpha t_0^2 t_1 t_2 + t_0 t_1^3 + t_0 t_2^3 + \beta t_1^2 t_2^2;$$

(iv) $(n = 4), (a, b) = (0, 1)$,

$$t_2^4 + g_4(t_0, t_1);$$

(v) $(n = 4), (a, b) = (1, 2)$,

$$t_0^4 + t_1^4 + t_2^4 + \alpha t_0^2 t_2^2 + \beta t_0 t_1^2 t_2;$$

(vi) $(n = 6), (a, b) = (2, 3)$,

$$t_0^4 + t_2^4 + \alpha t_0^2 t_2^2 + t_0 t_1^3;$$

(vii) $(n = 7), (a, b) = (1, 3)$,

$$t_0^3 t_2 + t_2^3 t_1 + t_0 t_1^3;$$

(viii) $(n = 8), (a, b) = (3, 7)$,

$$t_0^4 + t_1^3 t_2 + t_1 t_2^3;$$

(ix) $(n = 9), (a, b) = (2, 3)$,

$$t_0^4 + t_0 t_2^3 + t_1^3 t_2;$$

(x) $(n = 12), (a, b) = (3, 4)$,

$$t_0^4 + t_1^4 + t_0 t_2^3.$$

Here the subscripts in the polynomials g_i indicate their degree.

Proof Let us first choose coordinates such that σ acts by the formula

$$\sigma : [x_0, x_1, x_2] \mapsto [x_0, \zeta_n^a x_1, \zeta_n^b x_2],$$

where $a \leq b < n$. If $a = b$, we can scale the coordinates by ζ^{-a}, and then permute the coordinates to reduce the action to the case, where $0 \leq a < b$.

We will often use that f is of degree ≥ 3 in each variable. This follows from the assumption that f is nonsingular. A form f is invariant with respect to the action if all monomials entering in f with nonzero coefficients are eigenvectors of the action of σ on the space of quartic ternary forms. We denote by p_1, p_2, p_3 the points $[1, 0, 0], [0, 1, 0], [0, 0, 1]$.

Case 1: $a = 0$.

Write f in the form:

$$f = \alpha t_2^4 + t_2^3 g_1(t_0, t_1) + t_2^2 g_2(t_0, t_1) + t_2 g_3(t_0, t_1) + g_4(t_0, t_1). \quad (6.41)$$

Assume $\alpha \neq 0$. Since $g_4 \neq 0$, if $\alpha \neq 0$, we must have $4b \equiv 0 \mod n$. This implies that $n = 2$ or 4. In the first case $g_1 = g_3 = 0$, and we get case (i). If $n = 4$, we must have $g_1 = g_2 = g_3 = 0$, and we get case (iv).

If $\alpha = 0$, then $3b = 0 \mod n$. This implies that $n = 3$ and $g_2 = g_3 = 0$. This gives case (ii).

Case 2: $a \neq 0$.

The condition $a < b < n$ implies that $n > 2$.

Case 2a: The points p_1, p_2, p_3 lie on C.

This implies that no monomial t_i^4 enters f. We can write f in the form

$$f = t_0^3 a_1(t_1, t_2) + t_1^3 b_1(t_0, t_2) + t_2^3 c_1(t_0, t_1)$$

$$+ t_0^2 a_2(t_1, t_2) + t_1^2 b_2(t_0, t_2) + t_2^2 c_2(t_0, t_1),$$

where a_i, b_i, c_i are homogeneous forms of degree i. If one of them is zero, then we are in Case 1 with $\alpha = 0$. Assume that all of them are not zeros. Since f is invariant, it is clear that no t_i enters two different coefficients a_1, b_1, c_1. Without loss of generality, we may assume that

$$f = t_0^3 t_2 + t_2^3 t_1 + t_1^3 t_0 + t_0^2 a_2(t_2, t_3) + t_1^2 b_2(t_0, t_2) + t_2^2 c_2(t_0, t_1).$$

Now we have $b \equiv a + 3b \equiv 3a \mod n$. This easily implies $7a \equiv 0 \mod n$ and $7b \equiv 0 \mod n$. Since $n | g.c.m(a, b)$, this implies that $n = 7$, and $(a, b) =$

$(1,3)$. By checking the eigenvalues of other monomials, we verify that no other monomials enters f. This is case (vii).

Case 2b: Two of the points p_1, p_2, p_3 lie on the curve.

After scaling and permuting the coordinates, we may assume that the point $p_1 = [1,0,0]$ does not lie on C. Then we can write

$$f = t_0^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter in g_4.

Without loss of generality, we may assume that $t_1^3 t_2$ enters g_4. This gives $3a + b \equiv 0 \mod n$. Suppose $t_1 t_2^3$ enters g_4. Then $a + 3b \equiv 0 \mod n$. Then $8a \equiv n, 8b \equiv 0 \mod n$. As in the previous case this easily implies that $n = 8$. This gives case (viii). If $t_1 t_2^3$ does not enter in g_4, then t_2^3 enters g_3. This gives $3b \equiv 0 \mod n$. Together with $3a + b \equiv 0 \mod n$ this gives $n = 3$ and $(a,b) = (1,2)$, or $n = 9$ and $(a,b) = (2,3)$. These are cases (iii) and (ix).

Case 2c: Only one point p_i lies on the curve.

Again we may assume that p_1, p_3 do not lie on the curve. Then we can write

$$f = t_0^4 + t_1^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter in g_4. This immediately gives $4a \equiv 0 \mod n$. We know that either t_2^3 enters g_3, or $t_1 t_2^3$ enters g_4. In the first case, $3b \equiv 0 \mod n$ and together with $4a \equiv 0 \mod n$, we get $n = 12$ and $(a,b) = (3,4)$. Looking at the eigenvalues of other monomials, this easily leads to case (x). If $t_2^3 t_1$ enters g_4, we get $3b + a \equiv 0 \mod n$. Together with $4a \equiv 0 \mod n$, this gives $12b \equiv 0 \mod 12$. Hence $n = 12$ or $n = 6$. If $n = 12$, we get $a = b = 3$, this has been considered before. If $n = 6$, we get $a = 3, b = 1$. This leads to the equation $t_0^4 + t_1^4 + \alpha t_0^2 t_1^2 + t_1 t_2^3 = 0$. After permutation of coordinates $(t_0,t_1,t_2) \mapsto (t_2,t_0,t_1)$, we arrive at case (vi).

Case 2d: None of the reference point lies on the curve.

In this case we may assume that

$$f = t_0^4 + t_1^4 + t_2^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + t_1 t_2(\alpha t_1^2 + \beta t_2^2 + \gamma t_1 t_2).$$

Obviously, $4a = 4b = 0 \mod n$. If $n = 2$, we are in case (i). If $n = 4$, we get $(a,b) = (1,2), (1,3)$, or $(2,3)$. Permuting $(t_0,t_1,t_2) \mapsto (t_2,t_0,t_1)$, and multiplying the coordinates by ζ_4^2, we reduce the case $(1,2)$ to the case $(2,3)$. The case $(1,3)$ is also reduced to the case $(1,2)$ by multiplying coordinates by ζ_4 and then permuting them. Thus, we may assume that $(a,b) = (1,2)$. Checking the eigenvalues of the monomials entering in f, we arrive at case (v). □

6.5.2 Automorphism groups

We employ the notation from [125]: a cyclic group of order n is denoted by n, the semi-direct product $A \rtimes B$ is denoted by $A : B$, a central extension of a group A with kernel B is denoted by $B.A$. We denote by $L_n(q)$ the group $\mathrm{PSL}(n, \mathbb{F}_q)$.

Theorem 6.5.2 *The following Table is the list of all possible groups of automorphisms of a nonsingular plane quartic.*

Type	Order	Structure	Equation
I	168	$L_2(7)$	$t_0^3t_2 + t_0t_1^3 + t_1t_2^3$
II	96	$4^2 : \mathfrak{S}_3$	$t_0^4 + t_1^4 + t_2^4$
III	48	$4.\mathfrak{A}_4$	$t_0^4 + t_1^4 + 2\sqrt{-3}t_1^2t_2^2 + t_2^4$
IV	24	$\mathfrak{S}_4$	$t_0^4 + t_1^4 + t_2^4 + a(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2)$
V	16	4.2^2	$t_0^4 + t_1^4 + at_1^2t_2^2 + t_2^4$
VI	9	9	$t_0^4 + t_0t_2^3 + t_1^3t_2$
VII	8	D_8	$t_0^4 + t_1^4 + t_2^4 + at_0^2t_2^2 + bt_1^2t_0t_2$
VIII	6	6	$t_0^4 + t_2^4 + t_0t_1^3 + at_0^2t_2^2$
IX	6	$\mathfrak{S}_3$	$t_0^4 + t_0(t_1^3 + t_2^3) + at_0^2t_1t_2 + bt_1^2t_2^2$
X	4	2^2	$t_0^4 + t_1^4 + t_2^4 + at_0^2t_2^2 + bt_0^2t_1^2 + ct_1^2t_2^2$
XI	3	3	$t_2^3t_1 + t_0(t_1^3 + at_1^2t_0 + bt_1t_0^2 + ct_0^3)$
XII	2	2	$t_2^4 + t_2^2g_2(t_0, t_1) + t_0^4 + at_0^2t_1^2 + t_1^4$

Table 6.1 *Automorphisms of plane quartics*

Before we prove the theorem, let us comment on the parameters of the equations. First of all, their number is equal to the dimension of the moduli space of curves with the given automorphism group. The equations containing parameters may acquire additional symmetry for special values of parameters. Thus in Type IV, one has to assume that $a \neq \frac{3}{2}(-1 \pm \sqrt{-7})$, otherwise the curve becomes isomorphic to the Klein curve (see [226], vol. 2, p. 209, or [473]). In Type V, the special values are $a = 0, \pm 2\sqrt{-3}, \pm 6$. If $a = 0$, we get the Fermat quartic, if $a = \pm 6$, we again get Type II (use the identity

$$x^4 + y^4 = \frac{1}{8}((x+y)^4 + (x-y)^4 + 6(x+y)^2(x-y)^2).$$

If $a = \pm 2\sqrt{-3}$, we get Type III (the identity

$$x^4 + y^4 + ax^2y^2 = \frac{e^{-\pi i/3}}{4}((x+iy)^4 + (x-iy)^4 + a(x+iy)^2(x-iy)^2)$$

exhibits an additional automorphism of order 3). In Type VII, we have to assume $b \neq 0$, otherwise the curve is of Type V. In Type VIII, $a \neq 0$, otherwise

the curve is of type III. In Type IX, $a \neq 0$, otherwise the curve acquires an automorphism of order 4. In Type X, all the coefficients a, b, c are different. We leave the cases XI and XII to the reader.

Proof Suppose G contains an element of order $n \geq 6$. Applying Lemma 6.5.1, we obtain that C is isomorphic to a quartic of Type VIII ($n = 6$), I ($n = 7$), II ($n = 8$), VI ($n = 9$), and III ($n = 12$). Here we use that, in the case $n = 8$ (resp. $n = 12$) the binary form $t_1^3t_2+t_1t_2^3$ (resp. $t_1^4+t_0t_2^3$) can be reduced to the binary forms $t_1^4 + t_2^4$ (resp. $t_1^4 + 2\sqrt{-3}t_1^2t + t_2^4$) by a linear change of variables. It corresponds to a harmonic (resp. equianharmonic) elliptic curve.

Assume $n = 8$. Then C is a Fermat quartic. Obviously, G contains a subgroup $G' = 4^2 : \mathfrak{S}_3$ of order 96. If it is a proper subgroup, then the order of G is larger than 168. By *Hurwitz's Theorem*, the automorphism group of a nonsingular curve of genus g is of order $\leq 84(g - 1)$ (see [282], Chapter 5, Exercise 2.5). This shows that $G \cong 4^2 : \mathfrak{S}_3$, as in Type II.

Assume $n = 7$. Then the curve is projectively isomorphic to the *Klein curve*, which we will discuss in the next Subsection and will show that its automorphism group is isomorphic to $L_2(7)$. This deals with Type I.

Now we see that G may contain only Sylow 2-subgroups or 3-subgroups.

Case 1: G contains a 2-group.

First of all, the order $N = 2^m$ of G is less than or equal to 16. Indeed, by the above, we may assume that G does not contain cyclic subgroups of order 2^a with $a > 2$. By Hurwitz's formula

$$4 = N(2g' - 2) + N\sum(1 - \frac{1}{e_i}).$$

If $N = 2^m, m > 4$, then the right-hand side is divisible by 8.

So $N = 2^m, m \leq 4$. As is well-known, and is easy to prove, the center Z of G is not trivial. Pick up an element σ of order 2 in the center and consider the quotient $C \to C/(\sigma) = C'$. Since any projective automorphism of order 2 fixes pointwisely a line ℓ, g has a fixed point on C. By Hurwitz's formula, C' is a curve of genus 1, and the cover is ramified at four points. By choosing the coordinates such that $\sigma = \mathrm{diag}[-1, 1, 1]$, the equation of C becomes

$$t_2^4 + t_2^2g_2(t_0, t_1) + g_4(t_0, t_1) = 0. \tag{6.42}$$

If $G = (\sigma)$, we get Type XII. Suppose $G = 2^2$ and τ is another generator. After a linear change of variables t_1, t_2, we may assume that τ acts as $[t_0, t_1, t_2] \mapsto [t_0, t_1, -t_2]$. This implies that g_2 does not contain the monomial t_1t_2 and g_4 does not contain the monomials $t_1^3t_2, t_1t_2^3$. This leads to Type X.

If $G = (\tau) \cong \mathbb{Z}/4\mathbb{Z}$, there are two cases to consider corresponding to

items (iv) and (v) in Lemma 6.5.1. In the first case, we may assume that $\tau : [t_0, t_1, t_2] \mapsto [t_0, t_1, it_2]$. This forces $g_2 = 0$. It is easy to see that any binary quartic without multiple zeros can be reduced to the form $t_1^4 + at_1^2t_2^2 + t_2^4$. Now we see that the automorphism group of the curve

$$V(t_0^4 + t_1^4 + at_1^2t_2^2 + t_2^4),\ a \neq 0,$$

contains a subgroup generated by the transformations

$$\begin{aligned} g_1 &: [t_0, t_1, t_2] \mapsto [it_0, t_1, t_2], \\ g_2 &: [t_0, t_1, t_2] \mapsto [t_0, it_1, -it_2], \\ g_3 &: [t_0, t_1, t_2] \mapsto [it_0, it_2, it_1]. \end{aligned}$$

The element g_1 generates the center, and the quotient is isomorphic to $2^2 := (\mathbb{Z}/2\mathbb{Z})^2$. We denote this group by 4.2^2. It is one of nine non-isomorphic non-abelian groups of order 16. Other way to represent this group is $D_8 : 2$. The dihedral subgroup D_8 is generated by g_2 and g_1g_3. If $a = 0$, it is the Fermat curve of Type II.

In the second case we may assume that $\tau : [t_0, t_1, t_2] \mapsto [t_0, it_1, -t_2]$. In this case, we can reduce the equation to the form (v) from Lemma 6.5.1. It is easy to see that G contains the dihedral group D_8. If there is nothing else, we get Type VII. There are two isomorphism classes of group of order 16 that contain D_8. They are $D_8 \times 2$ or 4.2^2 from above. In the first case, the group contains a subgroup isomorphic to $2^3 := (\mathbb{Z}/2\mathbb{Z})^3$. This group does not embed in PGL(3). In the second case, the center is of order 4, hence commutes with τ but does not equal to (τ). The equation shows that this is possible only if the coefficient $b = 0$. Thus we get a curve of Type V.

Case 2: G contains a Sylow 3-subgroup.

Let Q be a Sylow 3-subgroup of G. Assume Q contains a subgroup Q' isomorphic to 3^2. By Hurwitz's formula, the quotient of C by a cyclic group of order 3 is either an elliptic curve or a rational curve. In the first case, the quotient map has two ramification points, in the second case it has five ramification points. In any case, the second generator of Q' fixes one of the ramification points. However, the stabilizer subgroup of any point on a nonsingular curve is a cyclic group. This contradiction shows that Q must be cyclic of order 3 or 9.

Case 2a: Q is of order 9.

If $Q = G$, we are getting Type VI. Thus, we may assume that G contains a Sylow 2-subgroup P of some order $2^m, m \leq 4$. By Sylow's Theorem, the number s_3 of Sylow 3-subgroups is equal to $1 + 3k$ and it divides 2^m. This gives $s_3 = 1, 4, 16$. If $m = 1$, the subgroup Q is normal. The cover $C \to C/Q$ is ramified at five points with ramification indices $(9, 9, 3)$. If $Q \neq G$, then P

contains a subgroup isomorphic to $9:2$. It does not contain elements of order 6. An element of order 2 in this group must fix one of the five ramification points and gives a stabilizer subgroup of order 6 or 18. Both is impossible.

Suppose Q is not a normal subgroup. The number n_3 of Sylow 3-subgroups is equal to 4 if $m = 2, 3$, or 16 if $m = 4$. Consider the action of G on the set of 28 bitangents. It follows from the normal form of an automorphism of order 9 in Lemma 6.5.1 that Q fixes a bitangent. Thus, the cardinality of each orbit of G on the set of bitangents divides 2^m and the number of orbits is equal to 4 or 16. It easy to see that this is impossible.

Case 2a: Q is of order 3.

If P contains an element of order 4 of type (v), then, by the analysis from Case 1, we infer that G contains D_8. If $P \cong D_8$, by Sylow's Theorem, the index of the normalizer $N_G(P)$ is equal to the number s_2 of Sylow 2-subgroups. This shows that $s_2 = 1$, hence P is normal in G. An element of order 4 in P must commute with an element of order 3, thus G contains an element of order 12, hence the equation can be reduced to the Fermat equation of Type II. Thus P must be of order 16. This leads to Type III.

So, we may assume that P does not contain an element of order 4 of type (v). If it contains an element of order 4, then it must have equation of Type V with $a = 0$. This leads again to the Fermat curve.

Finally, we arrive at the case when P has no elements of order 4. Then P is an abelian group $(\mathbb{Z}/2\mathbb{Z})^m$, where $m \leq 2$ (the group 2^3 does not embed in $\mathrm{Aut}(\mathbb{P}^2)$. If $m = 0$, we get Type XI, if $m = 1$, we get Type IX, if $m = 2$, we get Type IV.

□

6.5.3 The Klein quartic

Recall that a quartic curve admitting an automorphism of order 7 is projectively equivalent to the quartic

$$C = V(t_0t_1^3 + t_1t_2^3 + t_0^3t_2). \tag{6.43}$$

The automorphism S of order 7 acts by the formula

$$S : [t_0, t_1, t_2] \mapsto [\epsilon t_0, \epsilon^2 t_1, \epsilon^4 t_2], \quad \epsilon = e^{2\pi i/7},$$

where we scaled the action to represent the transformation by a matrix from $\mathrm{SL}(3)$.

As promised, we will show that the group of automorphisms of such a quartic is isomorphic to the simple group $L_2(7)$ of order 168. By Hurwitz's Theorem, the order of this group is the largest possible for curves of genus 3.

Observe that Equation (6.43) has a symmetry given by a cyclic permutation U of the coordinates. It is easy to check that

$$USU^{-1} = S^4, \tag{6.44}$$

so that the subgroup generated by S, U is a group of order 21 isomorphic to the semi-direct product $7:3$.

By a direct computation one checks that the following unimodular matrix defines an automorphism T of C of order 2:

$$\frac{i}{\sqrt{7}}\begin{pmatrix} \epsilon^2-\epsilon^5 & \epsilon^4-\epsilon^3 & \epsilon-\epsilon^6 \\ \epsilon-\epsilon^6 & \epsilon^2-\epsilon^5 & \epsilon^4-\epsilon^3 \\ \epsilon^4-\epsilon^3 & \epsilon-\epsilon^6 & \epsilon^2-\epsilon^5 \end{pmatrix}. \tag{6.45}$$

We have

$$TUT^{-1} = U^2, \tag{6.46}$$

so that the subgroup generated by U, T is the dihedral group of order 6. One checks that the 49 products S^aTS^b are all distinct. In particular, the cyclic subgroup (S) is not normal in the group G generated by S, T, U. Since the order of G is divisible by $2 \cdot 3 \cdot 7 = 42$, we see that $\#G = 42, 84, 126$ or 168. It follows from Sylow's Theorem that the subgroup (S) must be normal in the first three cases, so $\#G = 168$, and by Hurwitz's Theorem

$$\mathrm{Aut}(C) = G = \langle S, U, T\rangle.$$

One checks that $V = (TS)^{-1}$ satisfies $V^3 = 1$ and the group has the presentation

$$G = \langle S, T, V : S^7 = V^3 = T^2 = STV = 1\rangle.$$

Proposition 6.5.3 *The group* $\mathrm{Aut}(C)$ *is a simple group* G_{168} *of order 168.*

Proof Suppose H is a nontrivial normal subgroup of G. Assume that its order is divisible by 7. Since its Sylow 7-subgroup cannot be normal in H, we see that H contains all Sylow 7-subgroups of G. By Sylow's Theorem, their number is equal to 8. This shows that $\#H = 56$ or 84. In the first case H contains a Sylow 2-subgroup of order 8. Since H is normal, all its conjugates are in H, and, in particular, $T \in H$. The quotient group G/H is of order 3. It follows from (6.46) that the coset of U must be trivial. Since 3 does not divide 56, we get a contradiction. In the second case, H contains S, T, U and hence coincides with G. So, we have shown that H cannot contain an element of order 7. Suppose it contains an element of order 3. Since all such elements are conjugate, H contains U. It follows from (6.44) that the coset of S in G/H is

trivial, hence $S \in H$, contradicting the assumption. It remains for us to consider the case when H is a 2-subgroup. Then $\#G/H = 2^a \cdot 3 \cdot 7$, with $a \leq 2$. It follows from Sylow's Theorem that the image of the Sylow 7-subgroup in G/H is normal. Thus its preimage in G is normal. This contradiction finishes the proof that G is simple.

□

Remark 6.5.4 One can show that

$$G_{168} \cong \mathrm{PSL}(2, \mathbb{F}_7) \cong \mathrm{PSL}(3, \mathbb{F}_2).$$

The first isomorphism has a natural construction via the theory of automorphic functions. The Klein curve is isomorphic to a compactification of the modular curve $X(7)$, corresponding to the principal congruence subgroup of full level 7. The second isomorphism has a natural construction via considering a model of the Klein curve over a finite field of two elements (see [210]). We can see an explicit action of G on 28 bitangents via the geometry of the projective line $\mathbb{P}^1(\mathbb{F}_7)$ (see [136], [321]).

The group $\mathrm{Aut}(C)$ acts on the set of 36 even theta characteristics with orbits of cardinality $1, 7, 7, 21$ (see [177]. The unique invariant even theta characteristic θ gives rise to a unique G-invariant in $\mathbb{P}^3 = \mathbb{P}(V)$, where $V = H^0(C, \theta(1))$. Using the character table, one can decompose the linear representation $S^2(V)$ into the direct sum of the 3-dimensional representation $E = H^0(C, \mathcal{O}_C(1))^\vee$ and a 7-dimensional irreducible linear representation. The linear map $E \to S^2(V)$ defines the unique invariant net of quadrics. This gives another proof of the uniqueness of an invariant theta characteristic. The corresponding representation of C as a symmetric determinant is due to F. Klein [338] (see also [201]). We have

$$\det \begin{pmatrix} -t_0 & 0 & 0 & -t_2 \\ 0 & t_2 & 0 & -t_2 \\ 0 & 0 & t_2 & -t_0 \\ t_2 & -t_2 & -t_0 & 0 \end{pmatrix} = t_0^3 t_2 + t_2^3 t_0 + t_1^3 t_0. \tag{6.47}$$

The group $\mathrm{Aut}(C)$ has 3 orbits on C with nontrivial stabilizers of orders $2, 3, 7$. They are of cardinality $84, 56$ and 24, respectively.

The orbit of cardinality 24 consists of inflection points of C. They are the vertices of the eight triangles with inflection tangents as its sides. These are eight contact cubics corresponding to the unique invariant theta characteristic. The eight inflection triangles coincide with eight biscribed triangles. The group acts on the eight triangles with stabilizer subgroup of order 21. In fact, the coordinate triangle is one of the eight triangles. The subgroup generated by

S and U leaves it invariant. The element T of order 2 sends the coordinate triangle to the triangle with sides whose coordinates are the rows of the matrix (6.45). In fact, this is how the element T was found (see [338] or [226], vol. 2, p. 199).

We know that the inflection points are the intersection points of C and its Hessian given by the equation

$$\mathrm{He}(f) = 5t_0^2t_2^2t_2^2 - t_0t_2^5 - t_0^5t_1 - t_1t_1^5 = 0.$$

So the orbit of 24 points is cut out by the Hessian.

The orbit of cardinality 56 consists of the tangency points of 28 bitangents of C. An example of an element of order 3 is a cyclic permutation of coordinates. It has 2 fixed points $[1, \eta_3, \eta_3^2]$ and $[1, \eta_3^2, \eta_3]$ on C. They lie on the bitangent with equation

$$4t_0 + (3\eta_3^2 + 1)t_1 + (3\eta_3 + 1)t_2 = 0.$$

Define a polynomial of degree 14 by

$$\Psi = \det \begin{pmatrix} \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 t_1} & \frac{\partial^2 f}{\partial t_0 t_2} & \frac{\partial f}{\partial t_0} \\ \frac{\partial^2 f}{\partial t_1 t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 t_2} & \frac{\partial f}{\partial t_1} \\ \frac{\partial^2 f}{\partial t_2 t_0} & \frac{\partial^2 f}{\partial t_2 t_1} & \frac{\partial^2 f}{\partial t_2^2} & \frac{\partial f}{\partial t_2} \\ \frac{\partial f}{\partial t_0} & \frac{\partial f}{\partial t_1} & \frac{\partial f}{\partial t_2} & 0 \end{pmatrix}.$$

One checks that it is invariant with respect to G_{168} and does not contain f as a factor. Hence it cuts out in $V(f)$ a G-invariant positive divisor of degree 56. It must consists of a G_{168}-orbit of cardinality 56.

One can compute it explicitly (see [596], p. 524) to find that

$$\Psi = t_0^{14} + t_1^{14} + t_2^{14} - 34t_0t_1t_2(t_0^{10}t_1 + \cdots) - 250t_0t_1t_2(t_0^3t_2^8 + \cdots) +$$

$$375t_0^2t_1^2t_2^2(t_0^6t_1^2 + \cdots) + 18(t_0^7t_2^7 + \cdots) - 126t_0^3t_2^3t_1^3(t_0^3t_2^2 + \cdots).$$

Here the dots mean monomials obtained from the first one by permutation of variables.

The orbit of cardinality 84 is equal to the union of 21 sets, each consisting of four intersection points of C with the line of fixed points of a transformation of order 2. An example of such a point is

$$[(\epsilon^4 - \epsilon^3)(\epsilon - \epsilon^6)\epsilon^4, (\epsilon^2 - \epsilon^5)(\epsilon - \epsilon^6)\epsilon, (\epsilon^4 - \epsilon^3)(\epsilon^2 - \epsilon^5)\epsilon^2].$$

The product ξ of the equations defining the 21 lines defines a curve of degree 21 which coincides with the curve $V(J(f, H, \Psi))$, where $J(f, H, \Psi)$ is the Jacobian determinant of f, the Hesse polynomial, and Ψ. It is a G_{168}-invariant

polynomial of degree 21. Its explicit expression was given by P. Gordan in [254], p. 372:

$$\Xi = t_0^{21} + t_1^{21} + t_2^{21} - 7t_0t_1t_2(t_0^{17}t_1 + \cdots) + 217t_0t_1t_2(t_0^3t_2^{15} + \cdots) -$$

$$308t_0^2t_1^2t_2^2(t_0^{13}t_1^2 + \cdots) - 57(t_0^{14}t_1^7 + \cdots) - 289(t_0^7t_2^{14} + \cdots) +$$

$$4018t_0^3t_1^3t_2^3(t_0^2t_2^{10} + \cdots) + 637t_0^3t_2^3t_1^3(t_0^9t_2^3 + \cdots) +$$

$$1638t_0t_1t_2(t_0^{10}t_2^8 + \cdots) - 6279t_0^2t_1^2t_2^2(t_0^6t_2^9 + \cdots) +$$

$$7007t_0^5t_1^5t_2^5(t_0t_2^5 + \cdots) - 10010t_0^4t_1^4t_2^4(t_0^5t_2^4 + \cdots) + 3432t_0^7t_1^7t_2^7.$$

The group G_{168} admits a central extension $2.L_2(7) \cong \mathrm{SL}(2, \mathbb{F}_7)$. It has a linear representation in $\mathbb{C}^3$ where it acts as a complex reflection group. The algebra of invariants is generated by the polynomial f defining the Klein curve, the Hesse polynomial H, and the polynomials Ψ. The polynomial Ξ is a skew invariant, it is not invariant but its square is. We have (see [254],[226], vol. 2, p. 208)

$$\Xi^2 = \Phi^3 - 88f^2H\Psi^2 + 16(63fH^4\Psi + 68f^4H^2\Psi - 16f^7\Psi$$

$$+108H^7 - 3752f^3H^5 + 1376f^6H^3 - 128f^9H). \qquad (6.48)$$

(note that there is some discrepancy of signs in the formulas of Gordan and Fricke).

We have already mentioned that the Scorza quartic of the Klein quartic C coincides with C. The corresponding even theta characteristic is the unique invariant even theta characteristic θ. One can find all quartic curves X such that its Scorza quartic is equal to C (see [**?**], [177]).

The group G acts on the set of 63 Steiner complexes, or, equivalently, on the set of nontrivial 2-torsion divisor classes of the Jacobian of the curve. There is one orbit of length 28, an orbit of length 21, and two orbits of length 7. Also the group G_{168} acts on Aronhold sets with orbits of length $8, 168, 56$ and 56 [321]. In particular, there is no invariant set of seven points in the plane which defines C.

The variety $\mathrm{VSP}(C, 6)$ is a Fano threefold V_{22} admitting G_{168} as its group of automorphisms. It is studied in [383].

Exercises

6.1 Show that two syzygetic tetrads of bitangents cannot have two common bitangents.

6.2 Let $C_t = V(tf+q^2)$ be a family of plane quartics over $\mathbb{C}$ depending on a parameter t. Assume that $V(f)$ is nonsingular and $V(f)$ and $V(q)$ intersect transversally at eight points $p_1, \ldots, p_8$. Show that C_t is nonsingular for all t in some open neighborhood of 0 in usual topology and the limit of 28 bitangents when $t \to 0$ is equal to the set of 28 lines $\overline{p_i p_j}$.

6.3 Show that the locus of nonsingular quartics that admit an inflection bitangent is a hypersurface in the space of all nonsingular quartics.

6.4 Consider the Fermat quartic $V(t_0^4 + t_1^4 + t_2^4)$. Find all bitangents and all Steiner complexes. Show that it admits 12 inflection bitangents.

6.5 Show that a Fermat quartic has 12 inflection bitangents and this number is maximal possible [352].

6.6 Let $S = \{(\ell_1, \ell_1'), \ldots, (\ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents. Prove that the six intersection points $\ell_i \cap \ell_i'$ lie on a conic and all $\binom{28}{2} = 378$ intersection points of bitangents lie on 63 conics.

6.7 Show that the pencil of conics passing through the four contact points of two bitangents contains five members each passing through the contact points of a pair of bitangents.

6.8 Show that a choice of $\epsilon \in \mathrm{Jac}(C)[2] \setminus \{0\}$ defines a conic Q and a cubic B such that C is equal to the locus of points x such that the polar $P_x(B)$ is touching Q.

6.9 Let $C = V(a_{11}a_{22} - a_{12}^2)$ be a representation of a nonsingular quartic C as a symmetric quadratic determinant corresponding to a choice of a 2-torsion divisor class ϵ. Let $\tilde{C}$ be the unramified double cover of C corresponding to ϵ. Show that $\tilde{C}$ is isomorphic to a canonical curve of genus 5 given by the equations

$$a_{11}(t_0,t_1,t_2) - t_3^2 = a_{12}(t_0,t_1,t_2) - t_3t_4 = a_{22}(t_0,t_1,t_2) - t_4^2 = 0$$

in $\mathbb{P}^4$.

6.10 Show that the moduli space of bielliptic curves of genus 4 is birationally isomorphic to the moduli space of isomorphism classes of genus 3 curves together with a nonzero 2-torsion divisor class.

6.11 A plane quartic $C = V(f)$ is called a *Caporali quartic* if $\mathrm{VSP}(f,4)^o \neq \emptyset$.

(i) Show that the C admits a pencil of apolar conics.
(ii) Show that the Clebsch covariant quartic $\mathfrak{C}(C)$ is equal to the union of four lines.
(iii) Show that any Caporali quartic is projectively isomorphic to the curve

$$at_0(t_1^3 - t_2^3) + bt_1(t_2^3 - t_1^3) + ct_2(t_0^3 - t_1^3) = 0$$

([61]).

6.12 Let q be a nondegenerate quadratic form in three variables. Show that $\mathrm{VSP}(q^2,6)^o$ is a homogeneous space for the group $\mathrm{PSL}(2,\mathbb{C})$.

6.13 Show that the locus of lines $\ell = V(l)$ such that the anti-polar conic of l^2 with respect to a quartic curve $V(f)$ is reducible is a plane curve of degree 6 in the dual plane.

6.14 Classify automorphism groups of irreducible singular plane quartics.

6.15 For each nonsingular plane quartic curve C with automorphism group G describe the ramification scheme of the cover $C \to C/G$.

6.16 Let C be the Klein quartic. For any subgroup H of $\mathrm{Aut}(C)$ determine the genus of H and the ramification scheme of the cover $C \to C/H$.

6.17 Show that a smooth plane quartic admits an automorphism of order 2 if and only if among its 28 bitangents four form a syzygetic set of bitangents intersecting at one point.

6.18 Show that the set of polar conics $P_{x^2}(C)$ of a plane quartic C, where x belongs to a fixed line, form a family of contact conics of another plane quartic C'.

6.19 Show that the description of bitangents via the Cayley octad can be stated in the following way. Let $C = \det A$ be the symmetric determinantal representation of C with the Cayley octad O. Let P be the 8×4-matrix with columns equal to the coordinates of the points in O. The matrix $M = {}^tPAP$ is a symmetric 8×8-matrix, and its entries are the equations of the bitangents (the *bitangent matrix*, see [443]).

6.20 Show that the bitangents participating in each principal 4×4-minor of the bitangent matrix from the previous exercise is a syzygetic tetrad, and the minor itself defines the equation of the form (6.1).

6.21 Let C and K be a general conic and a general cubic. Show that the set of points a such that $P_a(C)$ is tangent to $P_a(K)$ is a Lüroth quartic. Show that the set of polar lines $P_a(C)$ which coincide with polar lines $P_a(K)$ is equal to the set of seven Aronhold bitangents of the Lüroth quartic ([28]).

6.22 Show that the set of 28 bitangents of the Klein quartic contains 21 subsets of four concurrent bitangents and each bitangent has 3 concurrency points.

6.23 Let $\mathsf{v}_3 : |E| \to |S^3(E^\vee)|$ be the Veronese embedding corresponding to the apolarity map $\mathrm{ap}^1_f : E \to S^3(E^\vee)$ for a general plane quartic $V(f) \subset |E|$. Show that the variety $\mathrm{VSP}(f, 6)$ is isomorphic to the variety of 6-secant planes of the projection of the Veronese surface $\nu_3(|E|)$ to $|S^3(E^\vee)/\mathrm{ap}^1_f(E)| \cong \mathbb{P}^6$ ([383]).

6.24 Find a symmetric determinant expression for the Fermat quartic $V(t_0^4 + t_1^4 + t_2^4)$.

Historical Notes

The fact that a general plane quartic curve has 28 bitangents was first proved in 1850 by C. Jacobi [318] although the number was apparently known to J. Poncelet. The proof used Plücker formulas and so did not apply to any nonsingular curve. Using contact cubics, O. Hesse extended this result to arbitrary nonsingular quartics [291].

The first systematic study of the configuration of bitangents began by O. Hesse [291],[292] and J. Steiner [544]. Steiner's paper does not contain proofs. They considered azygetic and syzygetic sets and Steiner complexes of bitangents although the terminology was introduced later by Frobenius [228]. Hesse's approach used the relationship between bitangents and Cayley octads. The notion of a Steiner group of bitangents was introduced by A. Cayley in [86]. Weber [595] changed it to a Steiner complex in order not to be confused with the terminology of group theory.

The fact that the equation of a nonsingular quartic could be brought to the form (6.1) was first noticed by J. Plücker [449]. Equation (6.2), arising from

a Steiner complex, appears first in Hesse's paper [292], §9. The determinantal identity for bordered determinants (6.29) appears in [291]. The number of hexads of bitangents with contact points on a cubic curve was first computed by O. Hesse [291] and by G. Salmon [492].

The equation of a quartic as a quadratic determinant appeared first in Plücker [446], p. 228, and in Hesse [292], §10, [293]. Both of them knew that it can be done in 63 different ways. Hesse also proves that the 12 lines of a Steiner complex, consider as points in the dual plane, lie on a cubic. More details appear in Roth's paper [481] and later, in Coble's book [121].

The relationship between bitangents of a plane quartic and seven points in the dual projective plane was first discovered by S. Aronhold [12]. The fact that Hesse's construction and Aronhold' construction are equivalent via the projection from one point of a Cayley octad was first noticed by A. Dixon [171].

The relation of bitangents to theta functions with odd characteristics goes back to B. Riemann [471] and H. Weber [595] and was developed later by A. Clebsch [107] and G. Frobenius [228], [230]. In particular, Frobenius had found a relationship between the sets of seven points or Cayley octads with theta functions of genus 3. Coble's book [121] has a nice exposition of Frobenius's work. The equations of bitangents presented in Theorem 6.1.9 were first found by Riemann, with more details explained by H. Weber. A modern treatment of the theory of theta functions in genus 3 can be found in many papers. We refer to [235], [249] and the references there.

The theory of covariants and contravariants of plane quartics was initiated by A. Clebsch in his fundamental paper about plane quartic curves [104]. In this paper he introduces his covariant quartic $\mathfrak{C}(C)$ and the catalecticant invariant. He showed that the catalecticant vanishes if and only if the curve admits an apolar conic. Much later G. Scorza [510] proved that the rational map S on the space of quartics is of degree 36 and related this number with the number of even theta characteristics. The interpretation of the apolar conic of a Clebsch quartic as the parameter space of inscribed pentagons was given by G. Lüroth [372]. In this paper (the first issue of Mathematische Annalen), he introduced the quartics that now bear his name. Darboux curves were first introduced by G. Darboux in [156]. They got a modern incarnation in a paper of W. Barth [24], where it was shown that the curves of jumping lines of a rank 2 vector bundle with trivial determinant is a Darboux curve. The modern exposition of works of F. Morley [393] and H. Bateman [28] on the geometry of Lüroth quartics can be found in papers of G. Ottaviani and E. Sernesi [420], [422], [423].

The groups of automorphisms of nonsingular plane quartic curves were clas-

sified by S. Kantor [327] and A. Wiman [602]. The first two curves from our table were studied earlier by F. Klein [338] and W. Dyck [197]. Of course, the Klein curve is the most famous of those and appears often in modern literature (see, for example, [534]).

The classical literature about plane quartics is enormous. We refer to Ciani's paper [97] for a nice survey of classical results, as well as to his own contributions to the study of plane quartics which are assembled in [99]. Other surveys can be found in [432] and [215].

7

Cremona transformations

7.1 Homaloidal linear systems

7.1.1 Linear systems and their base schemes

Recall that a *rational map* $f : X \dashrightarrow Y$ of algebraic varieties over a field $\mathbb{K}$ is a regular map defined on a dense open Zariski subset $U \subset X$. The largest such set to which f can be extended as a regular map is denoted by $\mathrm{dom}(f)$. A point $x \notin \mathrm{dom}(f)$ is called an *indeterminacy point*. Two rational maps are considered to be equivalent if their restrictions to an open dense subset coincide. A rational map is called *dominant* if $f : \mathrm{dom}(f) \to Y$ is a dominant regular map, i.e. the image is dense in Y. Algebraic varieties form a category with morphisms taken to be equivalence classes of dominant rational maps.

From now on we restrict ourselves to rational maps of irreducible varieties over $\mathbb{C}$. We use f_{d}

We will further assume that X is a smooth projective variety. It follows that the complement of $\mathrm{dom}(f)$ is of codimension ≥ 2. A rational map $f : X \dashrightarrow Y$ is defined by a linear system. Namely, we embed Y in a projective space $\mathbb{P}^r$ by a complete linear system $|V'| := |H^0(Y, \mathcal{L}')|$. Its divisors are hyperplane sections of Y. The invertible sheaf $f_{\mathrm{d}}^*\mathcal{L}'$ on $\mathrm{dom}(f)$ can be extended to a unique invertible sheaf $\mathcal{L}$ on all of X. Also we can extend the sections $f_{\mathrm{d}}^*(s), s \in V'$, to sections of $\mathcal{L}$ on all of X. The obtained homomorphism $f^* : V' \to H^0(X, \mathcal{L})$ is injective and its image is a linear subspace $V \subset H^0(X, \mathcal{L})$. The associated projective space $|V| \subset |\mathcal{L}|$ is the linear system defining a morphism $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y \hookrightarrow \mathbb{P}^r$.

The rational map f is given in the usual way. Evaluating sections of V at a point, we get a map $\mathrm{dom}(f) \to \mathbb{P}(V)$ and,, by restriction, the map $\mathrm{dom}(f) \to \mathbb{P}(V')$, which factors through the map $Y \hookrightarrow \mathbb{P}(V')$. A choice of a basis $(s_0, \ldots, s_r)$ in V and a basis in V' defines a rational map $f : X \dashrightarrow Y \subset \mathbb{P}^r$.

It is given by the formula

$$x \mapsto [s_0(x), \dots, s_r(x)].$$

For any rational map $f : X \dashrightarrow Y$ and any closed reduced subvariety Z of Y we denote by $f^{-1}(Z)$ the closure of $f_{\mathrm{d}}^{-1}(Z)$ in X. It is called the *inverse transform* of Z under the rational map f. Thus the divisors from $|V|$ are the inverse transforms of hyperplane sections of Y in the embedding $\iota : Y \hookrightarrow \mathbb{P}^r$. More generally, this defines the inverse transform of any linear system on Y.

Let $\mathcal{L}$ be a line bundle and $V \subset H^0(X, \mathcal{L})$. Consider the natural evaluation map of sheaves

$$\mathrm{ev} : V \otimes \mathcal{O}_X \to \mathcal{L}$$

defined by restricting global sections to stalks of $\mathcal{L}$. It is equivalent to a map

$$\mathrm{ev} : V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$$

whose image is a sheaf of ideals in $\mathcal{O}_X$. This sheaf of ideals is denoted $\mathfrak{b}(|V|)$ and is called the *base ideal* of the linear system $|V|$. The closed subscheme $\mathrm{Bs}(|V|)$ of X defined by this ideal is called the *base scheme* of $|V|$. The reduced scheme is called the *base locus*. In classical terminology, the base locus is the *F-locus*; its points are called *fundamental points*. We have

$$\mathrm{Bs}(|V|) = \cap_{D \in |V|} D = D_0 \cap \ldots \cap D_r \text{ (scheme-theoretically)},$$

where $D_0, \dots, D_r$ are the divisors of sections forming a basis of V. The largest positive divisor F contained in all divisors from $|V|$ (equivalently, in the divisors $D_0, \dots, D_r$) is called the *fixed component* of $|V|$. The linear system without fixed component is sometimes called *irreducible*. Each irreducible component of its base scheme is of codimension ≥ 2.

If $F = \mathrm{div}(s_0)$ for some $s_0 \in \mathcal{O}_X(F)$, then the multiplication by s_0 defines an injective map $\mathcal{L}(-F) \to \mathcal{L}$. The associated linear map $H^0(X, \mathcal{L}(-F)) \to H^0(X, \mathcal{L})$ defines an isomorphism from a subspace W of $H^0(X, \mathcal{L}(-F))$ onto V. The linear system $|W| \subset |\mathcal{L}(-F)|$ is irreducible and defines a rational map $f' : X \dashrightarrow \mathbb{P}(W) \cong \mathbb{P}(V)$.

The linear system is called *base-point-free*, or simply *free* if its base scheme is empty, i.e. $\mathfrak{b}(|V|) \cong \mathcal{O}_X$. The proper transform of such a system under a rational map is an irreducible linear system. In particular, the linear system $|V|$ defining a rational map $X \dashrightarrow Y$ as described in above, is always irreducible.

Here are some simple properties of the base scheme of a linear system.

(i) $|V| \subset |\mathcal{L} \otimes \mathfrak{b}(|V|)| := |H^0(X, \mathfrak{b}(|V|) \otimes \mathcal{L})|$.

(ii) Let $\phi : X' \to X$ be a regular map, and $V' = \phi^*(V) \subset H^0(X', \phi^*\mathcal{L})$. Then $\phi^{-1}(\mathfrak{b}(|V|)) = \mathfrak{b}(f^{-1}(|V|))$. Recall that, for any ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, its inverse image $\phi^{-1}(\mathfrak{a})$ is defined to be the image of $\phi^*(\mathfrak{a}) = \mathfrak{a} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ in $\mathcal{O}_{X'}$ under the canonical multiplication map.

(iii) If $\mathfrak{b}(|V|)$ is an invertible ideal (i.e. isomorphic to $\mathcal{O}_X(-F)$ for some effective divisor F), then $\mathrm{dom}(f) = X$ and f is defined by the linear system $|\mathcal{L}(-F)|$.

(iv) If $\mathrm{dom}(f) = X$, then $\mathfrak{b}(|V|)$ is an invertible sheaf and $\mathrm{Bs}(|V|) = \emptyset$.

7.1.2 Resolution of a rational map

Definition 7.1.1 *A* resolution *of a rational map $f : X \dashrightarrow Y$ of projective varieties is a pair of regular projective morphisms $\pi : X' \to X$ and $\sigma : X' \to Y$ such that $f = \sigma \circ \pi^{-1}$ and π is an isomorphism over* $\mathrm{dom}(f)$*:*

$$\begin{array}{ccc} & X' & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\sigma} \\ X & \overset{f}{- - - \succ} & Y. \end{array} \tag{7.1}$$

We say that a resolution is smooth (normal) if X' is smooth (normal).

Let $Z = V(\mathfrak{a})$ be the closed subscheme of a scheme X defined by an ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. We denote by

$$\sigma : \mathsf{Bl}_X(Z) = \mathrm{Proj} \bigoplus_{k=0}^{\infty} \mathfrak{a}^k \to X$$

the *blow-up* of Z (see [282], Chapter II, §7). We will also use the notation $\mathsf{Bl}_X(\mathfrak{a})$

Let $\nu : \mathsf{Bl}_X^+(Z) \to X$ denote the normalization of the blow-up $\mathsf{Bl}_X(Z)$ and E^+ be the scheme-theoretical inverse image of the exceptional divisor. It is the exceptional divisor of ν.

$$\nu_* \mathcal{O}_{\mathsf{Bl}_X^+(Z)}(-E^+) = \bar{\mathfrak{a}},$$

where $\bar{\mathfrak{a}}$ denotes the *integral closure* of the ideal sheaf $\mathfrak{a}$ (see [355], II, 9.6). A local definition of the integral closure of an ideal I in an integral domain A is the set of elements x in the fraction field of A such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $n > 0$ and $a_k \in I^k$ (pay attention to the power of I here). If $E^+ = \sum r_i E_i$, considered as a Weil divisor, then locally elements in $\bar{\mathfrak{a}}$ are functions ϕ such that $\mathrm{ord}_{E_i}(\nu^*(\phi)) \geq r_i$ for all i.

An ideal sheaf $\mathfrak{a}$ is called *integrally closed* (or *complete*) if $\bar{\mathfrak{a}} = \mathfrak{a}$. We have

$\mathsf{Bl}_X^+(Z) = \mathsf{Bl}_X(Z)$ if and only $\mathfrak{a}^m$ is integrally closed for $m \gg 0$. If X is nonsingular, and $\dim X = 2$, then $m = 1$ suffices [608], Appendix 5.

Proposition 7.1.2 *Let $f : X \dashrightarrow Y$ be a rational map of irreducible varieties defined by a linear system $|V|$ with base ideal $\mathfrak{b}$. Let $\pi : \mathsf{Bl}_X(\mathfrak{b}) \to X$ be the blow-up scheme of $\mathfrak{b}$. Then there exists a unique regular map $\sigma : \mathsf{Bl}_X(\mathfrak{b}) \to Y$ such that (π, σ) is a resolution of f. For any resolution (π', σ') of f there exists a unique morphism $\alpha : X' \to \mathsf{Bl}_X(\mathfrak{b})$ such that $\pi' = \pi \circ \alpha, \sigma' = \sigma \circ \alpha$.*

Proof By properties (ii) and (iii) from above, the linear system $\pi^{-1}(|V|) = |\pi^*(\mathcal{L}) \otimes \pi^{-1}(\mathfrak{b})|$ defines a regular map $\sigma : \mathsf{Bl}_X(\mathfrak{b}) \to Y$. It follows from the definition of maps defined by linear systems that $f = \sigma \circ \pi^{-1}$. For any resolution, (X', π', σ') of f, the base scheme of the inverse transform $\pi^{-1}(|V|)$ on X' is equal to $\pi^{-1}(\mathfrak{b})$. The morphism σ' is defined by the linear system ${\pi'}^{-1}(|V|)$ and hence its base sheaf is invertible. This implies that π' factors through the blow-up of $\mathrm{Bs}(|V|)$. □

Note that we also obtain that the exceptional divisor of π' is equal to the preimage of the exceptional divisor of the blow-up of $\mathrm{Bs}(|V|)$.

Theorem 7.1.3 *Assume that $f : X \dashrightarrow Y$ is a birational map of normal projective varieties and f is given by a linear system $|V| \subset |\mathcal{L}|$ equal to the inverse transform of a very ample complete linear system $|\mathcal{L}'|$ on Y. Let (X', π, σ) be a resolution of f and let E be the exceptional divisor of π. Then the canonical map*

$$V \to H^0(X', \pi^*\mathcal{L}(-E))$$

is an isomorphism.

Proof Set $\mathfrak{b} = \mathfrak{b}(|V|)$. We have natural maps

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}) \to H^0(X', \pi^*\mathcal{L} \otimes \pi^{-1}(\mathfrak{b}))$$

$$\stackrel{\cong}{\to} H^0(X', (\pi^*\mathcal{L})(-E)) \stackrel{\cong}{\to} H^0(X', \sigma^*\mathcal{O}_Y(1)) \stackrel{\cong}{\to} H^0(Y, \sigma_*\sigma^*\mathcal{L}')$$

$$\stackrel{\cong}{\to} H^0(Y, \mathcal{L}' \otimes \sigma_*\mathcal{O}_{X'}) \stackrel{\cong}{\to} H^0(Y, \mathcal{L}').$$

Here we used the Main Zariski Theorem that asserts that $\sigma_*\mathcal{O}_{X'} \cong \mathcal{O}_Y$ because σ is a birational morphism and Y is normal [282], Chapter III, Corollary 11.4. By definition of the linear system defining f, the composition of all these maps is a bijection. Since each map here is injective, we obtain that all the maps are bijective. One of the compositions is our map $V \to H^0(X', \pi^*\mathcal{L}(-E))$, hence it is bijective. □

Corollary 7.1.4 *Assume, additionally, that the resolution (X, π, σ) is normal. Then the natural maps*

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}(|V|)) \to H^0(X', \pi^*(\mathcal{L})(-E)) \to H^0(X, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)})$$

are bijective.

We apply Theorem 7.1.3 to the case when $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map, a *Cremona transformation*. In this case $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \geq 1$, called the (algebraic) *degree* of the Cremona transformation f. We take $|\mathcal{L}'| = |\mathcal{O}_{\mathbb{P}^n}(1)|$. The linear system $|V| = |\mathfrak{b}(|V|)(d)|$ defining a Cremona transformation is called a *homaloidal linear system*. Classically, members of $|V|$ were called *homaloids*. More generally, a *k-homaloid* is a proper transform of a k-dimensional linear subspace in the target space. They were classically called Φ-curves, Φ-surfaces, etc.

Proposition 7.1.5

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}) = 0.$$

Proof Let (X, π, σ) be the resolution of f defined by the normalization of the blow-up of $\mathrm{Bs}(\mathcal{H}_X)$. Let E be the exceptional divisor of $\pi : X \to \mathbb{P}^n$. We know that $\pi_*(\pi^*\mathcal{L}(-E)) = \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}$ and $\pi^*\mathcal{L}(-E) \cong \sigma^*\mathcal{O}_{\mathbb{P}^n}(1)$. The low degree exact sequence defined by the Leray spectral sequence, together with the projection formula, gives an exact sequence

$$0 \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(X, \sigma^*\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, R^1\sigma_*\mathcal{O}_{X'} \otimes \mathcal{O}_{\mathbb{P}^n}(1)). \tag{7.2}$$

Let $\nu : X' \to X$ be a resolution of singularities of X. Then, we have the spectral sequence

$$E_2^{pq} = R^p\sigma_*(R^q\nu_*\mathcal{O}_{X'}) \Rightarrow R^{p+q}(\pi \circ \nu)_*\mathcal{O}_{X'}.$$

It gives the exact sequence

$$0 \to R^1\pi_*(\nu_*\mathcal{O}_{X'}) \to R^1(\pi \circ \nu)_*\mathcal{O}_{X'} \to \pi_*R^1\nu_*\mathcal{O}_{X'}.$$

Since X is normal, $\nu_*\mathcal{O}_{X'} = \mathcal{O}_X$. Since the composition $\pi \circ \nu : X' \to \mathbb{P}^n$ is a birational morphism of nonsingular varieties, $R^1(\pi \circ \nu)_*\mathcal{O}_{X'} = 0$. This shows that

$$R^1\pi_*(\nu_*\mathcal{O}_{X'}) = 0.$$

Together with vanishing of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, (7.2) implies that

$$H^1(X, \pi^*(\mathcal{L})(-E)) = 0.$$

It remains for us to use that the canonical map

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}) \cong H^1(\mathbb{P}^n, \pi_*(\pi^*(\mathcal{L})(-E))) \to H^1(X, \pi^*(\mathcal{L})(-E))$$

is injective (use Čech cohomology, or the Leray spectral sequence). □

Using the exact sequence

$$0 \to \overline{\mathfrak{b}(\mathcal{H}_X)} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}/\overline{\mathfrak{b}(\mathcal{H}_X)} \to 0,$$

and tensoring it by $\mathcal{O}_{\mathbb{P}^n}(d)$, we obtain the following result, classically known as the *Postulation formula*.

Corollary 7.1.6 *Let $|V|$ be a homaloidal linear system. Then*

$$h^0(\mathcal{O}_{V(\overline{\mathfrak{b}(|V|)})}(d)) = \binom{n+d}{d} - n - 1.$$

7.1.3 The graph of a Cremona transformation

We define the *graph* Γ_f of a rational map $f : X \dashrightarrow Y$ as the closure in $X \times Y$ of the graph $\Gamma_{f_{\mathrm{d}}}$ of $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y$. Clearly, the graph, together with its projections to X and Y, defines a resolution of the rational map f.

By the universal property of the graph, we obtain that, for any resolution (X', π, σ) of f, the map $(\pi, \sigma) : X' \to X \times Y$ factors through the closed embedding $\Gamma_f \hookrightarrow X \times Y$. Thus the first projection $\Gamma_f \to X$ has the universal property for morphisms which invert $\mathfrak{b}(|V|)$, hence it is isomorphic to the blow-up scheme $\mathsf{Bl}_X(\mathfrak{b}((|V|))$.

Let us consider the case of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $(F_0, \ldots, F_n)$ is a basis of V defining the homaloidal linear system, then the graph is a closed subscheme of $\mathbb{P}^n \times \mathbb{P}^n$ which is an irreducible component of the closure of the subvariety of $\mathrm{dom}(f) \times \mathbb{P}^n$ defined by 2×2-minors of the matrix

$$\begin{pmatrix} F_0(x) & F_1(x) & \ldots & F_n(x) \\ y_0 & y_1 & \ldots & y_n \end{pmatrix}, \tag{7.3}$$

where $x = (t_0, \ldots, t_n)$ are projective coordinates in the first factor, and $y = (y_0, \ldots, y_n)$ are projective coordinates in the second factor.

In the usual way, the graph Γ_f defines the linear maps of cohomology

$$f_k^* : H^{2k}(\mathbb{P}^n, \mathbb{Z}) \to H^{2k}(\mathbb{P}^n, \mathbb{Z}), \quad \gamma \mapsto (\mathrm{pr}_1)_*([\Gamma_f] \cap (\mathrm{pr}_2)^*(\gamma)),$$

where $\mathrm{pr}_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projection maps. Since $H^{2k}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, these maps are defined by some numbers d_k, the vector $(d_0, \ldots, d_n)$ is called

the *multidegree* of f. In more details, we write the cohomology class $[\Gamma_f]$ in $H^*(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$ as

$$[\Gamma_f] = \sum_{k=0}^{n} d_i h_1^k h_2^{n-k},$$

where $h_i = \mathrm{pr}_i^*(h)$ and h is the class of a hyperplane in $\mathbb{P}^n$. Then

$$f_k^*(h^k) = (\mathrm{pr}_1)_*([\Gamma_f]) \cdot (\mathrm{pr}_2^*(h^k)) = (\mathrm{pr}_1)_*(d_k h_1^k) = d_k h^k.$$

The multidegree vector has a simple interpretation. The number d_k is equal to the degree of the proper transform under f of a general linear subspace of codimension k in $\mathbb{P}^n$. Since f is birational, $d_0 = d_n = 1$. Also $d_1 = d$ is the algebraic degree of f. Inverting f, we obtain that

$$\Gamma_{f^{-1}} = \tilde{\Gamma}_f,$$

where $\tilde{\Gamma}_f$ is the image of Γ_f under the self-map of $\mathbb{P}^n \times \mathbb{P}^n$ that switches the factors. In particular, we see that $(d_r, d_{r-1}, \ldots, d_0)$ is the multidegree of f^{-1}.

In the case when f is a birational map, we have $d_0 = d_n = 1$. We shorten the definition by saying that the multidegree of a Cremona transformation is equal to $(d_1, \ldots, d_{n-1})$.

The next result due to L. Cremona puts some restrictions on the multidegree of a Cremona transformation.

Proposition 7.1.7 (Cremona's inequalities) *For any $n \geq i, j \geq 0$,*

$$1 \leq d_{i+j} \leq d_i d_j, \quad d_{n-i-j} \leq d_{n-i} d_{n-j}.$$

Proof It is enough to prove the first inequality. The second one follows from the first one by considering the inverse transformation. Write a general linear subspace L_{i+j} of codimension $i + j$ as the intersection of a general linear subspace L_i of codimension i and a general linear subspace L_j of codimension j. Then, we have $f^{-1}(L_{i+j})$ is an irreducible component of the intersection $f^{-1}(L_i) \cap f^{-1}(L_j)$. By Bezout's Theorem,

$$d_{i+j} = \deg f^{-1}(L_{i+j}) \leq \deg f^{-1}(L_i) \deg f^{-1}(L_j) = d_i d_j.$$

□

Remark 7.1.8 There are more conditions on the multidegree which follow from the irreducibility of Γ_f. For example, by using the *Hodge type inequality* (see [355], Corollary 1.6.3), we get the inequality

$$d_i^2 \geq d_{i-1} d_{i+1} \tag{7.4}$$

for the multidegree of a Cremona transformation f. For example, if $n = 3$, the only nontrivial inequality following from the Cremona inequalities is $d_0 d_2 = d_2 \leq d_1^2$, and this is the same as the Hodge-type inequality. However, if $n = 4$, we get new inequalities besides the Cremona ones. For example, $(1, 2, 3, 5, 1)$ satisfies the Cremona inequalities, but does not satisfy the Hodge type inequality.

The following are natural questions related to the classification of possible multidegrees of Cremona transformations.

- Let $(1, d_1, \ldots, d_{n-1}, 1)$ be a sequence of integers satisfying the Cremona inequalities and the Hodge-type inequalities: Does there exist an irreducible reduced close subvariety Γ of $\mathbb{P}^n \times \mathbb{P}^n$ with $[\Gamma] = \sum d_k h_1^k h_2^{n-k}$?

- What are the components of the Hilbert scheme of this class containing an integral scheme?

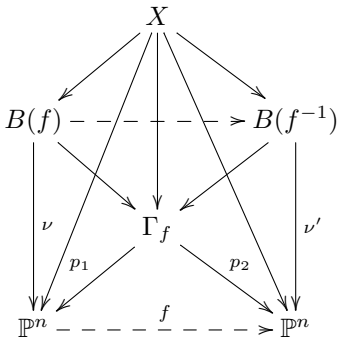
Note that any irreducible reduced closed subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ with multidegree $(1, d_1, \ldots, d_{n-1}, 1)$ is realized as the graph of a Cremona transformation.

7.1.4 F-locus and P-locus

The F-locus of a Cremona transformation f is the base locus of the linear system defining f. Its points are called the *fundamental points* or *indeterminacy points* (F-points, in classical language).

The P-locus of f is the union of irreducible hypersurfaces that are blown down to subvarieties of codimension ≥ 2. One can make this more precise and also give it a scheme-theoretical structure.

Let (X, π, σ) be any normal resolution of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by a homaloidal linear system $|V|$. The morphism π factors through the blow-up $B(f)$ of the integral closure of $\mathfrak{b}(|V|)$. The morphism σ factors through the blow-up $B(f^{-1})$ of the integral closure of the base ideal of the inverse transformation f^{-1}. So we have a commutative diagram, as follows.



Let $E = \sum_{i\in I} r_i E_i$ be the exceptional divisor of $\nu : B(f) \to \mathbb{P}^n$ and let $F = \sum_{j\in J} m_j F_j$ be the exceptional divisor of $\nu' : B(f^{-1}) \to \mathbb{P}^n$. Let J' be the largest subset of J such that the proper transform of $F_j, j \in J'$, in X is not equal to the proper transform of some E_i in X. The image of the divisor $\sum_{j\in J'} F_j$ under the composition map $B(f^{-1}) \to \Gamma_f \stackrel{p_1}{\to} \mathbb{P}^n$ is classically known as the *P-locus* of f. It is a hypersurface in the source $\mathbb{P}^n$. The image of any irreducible component of the P-locus is blown down under f (after we restrict ourselves to $\mathrm{dom}(f)$) to an irreducible component of the base locus of f^{-1}.

Let f be given by homogeneous polynomials $(F_0, \ldots, F_n)$. The same collection of polynomials defines a regular map $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Then the P-locus is the image in $\mathbb{P}^n$ of the locus of critical points of $\tilde{f}$. It is equal to the set of zeros of the determinant of the Jacobian matrix of $\tilde{f}$

$$J = \Big(\frac{\partial F_i}{\partial t_j}\Big)_{i,j=0,\ldots,n}.$$

So we expect that the P-locus is a hypersurface of degree $(d-1)^{n+1}$. Some of its components may enter with multiplicities.

Example 7.1.9 Consider the *standard quadratic transformation* given by

$$T_{\mathrm{st}} : [t_0, t_1, t_2] \mapsto [t_1 t_2, t_0 t_2, t_0 t_1]. \tag{7.5}$$

It has three fundamental points $p_1 = [1,0,0], p_2 = [0,1,0], p_3 = [0,0,1]$. The P-locus is the union of three coordinate lines $V(t_i)$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & 0 & t_0 \\ t_1 & t_0 & 0 \end{pmatrix}.$$

Its determinant is equal to $2t_0t_1t_2$. We may take $X = \mathsf{Bl}_{\mathbb{P}^2}(\{p_1, p_2, p_3\})$

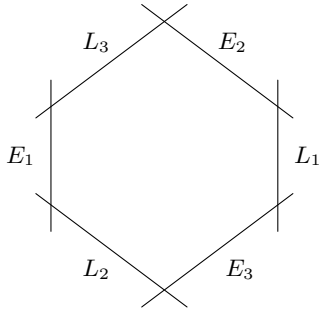


Figure 7.1 A resolution of T_{st}

as a smooth resolution of T_{st} (see Figure 7.1). Let E_1, E_2, E_3 be the exceptional divisors over the fundamental points p_1, p_2, p_3, and let $L_i, i = 1, 2, 3$, be the proper transform of the coordinate lines $V(t_0), V(t_1), V(t_2)$, respectively. Then the morphism $\sigma : X \to \mathbb{P}^2$ blows down L_1, L_2, L_3 to the points p_1, p_2, p_3, respectively. Note that $T_{\text{st}}^{-1} = T_{\text{st}}$, so there is no surprise here. Recall that the blow-up of a closed subscheme is defined uniquely only up to an isomorphism. The isomorphism τ between the blow-ups of the base scheme of T_{st} and T_{st}^{-1} that sends E_i to L_i is a lift of the Cremona transformation T_{st}^{-1}. The surface X is a *del Pezzo surface* of degree 6, a toric Fano variety of dimension 2. We will study such surfaces in the next Chapter. The complement of the open torus orbit is the hexagon of lines $E_1, E_2, E_3, L_1, L_2, L_3$ intersecting each other as in the picture. We call them lines because they become lines in the embedding $X \hookrightarrow \mathbb{P}^6$ given by the anticanonical linear system. The automorphism τ of the surface is the extension of the inversion automorphism $z \to z^{-1}$ of the open torus orbit to the whole surface. It defines the symmetry of the hexagon which exchanges its opposite sides.

Now let us consider the first *degenerate* standard quadratic transformation given by

$$T'_{\text{st}} : [t_0, t_1, t_2] \mapsto [t_2^2, t_0 t_1, t_0 t_2]. \tag{7.6}$$

It has two fundamental points $p_1 = [1, 0, 0]$ and $p_2 = [0, 1, 0]$. The P-locus consists of the line $V(t_0)$ blown down to the point p_1 and the line $V(t_2)$ blown down to the point p_2.

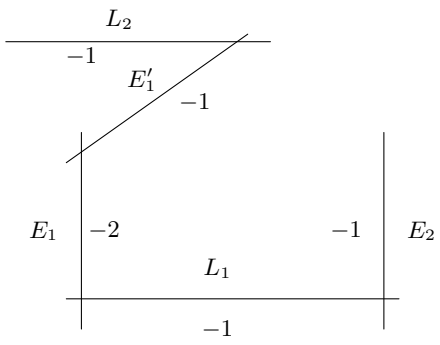


Figure 7.2 A resolution of T'_{st}

The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 0 & 2t_2 \\ t_1 & t_0 & 0 \\ t_2 & 0 & t_0 \end{pmatrix}.$$

Its determinant is equal to $-2t_0t_2^2$. Thus the line $V(t_2)$ enters with multiplicity 2. Let us see what is the resolution in this case. The base scheme is smooth at p_1 and locally isomorphic to $V(y^2, x)$ at the point p_2, where $y = t_2/t_1, x = t_0/t_1$. The blow-up $B(f)$ is singular over p_2 with the singular point p_2' corresponding to the tangent direction $t_0 = 0$. The singular point is locally isomorphic to the singularity of the surface $V(uv + w^2) \subset \mathbb{C}^3$ (a singularity of type A_1, see Example 1.2.3). Thus the exceptional divisor of $B(f) \to \mathbb{P}^2$ is the sum of two irreducible components E_1 and E_2, both isomorphic to $\mathbb{P}^1$, with the singular point p_2' lying on E_1 (see also Figure 7.2). The exceptional divisor of $B(f) = B(f^{-1}) \to \mathbb{P}^2$ is the union of two components, the proper transform L_1 of the line $V(t_1)$ and the proper transform L_2 of the line $V(t_0)$. When we blow-up p_2', we get a smooth resolution X of f. The exceptional divisor of $\pi : X \to \mathbb{P}^2$ is the union of the proper transforms of E_1 and E_2 on X and the exceptional divisor E_1' of the blow-up $X \to B(f)$. The exceptional divisor of $\sigma : X \to \mathbb{P}^2$ is the union of the proper transforms of L_1 and L_2 on X and the exceptional divisor E_1'. Note that the proper transforms of E_1, E_2 and L_1, L_2 are (-1)-curves and the curve E_1' is a (-2)-curve.[1]

Finally, we can consider the *second degenerate standard quadratic transfor-*

[1] A $(-n)$-curve is a smooth rational curve with self-intersection $-n$.

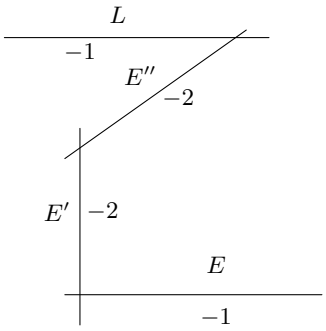


Figure 7.3 A resolution of T''_{st}

mation given by the formula

$$T''_{\text{st}} : [t_0, t_1, t_2] \mapsto [t_2^2 - t_0 t_1, t_1^2, t_1 t_2]. \tag{7.7}$$

Its unique base point is $p_1 = [1, 0, 0]$. In affine coordinates $x = t_1/t_0, y = t_2/t_0$, the base ideal is $(x^2, xy, y^2 - x) = (y^3, x - y^2)$. The blow-up of this ideal is singular. It has a singular point p'_1 locally isomorphic to the singular point of the affine surface $V(uv + w^3)$ in $\mathbb{C}^3$ (an A_2-singularity, see Example 1.2.3). The P-locus consists of one line $t_1 = 0$. A smooth resolution of the transformation is obtained by blowing up twice the singular point. The exceptional divisor of the morphism $\pi : X \to \mathbb{P}^2$ consists of three curves E, E', E'', where E is the proper transform of the exceptional divisor of $\mathsf{Bl}_{\mathbb{P}^2}(p_1) \to \mathbb{P}^2$ and $E' + E''$ is the exceptional divisor of the resolution of the singularity p'_1. The self-intersections of the exceptional curves are indicated on Figure 7.3.

Here L denotes the proper transform of the line $V(t_1)$. The Jacobian matrix of transformation T''_{st} is equal to

$$\begin{pmatrix} -t_1 & -t_0 & 2t_2 \\ 0 & 2t_1 & 0 \\ 0 & t_2 & t_1 \end{pmatrix}.$$

Its determinant is equal to $-2t_1^3$. So the P-locus consists of one line $V(t_1)$ taken with multiplicity 3.

7.1.5 Computation of the multidegree

The multidegree $(d_1, \ldots, d_{n-1})$ of a Cremona transformation can be computed using the intersection theory on algebraic varieties (see [231], [282], Appendix

A). For any closed subscheme Z of a scheme X of finite type over a field, the theory assigns the *Segre class* $s(Z, X)$ in the Chow group $A_*(X)$ of algebraic cycles on X.

One of the most frequently used properties of the Segre classes is the following one. Let $\pi : \tilde{X} = \mathsf{Bl}_X(Z) \to X$ be the blow-up of X and E be the exceptional divisor. Then

$$s(Z, X) = \sum_{i \geq 1} (-1)^{i-1} \pi_*([E]^i), \tag{7.8}$$

where $\pi_* : A_*(\tilde{X}) \to A_*(X)$ is the push-forward homomorphism.

The Segre classes are notoriously difficult to compute. However, in the special case when the embedding $j : Z \hookrightarrow X$ is a regular embedding with locally free normal sheaf $\mathcal{N}_{Z/X}$, they can be expressed in terms of the Chern classes of the normal sheaf $\mathcal{N}_{Z/X}$. We have

$$s(Z, X) = c(\mathcal{N}_{Z/X})^{-1},$$

where $c(\mathcal{E}) = \sum c_i(\mathcal{E})$ denote the total Chern class of a locally free sheaf $\mathcal{E}$.

In the case of a regular embedding, Chern classes of $\mathcal{N}_{Z/X}$ are computed by using the standard exact sequence of the sheaves of differentials

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to j^*\Omega_X^1 \to \Omega_Z \to 0. \tag{7.9}$$

By definition, the normal sheaf $\mathcal{N}_{Z/X}$ is $(\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$. We have

$$c(\mathcal{N}_{Z/X}) = j^*c((\Omega_X^1)^\vee)/c(\Omega_Z^\vee). \tag{7.10}$$

For example, when Z is a point on a smooth n-dimensional variety X, we have $s(Z, X) = [Z] \in A_0(X)$. Formula (7.8) gives

$$[E]^n = (-1)^{n-1}. \tag{7.11}$$

Of course, this can be computed without using Segre classes. We embed X in a projective space, take a smooth hyperplane section H passing through the point Z. Its full transform on $\mathsf{Bl}_X(Z)$ is equal to the union of E and the proper transform H_0 intersecting E along a hyperplane L inside E identified with $\mathbb{P}^{n-1}$. Replacing H with another hyperplane H' not passing through Z, we obtain

$$[H'] \cdot [E] = [H_0 + E] \cdot [E] = e + [E]^2 = 0,$$

where e is the class of a hyperplane in E. Thus $[E]^2 = -e$. This of course agrees with the general theory. The line bundle $\mathcal{O}(1)$ on the blow-up is isomorphic to $\mathcal{O}(-E)$. The normal sheaf $\mathcal{O}_E(E)$ is isomorphic $\mathcal{O}_E(-1)$.

We also have $[H_0] \cdot [E]^2 = -[H_0] \cdot e = -e^2$, hence

$$0 = [E]^2 \cdot [H'] = [E]^2 \cdot [H_0 + E] = [E]^2 \cdot [H_0] + [E]^3$$

gives $[E]^3 = e^3$. Continuing in this way, we find

$$[E]^k = (-1)^{n-1} e^k. \tag{7.12}$$

Another easy case is when we take $X = \mathbb{P}^n$ and Z be a linear subspace of codimension $k > 1$. We denote by h the class of a hyperplane section in $\mathbb{P}^n$. Since Z is a complete intersection of k hyperplanes,

$$\mathcal{N}_{Z/\mathbb{P}^n} \cong \mathcal{O}_Z(1)^{\oplus k},$$

hence

$$s(Z, X) = \frac{1}{(1+h)^k} = \frac{1}{(k-1)!}\Big(\frac{1}{1-x}\Big)^{(k-1)}_{|_{x=-h}} = \sum_{m=k-1}^{n-1} \tbinom{m}{k-1}(-h)^{m-k+1}.$$

(note that $h^i = 0, i > \dim Y = n - k$). In particular, we obtain

$$[E]^n = (-1)^{k-1}\tbinom{n-1}{k-1}. \tag{7.13}$$

For example, the self-intersection of the exceptional divisor of the blow-up of a line in $\mathbb{P}^n$ is equal to $(-1)^n(n-1)$.

Let us apply the intersection theory to compute the multidegree of a Cremona transformation.

Let (X, π, σ) be a resolution of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Consider the map $\nu = (\pi, \sigma) : X \to \mathbb{P}^n \times \mathbb{P}^n$. We have $\nu_*[X] = [\Gamma_f]$, and, by the projection formula,

$$\nu^*(h_1^k h_2^{n-k}) \cap [X] = [\Gamma_f] \cdot (h_1^k h_2^{n-k}) = d_k.$$

Let $s(Z, \mathbb{P}^n) \in A_*(Z)$ be the Segre class of a closed subscheme of $\mathbb{P}^n$. We write its image in $A_*(\mathbb{P}^n)$ under the canonical map $i_* : A_*(Z) \to A_*(\mathbb{P}^n)$ in the form $\sum s(Z, \mathbb{P}^n)_m h^{n-m}$, where h is the class of a hyperplane.

Proposition 7.1.10 *Let $(d_0, d_1, \dots, d_n)$ be the multidegree of a Cremona transformation. Let (X, π, σ) be its resolution and Z be the closed subscheme of $\mathbb{P}^n$ such that $\pi : X \to \mathbb{P}^n$ is the blow-up of a closed subscheme Z of $\mathbb{P}^n$. Then*

$$d_k = d^k - \sum_{i=1}^{k} d^{k-i}\tbinom{k}{i} s(Z, \mathbb{P}^n)_{n-i}.$$

Proof We know that $\sigma^*\mathcal{O}_{\mathbb{P}^n}(1) = \pi^*\mathcal{O}_{\mathbb{P}^n}(d)(-E) = \mathcal{O}_{X'}(dH - E)$, where $\mathcal{O}_X(H) = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and E is the exceptional divisor of π. We have $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ for each copy of $\mathbb{P}^n$. Thus

$$d_k = \pi_*[dH - E]^k \cdot h^{n-k} = \sum_{i=0}^{k}((-1)^i d^{k-i}\tbinom{k}{i}\pi_*([H]^{k-i} \cdot [E]^i)) \cdot h^{n-k}$$

$$= \sum_{i=0}^{k}(-1)^i d^{k-i}\tbinom{k}{i}h^{k-i}\cdot\pi_*([E]^i)\cdot h^{n-k} = \sum_{i=0}^{k}(-1)^i d^{k-i}\tbinom{k}{i}\cdot\pi_*([E]^i)\cdot h^{n-i}$$

$$= d^k + \sum_{i=1}^{k}(-1)^i d^{k-i}\tbinom{k}{i} \cdot \pi_*([E]^i) \cdot h^{n-i} = d^k - \sum_{i=1}^{k} d^{k-i}\tbinom{k}{i}s(Z, \mathbb{P}^n)_{n-i}.$$

□

Example 7.1.11 Assume Z is a smooth connected subscheme of $\mathbb{P}^n$. A Cremona transformation with a smooth connected base scheme is called *special Cremona transformation*. There are no such transformations in the plane and they are rather rare in higher-dimensional spaces and maybe classifiable. We start from 1-dimensional base schemes. In this case, (7.10) gives

$$c_1(\mathcal{N}_{Z/\mathbb{P}^n}) = j^*c_1((\Omega^1_{\mathbb{P}^n})^\vee) - c_1((\Omega^1_Z)^\vee) = (n+1)\deg Z + 2g - 2,$$

where g is the genus of Z. Thus $s(Z, \mathbb{P}^n) = \deg Z h - ((n+1)\deg Z + 2g - 2)[\text{point}]$. We have

$$d_n = d^n - dn\deg Z + (n+1)\deg Z + 2g - 2 = 1, \tag{7.14}$$

$$d_{n-1} = d^{n-1} - \deg C, \tag{7.15}$$

$$d_k = d^k, \quad k = 0, \ldots, n-2. \tag{7.16}$$

To get a Cremona transformation, we must have $d_n = 1$. Let $n = 3$. One uses the postulation formula and Riemann-Roch on Z to obtain

$$h^0(\mathcal{O}_Z(d)) = \binom{d+3}{3} - 4 = d\deg C + 1 - g.$$

Together with the previous equality $d^3 - 3d\deg Z + 4\deg Z + 2g - 2 = 1$, this easily gives $d \leq 3$, and then $\deg Z = 6, g = 3$. This is an example of a bilinear cubo-cubic transformation, which we will discuss later in this Chapter.

If $n = 4$, $g = 1, \deg C = 5, d = 2$, the formula gives $d_3 = 1$. This transformation is given by the homaloidal linear system of quadrics with base scheme equal to an elliptic curve of degree 5. The multidegree must be equal to $(2, 4, 3)$. This is an example of a quadro-cubic transformation in $\mathbb{P}^4$ discussed

in Semple and Roth's book [526]. It turns out that these two cases are the only possible cases with $\dim Z = 1$ (see [138], Theorem 2.2).

In the same paper [138], Theorem 3.3, one can find the classification in the case $\dim Z = 2$.

Theorem 7.1.12 *A Cremona transformation with 2-dimensional smooth connected base scheme Z is one of the following:*

(i) $n = 4, d = 3$, *Z is an elliptic scroll of degree 5, the base scheme of the inverse of the quadro-cubic transformation from above;*

(ii) $n = 4, d = 4$, *Z is a determinantal variety of degree 10 given by 4×4-minors of a 4×5-matrix of linear forms (a bilinear transformation, see later);*

(iii) $n = 5, d = 2$, *Z is a Veronese surface;*

(iv) $n = 6, d = 2$, *Z is an elliptic scroll of degree 7;*

(v) $n = 6, d = 2$, *Z is an octavic surface, the image of the projective plane under a rational map given by the linear system of quartics through eight points.*

In cases (ii) and (iii) the inverse transformation is similar, with isomorphic base scheme.

Example 7.1.13 There is no classification for higher-dimensional Z. However, we have the following nice results of L. Ein and N. Shepherd-Barron [207].

Recall that a *Severi-Zak variety* is a closed a subvariety Z of $\mathbb{P}^n$ of dimension $\frac{1}{3}(2n-4)$ such that the secant variety is a proper subvariety of $\mathbb{P}^{n+1}$. All such varieties are classified by F. Zak (see [354]). The list is as follows:

(i) Z is a Veronese surface in $\mathbb{P}^5$;

(ii) Z is the Grassmann variety $G_1(\mathbb{P}^5)$ embedded in the Plücker space $\mathbb{P}^{14}$;

(iii) Z is the Severi variety $s(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$;

(iv) Z is the E_6-variety, a 16-dimensional homogeneous variety in $\mathbb{P}^{26}$.

In all these cases the secant variety of the Severi variety Z is a cubic hypersurface X with the singular locus equal to Z.

A theorem of Ein and Shepherd-Barron asserts that a simple Cremona transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with base scheme of codimension 2 of degree 2 equal to the degree of T^{-1} is given by the linear system of the first polars of the cubic hypersurface X.

7.2 First examples

7.2.1 Quadro-quadratic transformations

Let us show that any vector $(2,\dots,2)$ is realized as the multidegree of a Cremona transformation. For $n = 2$, we can take the homaloidal linear system of conics through three non-collinear points. We can view a pair of the fundamental points as a 0-dimensional quadric in the line spanned by these points. This admits an immediate generalization to higher-dimensional spaces.

Consider the linear system of quadrics in $\mathbb{P}^n$ containing a fixed smooth quadric Q_0 of dimension $n-2$. It maps $\mathbb{P}^n$ to a quadric Q in $\mathbb{P}^{n+1}$. We may choose coordinates such that

$$Q_0 = V(z_0) \cap V(\sum_{i=1} z_i^2),$$

so that the hyperplane $H = V(z_0)$ is the linear span of Q_0. Then the linear system is spanned by the quadrics $V(\sum z_i^2), V(z_0 z_i), i = 0,\dots,n$. It maps the blow-up $\mathsf{Bl}_{\mathbb{P}^n}(Q_0)$ to the quadric Q in $\mathbb{P}^{n+1}$ with equation $t_0 t_{n+1} - \sum_{i=1}^n t_i^2 = 0$. The rational map $g : \mathbb{P}^n \dashrightarrow Q$ defined by a choice of a basis of the linear system, can be given by the formula

$$[t_0,\dots,t_n] \mapsto [\sum_{i=1}^n t_i^2, t_0t_1,\dots,t_0t_n,t_0^2].$$

Observe that the image of H is equal to the point $a = [1,0,\dots,0]$. The inverse of g is the projection map

$$p_a : Q \dashrightarrow \mathbb{P}^n, \quad [z_0,\dots,z_{n+1}] \to [z_0,\dots,z_n]$$

from the point a. It blows down the hyperplane $V(z_{n+1}) \subset \mathbb{P}^{n+1}$ to the quadric Q_0. Now consider the projection map $p_b : Q \dashrightarrow \mathbb{P}^n$ from a point $b \neq a$ not lying in the hyperplane $V(t_{n+1})$. Note that this hyperplane is equal to the embedded tangent hyperplane $\mathbb{T}_a(Q)$ of Q at the point a. The composition $f = p_b \circ p_a^{-1}$ of the two rational maps is a quadratic transformation defined by the homaloidal linear system of quadrics with the base locus equal to the union of Q_0 and the point $p_a(b)$. If we choose $b = [0,\dots,0,1]$ so that $p_a(b) = [1,0,\dots,0]$, then the Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ can be given by the formula

$$[t_0,\dots,t_n] \mapsto [\sum_{i=1}^n t_i^2, t_0t_1,\dots,t_0t_n]. \tag{7.17}$$

Note that $f^{-1} = p_a \circ p_b^{-1}$ must be given by similar quadratic polynomials.

So the degree of f^{-1} is equal to 2. This is the reason for the name quadro-quadratic transformation.

For example, if $n = 2$, if we rewrite the equation of Q_0 in the form $z_0 = z_1z_2 = 0$ and obtain the formula (7.5) for the standard quadratic transformation.

Let us compute the multidegree. For any general linear subspace L of codimension $k > 0$, its preimage under the projection $p_b : Q \dashrightarrow \mathbb{P}^n$ is the intersection of Q with the subspace $L' = \langle L, b\rangle$ spanned by L' and b. It is a quadric in this subspace. Since the point a does not belong to L', the projection of this quadric from the point a is a quadric in the projection of L' from the same point. Thus $d_k = 2$. This shows that the multidegree of the transformation is equal to $(2, \dots, 2)$.

Let us consider some degenerations of the transformation given by (7.17). Let us take two nonsingular points a, b on an arbitrary irreducible quadric $Q \subset \mathbb{P}^{n+1}$. We assume that b does not lie in the intersection of Q with the embedded tangent space $\mathbb{T}_a(Q)$ of Q at a. Let $f = p_a \circ p_b^{-1}$. The projection p_a blows down the intersection $\mathbb{T}_a(Q) \cap Q$ to a quadric Q_0 in the hyperplane $H = p_a(\mathbb{T}_a(Q))$. If $r =$ rank Q (i.e. $n + 1 - r$ is the dimension of the singular locus of Q), then rank $Q \cap \mathbb{T}_a(Q) = r - 1$. Its singular locus is spanned by the singular locus of Q and the point a. The projection Q_0 of $Q \cap \mathbb{T}_a(Q)$ is a quadric with singular locus of dimension $n+1-r$, thus, it is a quadric of rank equal to $n - 1 - (n + 1 - r) = r - 2$ in H. The inverse transformation $p_a^{-1} : \mathbb{P}^n \dashrightarrow Q$ is given by the linear system of quadrics in $\mathbb{P}^n$ which pass through Q_0. So, taking $a = [1, 0, \dots, 0]$ and $b = [0, \dots, 0, 1]$ as in the nondegenerate case, we obtain that f is given by

$$f : [t_0, \dots, t_n] \mapsto [\sum_{i=1}^{r-2} t_i^2, t_0t_1, \dots, t_0t_n]. \tag{7.18}$$

Note the following special cases. If $n = 2$, and Q is an irreducible quadric cone, then $r = 3$ and we get the formula for the first degenerate standard quadratic transformation (7.6). To get the second degenerate standard quadratic transformation, we should abandon the condition that $b \notin \mathbb{T}_a(Q)$. We leave the details to the reader.

Remark 7.2.1 A Cremona transformation T such that the degree of T and of T^{-1} is equal to 2 is called a *quadro-quadratic transformation*. It is not true that the multidegree of a quadro-quadratic transformation is always of the form $(2, \dots, 2)$. For example, if $n = 4$, applying Cremona's inequalities, we obtain $d_2 \leq 4$. A transformation of multidegree $(2, 3, 2)$ can be obtained by taking the homaloidal linear system of quadrics with the base scheme equal

to a plane and two lines intersecting the plane at one point. A transformation of multi degree $(2, 4, 2)$ is given by the homaloidal linear system of quadrics with rather complicated base scheme. The reduced base scheme consists of the union of a conic and a line intersecting at one point. The line supports a non-reduced scheme. All quadro-quadratic transformations in $\mathbb{P}^4$ were classified by A. Bruno and A. Verra.

7.2.2 Bilinear Cremona transformations

Here we encounter again the aCM sheaves that we use in Chapter 4.

Definition 7.2.2 *A closed subscheme Z of $\mathbb{P}^n$ of pure dimension r is called* arithmetically Cohen-Macaulay *(aCM for short) if its ideal sheaf $\mathcal{J}_Z$ is an aCM sheaf.*

Assume that codim $Z = 2$. Then, as in Chapter 4, we obtain a locally free resolution

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \bigoplus_{j=1}^{m+1} \mathcal{O}_{\mathbb{P}^n}(-b_j) \to \mathcal{J}_Z \to 0 \tag{7.19}$$

for some sequences of integers (a_i) and (b_j).

The numbers (a_i) and (b_j) are determined from the Hilbert polynomials of Z.

We will consider a special case of resolution of the form (4.14) which we used in the theory of linear determinantal representations of hypersurfaces:

$$0 \to \underline{U}^\vee(-n-1) \to \underline{V}(-n) \to \mathcal{J}_Z \to 0, \tag{7.20}$$

where U, V are linear spaces of dimensions n and $n+1$, respectively. By twisting the exact sequence, and taking cohomology, we obtain natural isomorhisms

$$U \cong H^{n-1}(\mathbb{P}^n, \mathcal{J}_Z), \quad V \cong H^0(\mathbb{P}^n, \mathcal{J}_Z(n)).$$

The resolution of $\mathcal{J}_Z$ allows one to compute the Hilbert polynomial of the subscheme Z. We get

$$\chi(\mathcal{O}_Z(k)) = \chi(\mathcal{O}_{\mathbb{P}^n}(k)) - \chi(\mathcal{J}_Z(k)) = \tbinom{n+k}{n} - \tbinom{k}{n} - n\tbinom{k-1}{n-1}. \tag{7.21}$$

It also defines an isomorphism between Z and the determinantal variety given by the linear map

$$\phi : E \to U \otimes V, \tag{7.22}$$

where $\mathbb{P}^n = |E|$. In coordinates, it is given by $n \times (n+1)$ matrix A with

linear functions on E as its entries. The maximal minors of A generate the homogeneous ideal of Z. Let

$$\mathfrak{r} : E \to V^\vee$$

be the right kernel maps. It defines the rational maps of projective spaces

$$|\mathfrak{r}| : |E| \dashrightarrow \mathbb{P}(V).$$

Remark 7.2.3 The Hilbert scheme of an aCM subscheme Z of $\mathbb{P}^n$ admitting a resolution (7.20) is isomorphic to an open subset of the projective space of $(n+1)\times n$ matrices $A(t)$ of linear forms such that the rank of $A(t)$ is equal to n for an open non-empty subset of $\mathbb{P}^n$. Modulo the action by $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$ by left and right multiplication. It is a connected smooth variety of dimension $n(n^2-1)$ (see [436] or [211]).

Theorem 7.2.4 *The map $T_\phi = |\mathfrak{r}| : |E| \dashrightarrow \mathbb{P}(V)$ is a birational map with base scheme Z. Its multidegree is equal to $(d_k) = (\binom{n}{k})$.*

Proof In coordinates, the map $|\mathfrak{r}|$ is defined by $n \times n$ minors of the matrix A. The subscheme Z is given scheme-theoretically by these minors. In particular, we already see that the degree of the map is equal to n. Let us view the linear map ϕ as an element of the tensor product $E^\vee \otimes U \otimes V$. Consider it as a linear map

$$\psi : E \otimes V^\vee \to U. \tag{7.23}$$

It may be considered as a collection of n bilinear forms on $E \otimes V^\vee$. It is immediate that $v^* = \mathfrak{r}(e)$ for some $v^* \in V^\vee$ and $e \in E$ if and only if $\psi(e \otimes v^*) = 0$. This relation is symmetric, so $v^* = \mathfrak{r}(e)$ if and only if $e = \mathfrak{r}'(v^*)$, where $\mathfrak{r}' : V^\vee \to E$ is the right kernel map for the linear map $\phi' : V^\vee \to U \otimes E^\vee$ defined by applying to the tensor ϕ the isomorphism $E^\vee \otimes U \otimes V \to V^\vee \otimes U \otimes E$. Thus, the map $T_{\phi'} = \mathfrak{r}'$ defines the inverse of T_ϕ.

In coordinates, if choose a basis $e_0, \dots, e_{n+1}$ in E, a basis $u_1, \dots, u_n$ in U and a basis $v_0, \dots, v_n$ in V, then the linear map ϕ can be written as a tensor

$$\phi = a_{ij}^k := \sum_{0\le k,j\le n; 1\le i\le n}^{n} a_{ij}^k t_k \otimes u_i \otimes v_j.$$

The matrix A is equal to $t_0A_0 + \cdots + t_nA_n$, where $A_k = (a_{ij}^k)$. The bilinear map ψ is given by n square matrices $X_i = (a_{ij}^k)$ of size $n+1$, where k is the index for the columns, and j is the index for the rows. The graph of the

Cremona map $|\mathfrak{r}|$ is given by n bilinear equations in $|E| \times \mathbb{P}(V)$

$$\sum_{j,k=0}^{n} t_k v_j a_{ij}^k = 0, \quad i = 1, \dots, n. \tag{7.24}$$

These equations define the graph of the transformation T_ϕ. Also note that the matrix B defining the linear map $\phi' : V^\vee \to U \otimes E^\vee$ is equal to $v_0 B_0 + \cdots + v_n B_n$, where $B_j = (a_{ij}^k)$. Here k is now the row index, and i is the column index.

It is easy to compute the cohomology class of the graph (7.24) of T_ϕ. It is equal to

$$(h_1 + h_2)^n = \sum_{k=0}^{n} \binom{n}{k} h_1^i h_2^{n-i}.$$

□

We can also see another determinantal variety, this time defined by the transpose of (7.23)

$$^t\psi : U^\vee \to E^\vee \otimes V. \tag{7.25}$$

Let $\mathsf{D}_k \subset \mathbb{P}(U)$ be the preimage of the determinantal variety of bilinear forms on $E \otimes V^\vee$ (or linear maps $E^\vee \to V$) of rank $\leq k$. We have regular kernel maps

$$\mathfrak{l}_\psi : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to |E|, \quad \mathfrak{r}_\psi : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to \mathbb{P}(V).$$

By definition, the image of the first map is equal to the base scheme Z of the rational map $|\mathfrak{r}|$ considered in the previous Theorem. The image of the second map is of course the base scheme of the inverse map. In particular, we see that the base schemes of T_ϕ and T_ϕ^{-1} are birationally isomorphic to the variety D_n.

Note the special case when $E = V^\vee$ and the image of $^t\psi$ is contained in the space of symmetric bilinear maps $E \times V^\vee \to \mathbb{C}$. In this case

$$T_\phi = T_\phi^{-1}.$$

Example 7.2.5 Consider the *standard Cremona transformation* of degree n in $\mathbb{P}^n$ given by

$$T_{\text{st}} : [t_0, \dots, t_n] \mapsto [\frac{t_0 \cdots t_n}{t_0}, \dots, \frac{t_0 \cdots t_n}{t_n}]. \tag{7.26}$$

In affine coordinates, $z_i = t_i/t_0$, it is given by the formula

$$(z_1, \dots, z_n) \mapsto (z_1^{-1}, \dots, z_n^{-1}).$$

The transformation T_{st} is an analog of the standard quadratic transformation of the plane in higher dimension.

The base ideal of T_{st} is generated by $t_1 \cdots t_n, \ldots, t_0 \cdots t_{n-1}$. It is equal to the ideal generated by the maximal minors of the $n \times n$ matrix

$$A(t) = \begin{pmatrix} t_0 & 0 & \cdots & 0 \\ 0 & t_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_{n-1} \\ -t_n & -t_n & \ldots & -t_n \end{pmatrix}.$$

The base scheme of T_{st} equal to the union of the coordinate subspaces of codimension 2.

It follows from the proof of Theorem 7.2.4 that the graph of T_{st} is isomorphic to the closed subvariety X of $\mathbb{P}^n \times \mathbb{P}^n$ given by n bilinear equations

$$t_i y_i - t_n y_n = 0, \quad i = 0, \ldots, n-1.$$

It is a smooth subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ isomorphic to the blow-up of the union of coordinate subspaces of codimension 2. The action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$ (by scaling the coordinates) extends to a biregular action on X. The corresponding toric variety is a special case of a toric variety defined by a fan formed by fundamental chambers of a root system of a semi-simple Lie algebra. In our case the root system is of type A_n, and the variety X is denoted by $X(A_n)$. In the case $n = 2$, the toric surface $X(A_2)$ is a del Pezzo surface of degree 6 isomorphic to the blow-up of 3 points in the plane, no three of which are collinear.

Example 7.2.6 Let $\alpha : U^\vee \to E^\vee \otimes V$ be a linear determinantal representation of a nonsingular plane quartic $C \subset \mathbb{P}(U) \cong \mathbb{P}^2$ given by the linear system $|K_C + a|$. The image Z of C in $|E|$ under the right kernel map $\mathfrak{r}$ is a curve Z of degree 6 and genus 3. Let $\phi : E \to U \otimes V$ be the linear map obtained from the tensor $\phi \in U \otimes E^\vee \otimes V$. Then the bilinear Cremona transformation $|E| \to \mathbb{P}(V)$ defined by this map is given by cubic polynomials generating the ideal of Z. Note that Z is an aCM subscheme of $|E| \cong \mathbb{P}^3$. Its Hilbert polynomial is $6t - 2$ in agreement with (7.21). Conversely, any irreducible and reduced curve of degree 6 and arithmetic genus 3 not lying on a quadric is arithmetically Cohen-Macaulay and admits a resolution of type (7.20) (see [211], p. 430). Assume Z is arithmetically Cohen-Macaulay. The bilinear Cremona transformation defined by such a curve is classically known as a *cubo-cubic transformations* (see [526]).

In fact, an example of a standard Cremona transformation in $\mathbb{P}^3$ shows that

one can often drop the assumption that Z is an integral curve. In this example, Z is the union of 6 coordinate lines, and is a curve of degree 6 and of arithmetic genus 3, and it does not lie on a quadric. Another example of this sort is when Z is the union of 4 skew lines and two lines intersecting them. There are examples when Z is not reduced, e.g. with the reduced scheme equal to a rational normal curve. I do not know whether any closed subscheme Z of degree 6 (in the sense that $[Z] = 6[\text{line}]$) with $h^0(\mathcal{O}_Z) = 1, h^1(\mathcal{O}_Z) = 3$, and not lying on a quadric surface, admits a resolution of type (7.20).

Assume Z is a smooth curve and let us describe the P-locus of the corresponding Cremona transformation. Obviously, any line intersecting Z at three distinct points (a *trisecant line*) must be blown down to a point (otherwise a general cubic in the linear system intersects the line at more than 3 points). Consider the surface $\mathrm{Tri}(Z)$ of Z, the closure in $\mathbb{P}^3$ of the union of lines intersecting Z at three points. Note that no line intersects Z at > 3 points because the ideal of Z is generated by cubic surfaces. Consider the linear system of cubics through Z. If all of them are singular, by Bertini's Theorem, there will be a common singular point at the base locus, i.e. at Z. But this easily implies that Z is singular, contradicting our assumption. Choose a nonsingular cubic surface S containing Z. By the adjunction formula, we have $Z^2 = -K_S \cdot Z + \deg K_Z = 6 + 4 = 10$. Take another cubic S' containing Z. The intersection $S \cap S'$ is a curve of degree 9, the residual curve A is of degree 3 and $Z + A \sim -3K_S$ easily gives $Z \cdot A = 18 - 10 = 8$. Note that the curves A are the proper transforms of lines under the Cremona transformation. So they are rational curves of degree 3. We know that the base scheme of the inverse transformation T^{-1} is a curve of degree 6 isomorphic to Z. Replacing T with T^{-1}, we obtain that the image of a general line ℓ under T is a rational curve of degree 3 intersecting Z' at eight points. These points are the images of eight trisecants intersecting ℓ. This implies that the degree of the trisecant surface $\mathrm{Tri}(Z)$ is equal to 8. Since the degree of the determinant of the Jacobian matrix of a transformation of degree 3 is equal to 8, we see that there is nothing else in the P-locus.

The linear system of planes containing a trisecant line ℓ cuts out on Z a linear series of degree 6 with moving part of degree 3. It is easy to see, by using Riemann-Roch, that any g^1_3 on a curve of genus 3 must be of the form $|K_Z - x|$ for a unique point $x \in Z$. Conversely, for any point $x \in Z$, the linear system $|\mathcal{O}_Z(1) - K_Z + x|$ is of dimension 0 and of degree 3 (here we use that $|\mathcal{O}_Z(1)| = |K_Z + \mathfrak{a}|$, where $\mathfrak{a}$ is not effective divisor class of degree 2). Thus it defines a trisecant line (maybe tangent at some point). This shows that the curve R parameterizing trisecant lines is isomorphic to Z. This agrees with the fact that R must be isomorphic to the base curve of the inverse

transformation. The Cremona transformation can be resolved by blowing up the curve Z and then blowing down the proper transform of the surface $\mathrm{Tri}(Z)$. The exceptional divisor is isomorphic to the minimal ruled surface with the base curve equal to Z. It is the universal family of lines parameterized by Z. Its image in the target $\mathbb{P}^3$ is surface $\mathrm{Tri}(Z')$, where Z' is the base locus of the inverse transformation (the same curve only re-embedded by the linear system $|K_Z + a'|$, where $a' \in |K_Z - a|$).

Remark 7.2.7 Let Z be a closed aCM subscheme of codimension 2 in $\mathbb{P}^5$ defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^3 \to \mathcal{O}_{\mathbb{P}^5}(-3)^4 \to \mathcal{I}_Z \to 0.$$

It is a determinantal variety in $\mathbb{P}^5$ with the right kernel map $\mathfrak{r} : Z \to \mathbb{P}^2$ isomorphic to a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 6$ (see [417], [419]). Thus Z is a scroll of lines in $\mathbb{P}^5$, called a *Bordiga scroll*. A general hyperplane section of Z is a surface S of degree 6 in $\mathbb{P}^4$ with ideal sheaf defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^3 \to \mathcal{O}_{\mathbb{P}^4}(-3)^4 \to \mathcal{I}_S \to 0.$$

It is a determinantal surface in $\mathbb{P}^4$ with the right kernel map $\mathfrak{r} : S \to \mathbb{P}^2$ isomorphic to the blow-up of 10 points in $\mathbb{P}^2$. The embedding of S in $\mathbb{P}^4$ is given by the linear system of quartic curves passing through the ten points. The surface S is of degree 6, classically known as a *Bordiga surface* [47]. Finally, a general hyperplane section of S is a sextic of genus 3 in $\mathbb{P}^3$ discussed in Example 7.2.6.

7.2.3 de Jonquières transformations

Let X be a reduced irreducible hypersurface of degree m in $\mathbb{P}^n$ that contains a linear subspace of points of multiplicity $m - 2$. Such a hypersurface is called *submonoidal* (a *monoidal hypersurface* is a hypersurface of degree m which contains a linear subspace of points multiplicity $m - 1$). For example, every smooth hypersurface of degree ≤ 3 is submonoidal.

Let X be a submonoidal hypersurface with a singular point $\mathfrak{o}$ of multiplicity $m - 2$. Let us choose the coordinates such that $\mathfrak{o} = [1, 0, \ldots, 0]$. Then X is given by an equation

$$F_m = t_0^2 a_{m-2}(t_1, \ldots, t_n) + 2t_0 a_{m-1}(t_1, \ldots, t_n) + a_m(t_1, \ldots, t_n) = 0, \tag{7.27}$$

where the subscripts indicate the degrees of the homogeneous forms. For a general point $x \in X$, let us consider the intersection of the line $\ell_x = \overline{\mathfrak{o}x}$ with

X. It contains $\mathfrak{o}$ with multiplicity $m-2$ and the residual intersection is a set of two points a, b in ℓ_x. Define $T(x)$ to be the point on ℓ_x such that the pairs $\{a, b\}$ and $\{x, T(x)\}$ are harmonically conjugate. We call it a *de Jonquières involution* (observe that $T = T^{-1}$).

Let us find an explicit formula for the de Jonquières involution which we have defined. Let $x = [\alpha_0, \dots, \alpha_n]$ and let $[u + v\alpha_0, v\alpha_1, \dots, v\alpha_n]$ be the parametric equation of the line ℓ_x. Plugging in (7.27), we find

$$(u+v\alpha_0)^2 v^{m-2} a_{m-2}(\alpha_1, \dots, \alpha_n) + 2(u+v\alpha_0) v^{m-1} a_{m-1}(\alpha_1, \dots, \alpha_n)$$
$$+ v^m a_m(\alpha_1, \dots, \alpha_n) = 0.$$

Canceling v^{m-2}, we see that the intersection points of the line ℓ_x with X are the two points corresponding to the zeros of the binary form $Au^2 + 2Buv + Cv^2$, where

$$(A, B, C) = (a_{m-2}(x), \alpha_0 a_{m-2}(x) + a_{m-1}(x), F_m(x)).$$

The points x and $T(x)$ correspond to the parameters satisfying the quadratic equation $A'u^2 + 2B'uv + C'v^2 = 0$, where $AA' + CC' - 2BB' = 0$. Since x corresponds to the parameters $[0, 1]$, we have $C' = 0$. Thus $T(x)$ corresponds to the parameters $[u, v] = [-C, B]$, and

$$T(x) = [-C + B\alpha_0, B\alpha_1, \dots, B\alpha_n].$$

Plugging in the expressions for C and B, we obtain the following formula for the transformation T

$$t_0' = -t_0 a_{m-1}(t_1, \dots, t_n) - a_m(t_1, \dots, t_n),$$
$$t_i' = t_i(a_{m-2}(t_1, \dots, t_n)t_0 + a_{m-1}(t_1, \dots, t_n)), \quad i = 1, \dots, n.$$

In affine coordinates $z_{i+1} = t_i/t_n,\ i = 0, \dots, n-1$, the formulas are

$$z_1' = -\frac{a_{m-1}(z_2, \dots, z_n)' z_1 + a_m(z_2, \dots, z_n)'}{a_{m-2}(z_2, \dots, z_n)' z_1 + a_{m-1}(z_2, \dots, z_n)'},$$
$$z_i' = z_i,\ i = 2, \dots, n.$$

A de Jonquières involution is an example of a *dilated Cremona transformation*. Starting from a Cremona transformation T in $\mathbb{P}^{n-1}$ we seek to extend it to a Cremona transformation in $\mathbb{P}^n$. More precisely, if $p_{\mathfrak{o}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is a projection map from a point $\mathfrak{o}$, we want to find a Cremona transformation $\overline{T} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ such that $p_{\mathfrak{o}} \circ \bar{T} = T \circ p_{\mathfrak{o}}$. Suppose that T is given by a sequence of degree d homogeneous polynomials $(G_1, \dots, G_n)$. Composing with a projective transformation in $\mathbb{P}^n$, we may assume that $\mathfrak{o} = [1, 0, \dots, 0]$. Thus the transformation $\overline{T}$ must be given by $(F_0, QG_1, \dots, QG_n)$, where Q

and F_0 are coprime polynomials of degrees r and $d+r$. The following result can be found in [428].

Proposition 7.2.8 *Let $(G_1,\dots,G_n)$ be homogeneous polynomials of degree d in $t_1,\dots,t_n$. Let $F_0 = t_0A_1 + A_2, Q = t_0B_1 + B_2$, where A_1, A_2, B_1, B_2 are homogeneous polynomials in $t_1,\dots,t_n$ of degrees $d+r-1, d+r, r-1, r$, respectively. Assume that F_0 and Q are coprime and $A_1B_2 \neq A_2B_1$. Then the polynomials $(F_0, QG_1,\dots,QG_n)$ define a Cremona transformation of $\mathbb{P}^n$ if and only if $(G_1,\dots,G_n)$ define a Cremona transformation of $\mathbb{P}^{n-1}$.*

Proof Let $F'(z_1,\dots,z_n)$ denote the dehomogenization of a homogeneous polynomial $F(t_0,\dots,t_n)$ in the variable t_1. It is obvious that $(F_0,\dots,F_n)$ defines a Cremona transformation if and only if the field

$$\mathbb{C}(F_1/F_0,\dots,F_n/F_0) := \mathbb{C}(F_1'/F_0',\dots,F_n'/F_0') = \mathbb{C}(z_1,\dots,z_n).$$

Consider the ratio $F_0/QG_1 = \frac{t_0A_1+A_2}{t_0GB_1+GB_2}$. Dehomogenizing with respect to t_1, we can write the ratio in the form $\frac{az_1+b}{cz_1+d}$, where $a,b,c,d \in \mathbb{C}(z_2,\dots,z_n)$. By our assumption, $ad-bc \neq 0$. Then

$$\mathbb{C}(F_1/F_0,\dots,F_n/F_0) = \mathbb{C}(F_0/QG_1, G_2/G_1,\dots,G_n/G_1)$$

$$= \mathbb{C}(G_2/G_1,\dots,G_n/G_1)(F_0/QG_1) = \mathbb{C}(G_2/G_1,\dots,G_n/G_1)(\frac{az_1+b}{cz_1+d}).$$

This field coincides with $\mathbb{C}(z_1,\dots,z_n)$ if and only if $\mathbb{C}(G_2/G_1,\dots,G_n/G_1)$ coincides with $\mathbb{C}(z_2,\dots,z_n)$.

□

Taking $G_i = t_i, i = 1,\dots,n$, and

$$F_0 = -t_0a_{m-1}(t_1,\dots,t_n) - a_m(t_1,\dots,t_n),$$
$$Q = a_{m-2}(t_1,\dots,t_n)t_0 + a_{m-1}(t_1,\dots,t_n),$$

we see that a de Jonquières involution is dilated from the identity transformation of $\mathbb{P}^{n-1}$. If we replace F_0 with $t_0b_{m-1}(t_1,\dots,t_n) + b_m(t_1,\dots,b_m)$, where b_{m-1}, b_m are any polynomials of indicated degrees such that F_0 and Q still satisfy the assumptions of Proposition 7.2.8, then we get a Cremona transformation, not necessarily involutive. In fact, one defines a general de Jonquières transformation as follows.

Definition 7.2.9 *A Cremona transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called a* de Jonquières transformation *if there exists a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ birationally isomorphic to the projection map $\mathrm{pr}_2 : \mathbb{P}^{n-k} \times \mathbb{P}^k \to \mathbb{P}^k$ and a Cremona transformation $T' : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ such that $f \circ T = T' \circ f$.*

In algebraic language, this definition is equivalent to T defining an automorphism Φ of the field of rational functions in $z_1, \dots, z_n$ of the form

$$(z_1, \dots, z_n) \mapsto (R_1, \dots, R_k, R_{k+1}, \dots, R_n),$$

where $R_1, \dots, R_k$ are rational functions in variables $z_1, \dots, z_k$ with coefficients in $\mathbb{C}$ and $R_{k+1}, \dots, R_n$ are rational functions in variables $z_{k+1}, \dots, z_n$ with coefficients in the field $\mathbb{C}(z_1, \dots, z_k)$.

A de Jonquières transformation obtained by dilating the identity map of $\mathbb{P}^{n-1}$ is the special case when $k = n-1$ and T' is the identity. It is easy to compute its multidegree. Take a general linear k-codimensional subspace L of $\mathbb{P}^n$. We can write L as the intersection of $k-1$ hyperplanes $H_i = V(l_i(t_1, \dots, t_n))$ containing the point $\mathfrak{o}$ and one hyperplane $H_k = V(l_k(t_0, \dots, t_n))$ which does not contain $\mathfrak{o}$. The preimage of the first $k-1$ hyperplanes H_i are reducible hypersurfaces $D_i = V(t_i Q)$ of degree m. The preimage of H_k is a hypersurface D_k of degree m. The intersection of the hypersurface $V(Q)$ with D_k is contained in the base scheme of T. Thus the degree of the intersection $D_1 \cdots D_k$ outside the base locus is equal to m. This shows that the multidegree of T is equal to $(m, \dots, m)$. Note that the case $m = 2$ corresponds to quadratic transformations we studied in Subsection 7.2.1. In the notation from this Subsection, the point $\mathfrak{o}$ is the isolated base point and the submonoidal hypersurface in this case is a quadric hypersurface Q such that the quadric component Q_0 of the base locus is equal to the intersection $Q \cap P_{\mathfrak{o}}(Q)$.

7.3 Planar Cremona transformations

7.3.1 Exceptional configurations

From now on we will study birational maps between algebraic surfaces. We know (see [282], Chapter V, §5) that any birational morphism $\pi : Y \to X$ of nonsingular projective surfaces can be factored into a composition of blow-ups with centers at closed points. Let

$$\pi : Y = Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = X \tag{7.28}$$

be such a factorization. Here $\pi_i : Y_i \to Y_{i-1}$ is the blow-up of a point $x_i \in Y_{i-1}$. Set

$$\pi_{ki} := \pi_{i+1} \circ \dots \circ \pi_k : Y_k \to Y_i, \quad k > i.$$

Let

$$E_i = \pi_i^{-1}(x_i), \quad \mathcal{E}_i = \pi_{Ni}^*(E_i). \tag{7.29}$$

The divisors $\mathcal{E}_i$ are called the *exceptional configurations* of the birational morphism $\pi : Y \to X$. Note that $\mathcal{E}_i$ should be considered as an effective divisor, not necessarily reduced.

For any effective divisor $D \neq 0$ on X let $\mathrm{mult}_{x_i} D$ be defined inductively in the following way. We set $\mathrm{mult}_{x_1} D$ to be the usual multiplicity of D at x_1. It is defined as the largest integer m such that the local equation of D at x_1 belongs to the m-th power of the maximal ideal $\mathfrak{m}_{X,x_1}$. So, the multiplicity $\mathrm{mult}_{x_i} D$ is defined. Next, we take the proper inverse transform $\pi_i^{-1}(D)$ of D in X_i and define $\mathrm{mult}_{x_{i+1}}(D) = \mathrm{mult}_{x_{i+1}} \pi_i^{-1}(D)$. It follows from the definition that

$$\pi^{-1}(D) = \pi^*(D) - \sum_{i=1}^{N} m_i \mathcal{E}_i,$$

where $m_i = \mathrm{mult}_{x_i} D$. Now suppose $\pi : Y \to X$ is a resolution of a rational dominant map $f : X \to X'$ of algebraic surfaces given by the linear system $|V| \subset |\mathcal{L}|$, the inverse image of the complete linear system $|\mathcal{L}'|$ defining a closed embedding $X' \hookrightarrow \mathbb{P}^r$. Let

$$m_i = \min_{D \in |V|} \mathrm{mult}_{x_i} D, \ i = 1, \ldots, N.$$

If $D_0, \ldots, D_k$ are divisors corresponding to a basis of V, then

$$m_i = \min\{\mathrm{mult}_{x_i} D_0, \ldots, \mathrm{mult}_{x_i} D_k\}, \ i = 1, \ldots, N.$$

It is clear that

$$\pi^{-1}(|V|) = \pi^*(|V|) - \sum_{i=1}^{N} m_i \mathcal{E}_i. \tag{7.30}$$

Let

$$E = \sum_{i=1}^{N} m_i \mathcal{E}_i.$$

Then $\pi^{-1}(|V|) \subset |\pi^*(\mathcal{L})(-E)|$. Let $\mathfrak{b} = \mathfrak{b}(|V|)$. The ideal sheaf $\pi^{-1}(\mathfrak{b}) = \mathfrak{b} \cdot \mathcal{O}_Y$ is the base locus of $\pi^{-1}(|V|)$ and hence coincides with $\mathcal{O}_Y(-E)$. The complete linear system $|\pi^*(\mathcal{L}) \otimes \mathcal{O}_Y(-E)|$ has no base points and defines a morphism $\sigma : Y \to X'$. The preimage of a general $m-2$-dimensional linear space in Y consists of $m \deg X'$ points, where m is the degree of the rational map $f : X \to X'$ and $\deg X'$ is the degree of X' in the embedding $X' \hookrightarrow \mathbb{P}^r$. It is also equal to the self-intersection $[D - E]^2$, where $\mathcal{L} \cong \mathcal{O}_X(D)$. Thus, we obtain that f is a birational map onto X' if and only if

$$D^2 - E^2 = \deg X'. \tag{7.31}$$

From now on we use the intersection theory on smooth surfaces and use the notation $D \cdot D'$ for the intersection of the divisor classes $[D] \cdot [D']$.

Lemma 7.3.1 *Let $\pi : Y \to X$ be a birational morphism of nonsingular surfaces and let $\mathcal{E}_i, i = 1, \ldots, N$, be its exceptional configurations. Then*

$$\mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij},$$

$$\mathcal{E}_i \cdot K_Y = -1.$$

Proof This follows from the standard properties of the intersection theory on surfaces. For any morphism of nonsingular projective surfaces $\phi : X' \to X$ and two divisors D, D' on X, we have

$$\phi^*(D) \cdot \phi^*(D') = \deg(\phi) D \cdot D'. \tag{7.32}$$

Also, if C is a curve such that $\phi(C)$ is a point, we have

$$C \cdot \phi^*(D) = 0. \tag{7.33}$$

Applying (7.32), we have

$$-1 = E_i^2 = \pi_{Ni}^*(E_i)^2 = \mathcal{E}_i^2.$$

Assume $i < j$. Applying (7.33) by taking $C = E_j$ and $D = E_i$, we obtain

$$0 = E_j \cdot \pi_{ji}^*(E_i) = \pi_{Nj}^*(E_j) \cdot \pi_{Ni}^*(E_i) = \mathcal{E}_j \cdot \mathcal{E}_i.$$

This proves the first assertion.

To prove the second assertion, we use that

$$K_{Y_{i+1}} = \pi_i^*(K_{Y_i}) + E_i.$$

By induction, this implies that

$$K_Y = \pi^*(K_{Y_0}) + \sum_{i=1}^{N} \mathcal{E}_i. \tag{7.34}$$

Intersecting with both sides and using (7.33), we get

$$K_Y \cdot \mathcal{E}_j = (\sum_{i=1}^{N} \mathcal{E}_i) \cdot E_j = \mathcal{E}_j^2 = -1.$$

□

Assume now that $f : X \dashrightarrow X'$ is a birational map of nonsingular projective algebraic surfaces. By Bertini's Theorem ([282], Chapter II, Theorem 8.18), a general hyperplane section H' of X' is a nonsingular irreducible curve of some genus g. Since $f^{-1}(|V|)$ has no base points, by another Bertini's Theorem

([282], Chapter II, Corollary 10.9), its general member H is a nonsingular irreducible curve. Since $H \in |\sigma^*(H')|$, we obtain that H is of genus g and the map $\sigma : H \to \sigma(H)$ is an isomorphism. Using the adjunction formula, we obtain

$$H \cdot K_Y = 2g - 2 - H^2 = H'^2 + H' \cdot K_{X'} - H^2.$$

Write $H = \pi^*(D) - E$ and apply the projection formula, to obtain

$$H \cdot K_Y = D \cdot K_X - E \cdot K_Y.$$

Applying (7.31) and the previous Lemma, we obtain the following.

Proposition 7.3.2 *Suppose $f : X \dashrightarrow X'$ is a birational rational map of nonsingular projective algebraic surfaces. Let $D \in |\mathcal{L}|$. Then*

(i) $D^2 - \sum_{i=1}^N m_i^2 = H'^2 = \deg X'$;
(ii) $D \cdot K_X - \sum_{i=1}^N m_i = H' \cdot K_{X'}$.

Let us apply all of this to the case of a Cremona transformation $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$. Of course, we take $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^2}(1)$, then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$, where d is the algebraic degree of T. By the previous Proposition,

$$\begin{aligned} 1 &= d^2 - \sum_{i=1}^N m_i^2, \\ 3 &= 3d - \sum_{i=1}^N m_i. \end{aligned} \tag{7.35}$$

Let $\mathfrak{b}$ be the base ideal of $|V|$. We know that $\pi_*(\mathcal{O}_Y(-E))$ is equal to the integral closure $\bar{\mathfrak{b}}$ of $\mathfrak{b}$, and we have a bijection

$$H^0(\mathbb{P}^2, \mathcal{L} \otimes \bar{\mathfrak{b}}) \cong H^0(Y, \pi^*\mathcal{O}_{\mathbb{P}^2}(d)(-E)) \cong H^0(Y, \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C}^3.$$

Subtracting the two equalities from (7.35), and applying the Postulation formula from Corollary 7.1.6, we find

$$h^0(\mathcal{O}_{\mathbb{P}^2}/\bar{\mathfrak{b}}) = \tfrac{1}{2}\sum_{i=1}^N m_i(m_i+1). \tag{7.36}$$

This also follows from the *Hoskin-Deligne formula* (see [158], Th'eorème 2.13 and [303]). The vector $(d; m_1, \ldots, m_N)$ is called the *characteristic* of the homaloidal net, or, of a Cremona transformation defined by this net.

Of course, not every vector $(d; m_1, \ldots, m_N)$ satisfying equalities (7.35) is realized as the characteristic vector of a homaloidal net. There are other necessary conditions for a vector to be realized as the characteristic $(d; m_1, \ldots, m_N)$

for a homaloidal net. For example, if m_1, m_2 correspond to points of largest multiplicity, a line through the points should intersect a general member of the net non-negatively. This gives the inequality

$$d \geq m_1 + m_2.$$

Next we take a conic through five points with maximal multiplicities. We get

$$2d \geq m_1 + \cdots + m_5.$$

Then we take cubics through none points, quartics through 14 points and so on. The first case that can be ruled out in this way is $(5; 3, 3, 1, 1, 1, 1, 1)$. It satisfies the equalities from the Theorem but does not satisfy the condition $m \geq m_1 + m_2$. We will discuss the description of characteristic vectors later in this Chapter.

7.3.2 The bubble space of a surface

Consider a factorization (7.28) of a birational morphism of nonsingular surfaces. Note that, if the morphism $\pi_1 \circ \cdots \circ \pi_i : Y_i \to X$ is an isomorphism on a Zariski open neighborhood of the point x_{i+1}, the points x_i can be identified with its image in X. Other points are called *infinitely near* points in X. To make this notion more precise one introduces the notion of the *bubble space* of a surface X.

Let $\mathcal{B}_X$ be the category of birational morphisms $\pi : X' \to X$ of nonsingular projective surfaces. Recall that a morphism from $(X' \stackrel{\pi'}{\to} X)$ to $(X'' \stackrel{\pi''}{\to} X)$ in this category is a regular map $\phi : X' \to X''$ such that $\pi'' \circ \phi = \pi'$.

Definition 7.3.3 *The* bubble space X^{bb} *of a nonsingular surface X is the factor set*

$$X^{\mathrm{bb}} = \Big(\bigcup_{(X' \stackrel{\pi'}{\to} X) \in \mathcal{B}_X} X' \Big) / R,$$

where R is the following equivalence relation: $x' \in X'$ is equivalent to $x'' \in X''$ if the rational map $\pi''^{-1} \circ \pi' : X' \dashrightarrow X''$ maps isomorphically an open neighborhood of x' to an open neighborhood of x''.

It is clear that for any $\pi : X' \to X$ from $\mathcal{B}_X$ we have an injective map $i_{X'} : X' \to X^{\mathrm{bb}}$. We will identify points of X' with their images. If $\phi : X'' \to X'$ is a morphism in $\mathcal{B}_X$ which is isomorphic in $\mathcal{B}_{X'}$ to the blow-up of a point $x' \in X'$, any point $x'' \in \phi^{-1}(x')$ is called a point *infinitely near* x' of the first order. This is denoted by $x'' \succ_1 x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This puts a partial order on X^{bb},

where $x > y$ if x is infinitely near y. When we do not specify the order of an infinitely near point we write $x' \succ x$.

We say that a point $x \in X^{\mathrm{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in X$. This defines the *height function* on the bubble space

$$\mathrm{ht} : X^{\mathrm{bb}} \to \mathbb{N}.$$

Clearly, $X = \mathrm{ht}^{-1}(0)$. Points of height zero are called *proper points* of the bubble space. They will be identified with points in X. They are minimal points with respect to the partial order on X^{bb}.

Let $\mathbb{Z}^{X^{\mathrm{bb}}}$ be the free abelian group generated by the set X^{bb}. Its elements are integer valued functions on X^{bb} with finite support. They added up as functions with values in $\mathbb{Z}$. We write elements of $\mathbb{Z}^{X^{\mathrm{bb}}}$ as finite linear combinations $\sum m(x)x$, where $x \in X^{\mathrm{bb}}$ and $m(x) \in \mathbb{Z}$ (similar to divisors on curves). Here $m(x)$ is the value of the corresponding function at x.

Definition 7.3.4 *A* bubble cycle *is an element $\eta = \sum m(x)x$ of $\mathbb{Z}^{X^{\mathrm{bb}}}$ satisfying the following additional properties:*

(i) $m(x) \geq 0$ *for any* $x \in X^{\mathrm{bb}}$*;*
(ii) $\sum_{x' \succ x} m_{x'} \leq m_x$.

We denote the subgroup of bubble cycles by $\mathcal{Z}_+(X^{\mathrm{bb}})$.

Clearly, any bubble cycle η can be written in a unique way as a sum of bubble cycles Z_k such that the support of η_k is contained in $\mathrm{ht}^{-1}(k)$.

We can describe a bubble cycle by a weighted oriented graph, called the *Enriques diagram*, by assigning to each point from its support a vertex, and joining two vertices by an oriented edge if one of the points is infinitely near another point of the first order. The arrow points to the vertex of lower height. We weight each vertex by the corresponding multiplicity.

Let $\eta = \sum m_x x$ be a bubble cycle. We order the points from the support of η such that $x_i \succ x_j$ implies $j < i$. We refer to such an order as an *admissible order*. We write $\eta = \sum_{i=1}^{N} m_i x_i$. Then we represent x_1 by a point on X and define $\pi_1 : X_1 \to X$ to be the blow-up of X with center at x_1. Then x_2 can be represented by a point on X_1 as either infinitely near of order 1 to x_1 or as a point equivalent to a point on X. We blow up x_2. Continuing in this way, we get a sequence of birational morphisms:

$$\pi : Y_\eta = Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = X, \tag{7.37}$$

where $\pi_{i+1} : Y_{i+1} \to Y_i$ is the blow-up of a point $x_i \in Y_{i-1}$. Clearly, the bubble cycle η is equal to the bubble cycle $\sum_{i=1}^{N} m_i x_i$.

Let $\mathcal{E}_i = \pi_{Ni}^*(E_i)$ be the exceptional configurations and $E = \sum m_i \mathcal{E}_i$. The ideal $\mathfrak{a} = \pi_*(\mathcal{O}_Y(-E))$ is an integrally closed ideal associated to η. The *Deligne-Hoskin formula* we mentioned before asserts that

$$h^0(\mathcal{O}_X/\mathfrak{a}) = \tfrac{1}{2}\sum m_i(m_i+1). \tag{7.38}$$

Conversely, any integrally closed ideal $\mathfrak{a}$ defining a 0-dimensional closed subscheme of X defines a bubble cycle $\eta_{\mathfrak{a}}$ as follows. First we blow-up $\mathfrak{a}$ to get a morphism $X' \to X$. By a result of O. Zariski (see [608], Appendix 5), the blow-up of an integrally closed ideal on a smooth surface is a normal surface (this is specific to the 2-dimensional case). Then we take a minimal resolution of singularities $Y \to X'$. Then we factor the composition $\pi : Y \to X' \to X$ as in (7.28). The corresponding bubble cycle is $\mathfrak{a}_\eta$.

Definition 7.3.5 *The bubble cycle η corresponding to the integral closure of the base ideal of the linear system $|V|$ defining a Cremona transformation $T : \mathbb{P}^2 \to \mathbb{P}^2$ is called the* fundamental bubble cycle*. Its points are called* fundamental points *of T.*

Let η be the bubble cycle corresponding to $\mathfrak{b}(|V|)$. We set

$$|dh - \eta| := |\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathfrak{b}(|V|)|.$$

Example 7.3.6 Suppose $\eta = \sum m_i x_i$, where all points x_i are proper. Then the integrally closed ideal corresponding to η is equal to the product of the ideal sheaves $\mathfrak{m}_{x_i}^{m_i}$. In fact, the blow-up of this ideal has the exceptional divisor $\sum m_i E_i$, and the same exceptional divisor is defined by η. One immediately checks the Deligne-Hoskin formula in this case. If η is the fundamental bubble cycle of a homaloidal linear system $|V|$, then $\mathfrak{b} = \mathfrak{b}(|V|)$ is generated at each point x_i by three elements g_1, g_2, g_3, the local equations of a basis of the linear system. Certainly, $\mathfrak{b}$ is not integrally closed if $m_i \geq 3$, and its integral closure is equal to $\mathfrak{m}_{x_i}^{m_i}$.

Remark 7.3.7 An ideal I in the formal power series ring $\mathbb{C}[[x, y]]$ of length n such that I is not contained in the square of the maximal ideal can be all explicitly described (see [312]). Every such ideal has one or both of the following forms:

$$I = (x + a_0 y + \cdots + a_{n-1} y^{n-1}, y^n),$$

or,

$$I = (y + b_0 x + \cdots + a_{n-1} x^{n-1}, x^n).$$

If x is a base point of a homaloidal linear system of multiplicity 1, then the completion of the localization $\mathfrak{b}_x$ of the base ideal is an ideal of the above

form. The number n is equal to the number of irreducible components in the exceptional curve over x. The coefficients (a_i, b_i) determine the corresponding pointt of the bubble cycle. Thus the first point $x_1 \succ_1 x$ infinitely near x corresponds to the tangent direction defined by the ideal $(x + a_0 y)$ or $(y + b_0)$. The next infinitely near point $x_1 \succ_1 x_1$ is determined by the coefficient a_1 or b_1, and so on.

It follows from this that the blow-up of the ideal I is a normal surface with one singularity locally isomorphic to the singularity at the origin of the surface $V(uv + w^n) \subset \mathbb{C}^3$.

7.3.3 Nets of isologues and fixed points

Let $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a Cremona transformation. Let p be a point in the plane. Consider the locus of points $C_T(p)$ such that $x, \phi(x), p$ are collinear. This locus is called the *isologue* of p, the point p is called its *center*. In terms of equations, if T is given by polynomials $(f_0(t), f_1(t), f_2(t))$ of degree d and $p = [a_0, a_1, a_2]$, then $C_T(p)$ is given by equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ t_0 & t_1 & t_2 \\ f_0(t) & f_1(t) & f_2(t) \end{pmatrix} = 0. \tag{7.39}$$

It follows immediately that $\deg C_T(p) = d + 1$ unless $C_T(p) = \mathbb{P}^2$. This always happens for de Jonquières transformation if we take p to be the base point $\mathfrak{o}$ of maximal multiplicity.

From now on we assume that $C_T(p) \neq \mathbb{P}^2$ for any point p. Then $C_T(p)$ is a curve of degree $d + 1$. It passes through the fundamental points of T (because the last row in the determinant is identically zero for such point) and it passes through the *fixed points* of T, i.e. points $x \in \mathrm{dom}(T)$ such that $T(x) = x$ (because the last two rows are proportional). Also $C_T(p)$ contains its center p (because the first two rows are proportional).

One more observation is that

$$C_T(p) = C_{T^{-1}}(p).$$

When p varies in the plane we obtain a *net of isologues*. If F is the 1-dimensional component of the set of fixed points, then F is a fixed component of the net of isologues.

Remark 7.3.8 It follows from the definition that the isologue curve $C_T(p)$ is *projectively generated* by the pencil of lines ℓ through p and the pencil of curves $T^{-1}(\ell)$. Recall that given two pencils $\mathcal{P}$ and $\mathcal{P}'$ of plane curves of

degree d_1 and d_2 and a projective isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$, the union of points $Q \cap \alpha(Q), Q \in \mathcal{P}$, is a plane curve C. Assuming that the pencils have no common base points, C is a plane curve of degree $d_1 + d_2$. To see this we take a general line ℓ and restrict $\mathcal{P}$ and $\mathcal{P}'$ to it. We obtain two linear series g^1_d and $g^1_{d'}$ on ℓ. The intersection $C \cap \ell$ consists of points common to divisors from g^1_d and $g^1_{d'}$. The number of such points is equal to the intersection of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ with a curve of bidegree (d, d'), hence it is equal to $d+d'$. It follows from the definition that C contains the base points of the both pencils.

Proposition 7.3.9 *Assume that T has no infinitely near fundamental points. Then the multiplicity of a general isologue curve at a fundamental point x of multiplicity m is equal to m.*

Proof Let u, v be local affine parameters at x. For each homogeneous polynomial $\phi(t_0, t_1, t_2)$ vanishing at x with multiplicity $\geq m$, let $[\phi]_k := [\phi]_k(u, v)$ be the degree k homogeneous term in the Taylor expansion at x. If $V(f)$ is a general member of the homaloidal net, then $[f]_k = 0$ for $k < m$ and $[f_m] \neq 0$. Let B_m be the space of binary forms of degree m in variables u, v. Consider the linear map $\alpha : \mathbb{C}^3 \to B_m$ defined by

$$(a, b, c) \mapsto [(bt_2 - ct_1)f_0(t) + (ct_0 - at_2)f_1(t) + (at_1 - bt_0)f_2(t)]_m.$$

The map is the composition of the linear map $\mathbb{C}^3 \to \mathbb{C}^3$ defined by $(a, b, c) \mapsto ([bt_2 - ct_1]_0, [ct_0 - at_2]_0, [at_1 - bt_0]_0)$ and the linear map $\mathbb{C}^3 \to B_m$ defined by $(a, b, c) \mapsto [af_0 + bf_1 + cf_2]_m$. The rank of the first map is equal to 2, the kernel is generated by $[t_0]_0, [t_1]_0, [t_2]_0)$. Since no infinitely near point is a base point of the homaloidal net, the rank of the second map is greater than or equal to 2. This implies that the map α is not the zero map. Hence there exists an isologue curve of multiplicity equal to m. □

Remark 7.3.10 Coolidge claims in [129], p. 460, that the assertion is true even in the case of infinitely near points. By a direct computation, one checks that the multiplicity of isologue curves of the degenerate standard Cremona transformation (7.7) at the unique base point is equal to 2.

Corollary 7.3.11 *Assume that the homaloidal net has no infinitely near base points and the net of isologues has no fixed component. Then the number of fixed points of T is equal to $d + 2$.*

Proof Take two general points p, q in the plane. In particular, we may assume that the line $\ell = \overline{pq}$ does not pass through the base points of the homaloidal net and the fixed points. Also $p \notin C_T(q)$ and $q \notin C_T(p)$. Consider a point x in the intersection $C_T(p) \cap C_\phi(q)$. Assume that it is neither a base point

nor a fixed point. Then $p, q \in \overline{xT(x)}$, hence $x, T(x), p, q$ lie on ℓ. Conversely, if $x \in \ell \cap C_T(p)$ and $x \neq p$, then the points $x, T(x), p$ are collinear and, since $q \in \ell$, we get that $x, T(x), q$ are collinear. This implies that $x \in C_T(q)$. This shows that the base points of the pencil of isologue curves $C_T(p), p \in \ell$, consists of base points of the homaloidal net, fixed points and d points on ℓ (counted with multiplicities). The base points of the homaloidal net contribute $\sum_{i=1}^N m_i^2$ to the intersection. Applying Proposition 7.3.9, we obtain that fixed points contribute $d + 2 = (d+1)^2 - d - \sum_{i=1}^N m_i^2$ to the intersection. □

Note that the transformation from Remark 7.3.10 has no fixed points.

Remark 7.3.12 The assumption that T has no infinitely near points implies that the graph Γ of T is a nonsingular surface in $\mathbb{P}^2 \times \mathbb{P}^2$ isomorphic to the blow-up of the base scheme of the homaloidal net. Let h_1, h_2 be the preimages of the cohomology classes of lines under the projections. They generate the cohomology ring $H^*(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z})$. Let $[\Gamma]$ be the cohomology class of Γ and $[\Delta]$ be the cohomology class of the diagonal Δ. Write $[\Gamma] = ah_1^2 + bh_1h_2 + ch_2^2$. Since the preimage of a general point under T is a point, we have $[\Gamma] \cdot h_2^2 = 1$. Replacing ϕ with T^{-1}, we get $[\Gamma] \cdot h_1^2 = 1$. Since a general line intersects the preimage of a general line at d points we get $[\Gamma] \cdot h_1 \cdot h_2 = d$. This gives

$$[\Gamma] = h_1^2 + dh_1h_2 + h_2^2. \tag{7.40}$$

Similarly, we get

$$[\Delta] = h_1^2 + h_1h_2 + h_2^2. \tag{7.41}$$

This implies that

$$[\Gamma] \cdot [\Delta] = d + 2.$$

This confirms the assertion of the previous Corollary. In fact, one can use the argument for another proof of the Corollary if we assume (that follows from the Corollary) that no point in the intersection $\Gamma \cap \Delta$ lies on the exceptional curves of the projections.

The net of isologue curves without fixed component is a special case of a *Laguerre net*. It is defined by one of the following three equivalent properties.

Theorem 7.3.13 *Let $|V|$ be an irreducible net of plane curves of degree d. The following properties are equivalent.*

(i) There exists a basis f_0, f_1, f_2 such that

$$t_0f_0(t) + t_1f_1(t) + t_2f_2(t) = 0. \tag{7.42}$$

(ii) For any basis f_0, f_1, f_2 of V, there exist three linearly independent linear forms l_1, l_2, l_3 such that

$$l_0 f_0 + l_1 f_1 + l_2 f_2 = 0.$$

(iii) There exists a basis f_0, f_1, f_2 of V such that

$$f_0 = t_1 g_2 - t_2 g_1, \quad f_1 = t_2 g_0 - t_0 g_2, \quad f_2 = t_0 g_1 - t_1 g_0,$$

where g_0, g_1, g_2 are homogeneous forms of degree $d-1$.

(iv) The base locus of a general pencil in $|V|$ is the union of the base locus of $|V|$ and a set of $d-1$ collinear points.

Proof The equivalence of the fist two properties is obvious. Also property (iii) obviously implies property (i). Suppose (i) holds. The Koszul complex in the ring of polynomials $S = \mathbb{C}[t_0, t_1, t_2]$ is an exact sequence

$$0 \to S \xrightarrow{\alpha} S^3 \xrightarrow{\beta} S^3 \xrightarrow{\gamma} S \to S/(t_0, t_1, t_2) \to 0,$$

where α is defined by $a \mapsto a(t_0, t_1, t_2)$. The map β is defined by the matrix

$$\begin{pmatrix} 0 & -t_2 & t_1 \\ t_2 & 0 & -t_0 \\ -t_1 & t_0 & 0 \end{pmatrix},$$

and the map γ is defined by $(a, b, c) \mapsto at_0 + bt_1 + ct_2$ (see [208], 17.2). Property (i) says that (f_0, f_1, f_2) belongs to the kernel of γ. Thus it belongs to the image of β, and hence (iii) holds.

(i) $\Rightarrow$ (iv) Take two general curves $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2)$ and $C_\mu = V(\mu_0 f_0 + \mu_1 f_1 + \mu_2 f_2)$ from the net. They intersect with multiplicity ≥ 2 at a point x if and only if x belongs to the Jacobian curve of the net. This shows that the set of pencils which intersect non-transversally outside the base locus is a proper closed subset of $|V^\vee|$. So, we may assume that $C(\mu)$ and $C(\nu)$ intersect transversally outside the base locus of the net. Let $p = [a]$ belong to $C_\lambda \cap C_\mu$ but does not belong to the base locus of $|V|$. Then $(f_0(a), f_1(a), f_2(a))$ is a nontrivial solution of the system of linear equations with the matrix of coefficients equal to

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \mu_0 & \mu_1 & \mu_2 \\ a_0 & a_1 & a_2 \end{pmatrix}.$$

This implies that the line spanned by the points $\lambda = [\lambda_0, \lambda_1, \lambda_2]$ and $\mu =$

$[\mu_0, \mu_1, \mu_2]$ contains the point p. Thus all base points of the pencil different from the base points of the net are collinear. Conversely, suppose a non-base point $[a] \neq \lambda, \mu$ lies on a line $\overline{\lambda\mu}$ and belongs to the curve C_λ. Then $(f_0(a), f_1(a), f_2(a))$ is a nontrivial solution of

$$\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2 = 0,\ a_0 t_0 + a_1 t_1 + a_2 t_2 = 0,$$

and hence satisfies the third equation $\mu_0 t_0 + \mu_1 t_1 + \mu_2 t_2 = 0$. This shows that $a \in C_\lambda \cap C_\mu$. Thus we see that the intersection $C_\lambda \cap C_\mu$ consists of $d-1$ non-base points.

(iv) $\Rightarrow$ (ii) We follow the proof from [129], p. 423. Let $V(f_0), V(f_1)$ be two general members intersecting at $d-1$ points on a line $V(l)$ not passing through the base points. Let p_i be the residual point on $V(f_i)$. Choose a general line $V(l_0)$ passing through p_2 and a general line $V(l_1)$ passing through p_1. Then $V(l_0 f_0)$ and $V(l_1 f_1)$ contain the same set of $d+1$ points on the line $V(l)$, hence we can write

$$l_0 f_0 + c l_1 f_1 = l f_2 \tag{7.43}$$

for some polynomial f_2 of degree d and some constant c. For any base point q of the net, we have $l_0(q) f_0(q) + c l_1(q) f_1(q) = l(q) f_2(q)$. Since $l(q) \neq 0$ and $f_0(q) = f_1(q) = 0$, we obtain that $f_2(q) = 0$. Thus the curve $V(f_2)$ passes through each base point and hence belongs to the net $|V|$. This shows that f_0, f_1 and f_2 define a basis of $|V|$ satisfying property (ii). □

Corollary 7.3.14 *Let $\mathfrak{b}$ be the base ideal of a Laguerre net of curves of degree d. Then $h^0(\mathcal{O}_{\mathbb{P}^2}/\mathfrak{b}) = d^2 - d + 1$.*

Proof It is clear that, any base-point, $\mathfrak{b}$ is generated by two general members of the net. By Bezout's Theorem $h^0(\mathcal{O}_{\mathbb{P}^2}/\mathfrak{b}) = d^2 - (d-1)$. □

Example 7.3.15 Take an irreducible net of cubic curves with seven base points. Then it is a Laguerre net since two residual intersection points of any two general members are on a line. Thus it is generated by the minors of the matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 \\ g_0 & g_1 & g_2 \end{pmatrix},$$

where g_0, g_1, g_2 are quadratic forms. Recall that the linear system of cubics with seven base points satisfying the conditions in Section 6.3.3 define a non-singular quartic curve. It is known that the quartic curve is a Lüroth quartic if and only if there exists a cubic curve $V(f)$ such that $g_i = \frac{\partial f}{\partial t_i}$ (see [28], [422]).

7.3.4 Quadratic transformations

Take $d = 2$. Formulas (7.35) give $m_1 = m_2 = m_3 = 1, N = 3$. The birational transformation of this type is called a *quadratic transformation*. We have already encountered these transformations before, the standard quadratic transformations $T_{\text{st}}, T'_{\text{st}}, T''_{\text{st}}$.

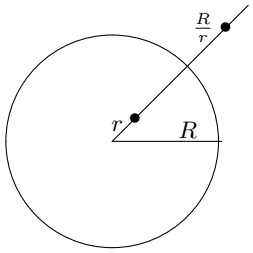
The homaloidal linear system is the net of conics $|2h - \eta|$. Since the net is irreducible $|h-\eta| = \emptyset$. Assume first that η consists of proper points x_1, x_2, x_3. Let $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a projective automorphism which sends the points $p_1 = [1,0,0], p_2 = [0,1,0], p_3 = [0,0,1]$ to the points x_1, x_2, x_3. Then the composition $T \circ g$ has base points at p_1, p_2, p_3. The linear system of conics through these points consists of conics $V(at_0t_1 + bt_0t_1 + ct_1t_2)$. If we choose a basis formed by the conics $V(t_1t_2), V(t_0t_2), V(t_0t_1)$, this amounts to composing $T \circ g$ with some $g' \in \text{Aut}(\mathbb{P}^2)$ on the left, and we obtain the standard quadratic transformation T_{st} from Example 7.1.9. It has the special property that $T_{\text{st}} = T_{\text{st}}^{-1}$.

Next we assume that the bubble cycle η has two proper points x_1, x_2 and an infinitely near point $x_3 \succ x_2$. The base ideal $\mathfrak{b}$ of the homaloidal linear system is integrally closed and coincides with the ideal $\mathfrak{m}_{x_1} \cap \mathfrak{a}_2$, where $\mathfrak{a}_{x_2} \subset \mathfrak{m}_{x_2}$ with $\dim \mathfrak{m}_{x_2}/\mathfrak{a}_{x_2} = 1$. If we choose local parameters u, v at x_2, then $\mathfrak{a}_{x_2} = (au + bv, v^2)$, where $au + bv = 0$ is the tangent direction defining the point x_3. Let g be an automorphism of $\mathbb{P}^2$ which sends p_1 to x_1, p_2 to x_2 and the tangent direction $t_1 = 0$ at p_2 to the tangent direction $au + bv = 0$ at p_2. Then the composition $T \circ g$ is given by the linear system of conics passing through the points p_1 and p_2 and tangent to the line $V(t_0)$ at the point p_2. The equation of a conic from this linear system is $at_2^2 + bt_0t_1 + ct_0t_2 = 0$. A choice of a basis of this linear system defines a quadratic Cremona transformation. The special choice of a basis formed by $V(t_1^2), V(t_0t_1), V(t_0t_2)$ defines the standard quadratic transformation T'_{st} from Example 7.1.9 with $T'_{\text{st}} = T'^{-1}_{\text{st}}$.

Assume now that $x_3 \succ x_2 \succ x_1$. Let $\mathfrak{b}$ be the ideal of the base scheme. Applying a linear transformation g, we obtain that the base point of $T \circ g$ is equal to p_1. Let $x = t_1/t_0, y = t_2/t_0$ be local parameters at x_1. By Remark 7.3.7, we may assume that $\mathfrak{b} = (x + ay + by^2, y^3)$ or $(y + ax + bx^2, x^3)$. It is easy to see that one can find another linear transformation g' which fixes p_1 such that the base ideal of $T \circ g \circ g'$ is equal to $(-x + y^2, y^3)$. The homaloidal linear system generated by conics $V(t_2^2 - t_0t_1), V(t_1^2), V(t_1t_2)$. In this basis, the transformation coincides with the involutorial standard quadratic transformation T''_{st} from Example 7.1.9.

Example 7.3.16 The first historical example of a Cremona transformation is the *inversion* map. Recall the inversion transformation from the plane geome-

try. Given a circle of radius R, a point $x \in \mathbb{R}^2$ with distance r from the center of the circle is mapped to the point on the same ray at the distance R/r (as in the picture below).



In the affine plane $\mathbb{C}^2$ the transformation is given by the formula

$$(x, y) \mapsto \Big(\frac{Rx}{x^2 + y^2}, \frac{Ry}{x^2 + y^2}\Big).$$

In projective coordinates, the transformation is given by the formula

$$(t_0, t_1, t_2) \mapsto (t_1^2 + t_2^2, Rt_1t_0, Rt_2t_0).$$

Note that the transformation has three fundamental points $[1, 0, 0]$, $[0, 1, i]$, and $[0, 1, -i]$. It is an involution and transforms lines not passing through the fundamental points to conics (circles in the real affine plane). The lines passing though one of the fundamental points are transformed to lines. The lines passing through the origin $(1, 0, 0)$ are invariant under the transformation. The conic $t_1^2 + t_2^2 - Rt_0^2 = 0$ is the closure of the set of fixed points.

Example 7.3.17 Let C_1 and C_2 be two conics intersecting at four distinct points. For each general point x in the plane let $T(x)$ be the intersection of the polar lines $P_x(C_1)$ and $P_x(C_2)$. Let us see that this defines an involutorial quadratic transformation with fundamental points equal to the singular points of three reducible conics in the pencil generated by C_1 and C_2. It is clear that the transformation T is given by three quadratic polynomials. Since $P_x(C_1) \cap P_x(C_2)$ is equal to $P_x(C) \cap P_x(C')$ for any two different members of the pencil, taking C to be a reducible conic and x to be its singular point, we obtain that T is not defined at x. Since the pencil contains three reducible members, we obtain that T has three base points, hence T is given by a homaloidal net and hence is a birational map. Obviously, $x \in P_{T(x)}(C_1) \cap P_{T(x)}(C_2)$, hence T is an involution.

7.3.5 Symmetric Cremona transformations

Assume that the fundamental bubble cycle η of a Cremona transformation T consists of points taken with equal multiplicity m. In this case the Cremona transformations is called *symmetric*. We must have

$$d^2 - Nm^2 = 1, \quad 3d - Nm = 3.$$

Multiplying the second equality by m and subtracting from the first one, we obtain $d^2 - 3dm = 1 - 3m$. This gives $(d-1)(d+1) = 3m(d-1)$. The case $d = 1$ corresponds to a projective transformation. Assume $d > 1$. Then we get $d = 3m - 1$ and hence $3(3m-1) - Nm = 3$. Finally, we obtain

$$(9 - N)m = 6, \quad d = 3m - 1.$$

This gives us four cases:

(1) $m = 1, N = 3, d = 2$;
(2) $m = 2, N = 6, d = 5$;
(3) $m = 3, N = 7, d = 8$;
(4) $m = 6, N = 8, d = 17$.

The first case is obviously realized by a quadratic transformation with three fundamental points.

The second case is realized by the linear system of plane curves of degree 5 with six double points. Take a bubble cycle $\eta = 2x_1 + \cdots + 2x_6$, where the points x_i in the bubble space do not lie on a proper transforms of a conic and no three lie on the proper transforms of a line. I claim that the linear system $|V| = |\mathcal{O}_{\mathbb{P}^2}(2) - \eta|$ is homaloidal. The space of plane quintics is of dimension 20. The number of conditions for passing through a point with multiplicity ≥ 2 is equal to 3. Thus $\dim |\mathcal{O}_{\mathbb{P}^2}(2) - \eta| \geq 2$. It is easy to see that the linear system does not have fixed components. For example, if the fixed component is a line, it cannot pass through more than two points, hence the residual components are quartics with four double points, obviously reducible. If the fixed component is a conic, then it passes through at most five points, hence the residual components are cubics with at least one double point and passing through the remaining points. It is easy to see that the dimension of such linear system is at most 1. If the fixed component is a cubic, then by the previous analysis we may assume that it is irreducible. Since it has at most one singular point, the residual conics pass through at least five points and the dimension of the linear system is equal to zero (or it is empty). Finally, if the fixed component is a quartic, then the residual components are lines passing through three points, again a contradiction.

Applying Bezout's Theorem, we see that two general members of our linear system intersect at one point outside the base locus, also their genus is equal to 0. Thus the linear system is a homaloidal.

Assume that all base points are proper points in the plane. Then the P-locus of the transformation consists of six conics, each passing through five of the six base points.

The third case is realized by a *Geiser involution*. We consider an irreducible net $\mathcal{N}$ of cubic curves through seven points $x_1, \ldots, x_7$ in the plane. The existence of such a net puts some conditions on the seven points. For example, no four points must be collinear, and no seven points lie on a conic. We leave it to the reader to check that these conditions are sufficient that such a net exists. Now consider the transformation $\mathfrak{G}$ that assigns to a general point x in the plane the base point of the pencil of cubics from the net which pass through x. If points $x_1, \ldots, x_7$ satisfy condition $(*)$ from Subsection 6.3.3, then the net of cubics defines a rational map of degree 2 to the plane with a nonsingular quartic curve as the branch curve. The Geiser involution G is the rational deck transformation of this cover. Under weaker conditions on the seven points, the same is true. The only difference is that the branch curve may acquire simple singularities.

Let us confirm that the degree of the transformation $\mathfrak{G}$ is equal to 8. The image of a general line ℓ under the map given by $\mathcal{N}$ is a cubic curve L. Its preimage is a curve of degree 9 passing through the points x_i with multiplicity 3. Thus the union $\ell + L$ is invariant under T, hence $T(\ell) = L$. Since $T = T^{-1}$, this shows that the degree of T is equal to 8. It also shows that the homaloidal linear system consists of curves of degree 8 passing through the base points with multiplicities ≥ 3. In other words, the homaloidal linear system is $|8h - 3\eta|$, where $\eta = x_1 + \cdots + x_7$. If one composes $\mathfrak{G}$ with a projective transformation we obtain a transformation given by the same homaloidal linear system but not necessarily involutorial. Also, the bubble cycle η may not consist of only proper points, as soon as we continue to require that the linear system $|3h - \eta|$ has no fixed components. All admissible η's will be classified in the next Chapter.

The last case is realized by a *Bertini involution*. We consider an irreducible pencil of cubic curves through a general set of 8 points $x_1, \ldots, x_8$. Let q be its ninth base point (it could be infinitely near one of the points x_i). For any general point x in the plane, let $F(x)$ be the member of the pencil containing x. Let $\mathfrak{B}(x)$ be the residual point in the intersection of $F(x)$ with the line $\overline{xq}$ (it could be the tangent line to $F(x)$ if $q \succ x_i$). The transformation $x \to \mathfrak{B}(x)$ is the Bertini involution. If we take q as the origin in the group law on a nonsingular cubic $F(p)$, then $\mathfrak{B}(x) = -x$.

Consider the web $\mathcal{N}$ of curves of degree 6 and genus 2 whose general member passes through each point x_i with multiplicity 2. The restriction of $\mathcal{N}$ to any $F(x)$ is a pencil with a fixed part $2x_1 + \cdots + 2x_8$ and a moving part g^1_2. One of the members of this g^1_2 is the divisor $2q$ cut out by $2F(x'), x \neq x'$. As we have seen in Subsection 6.3.3, the members of this pencil are cut out by lines through the coresidual point on $F(x)$. This point must coincide with the point q. Thus members of the g^1_2 are divisors $x + \mathfrak{B}(x)$. We will see in the next Chapter that the net $\mathcal{N}$ defines a degree 2 rational map $f : \mathbb{P}^2 \dashrightarrow Q \subset \mathbb{P}^3$, where Q is a singular irreducible quadric in $\mathbb{P}^3$. The image of q is the vertex of the cone. The images of the curves $F(x)$ are lines on Q. Consider a general line ℓ in the plane. It is mapped to a curve of degree 6 on Q not passing through the vertex of Q. A curve on Q not passing through the vertex is always cut out by a cubic surface. In our case the curve $f(\ell)$ is cut out by a cubic surface. The preimage of this curve is a curve of degree 18 passing through the points x_i with multiplicities 6. As in the case of the Geiser involution, this shows that $\mathfrak{B}(\ell)$ is a curve of degree 17 with 6-tuple points $x_1, \ldots, x_8$. Thus the homaloidal linear system defining the Bertini involution is equal to $|17h-6\eta|$, where $\eta = x_1 + \cdots + x_8$. Again, we may consider η not necessarily consisting of proper points. All admissible η's will be classified in the next Chapter.

7.3.6 de Jonquières transformations and hyperelliptic curves

A planar de Jonquières transformation J is obtained by dilation from the identity transformation of $\mathbb{P}^1$. It follows from Subsection 7.2.3 that such a transformation is given by the formulae

$$\begin{aligned} t'_0 &= t_0 b_{m-1}(t_1, t_2) + b_m(t_1, t_2), \\ t'_1 &= t_1(t_0 a_{m-2}(t_1, t_2) + a_{m-1}(t_1, t_2)), \\ t'_2 &= t_2(t_0 a_{m-2}(t_1, t_2) + a_{m-1}(t_1, t_2)). \end{aligned} \tag{7.44}$$

Here it is assumed that the polynomials $F_m = t_0 b_{m-1} + b_m$ and $Q_{m-1} = t_0 a_{m-2} + a_{m-1}$ are coprime and $b_{m-1}a_{m-1} \neq b_m a_{m-2}$.

In affine coordinates $x = t_1/t_2, y = t_0/t_2$, the transformation is given by

$$(x', y') = \Big(x, \frac{y b'_{m-1}(x) + b'_m(x)}{y a'_{m-2}(x) + a'_{m-1}(x)}\Big). \tag{7.45}$$

Let us consider the closure of fixed points of this transformation. It is given by the affine equation

$$y b'_{m-1}(x) + b'_m(x) = y(y a'_{m-2}(x) + a'_{m-1}(x)).$$

Going back to our projective coordinates, the equation becomes

$$t_0^2 a_{m-2}(t_1,t_2) + t_0(a_{m-1}(t_1,t_2) - b_{m-1}(t_1,t_2)) - b_m(t_1,t_2) = 0. \quad (7.46)$$

This a plane curve H_m of degree m with the point $\mathfrak{o} = [1,0,0]$ of multiplicity $m-2$. The pencil of lines through x_1 defines a g_2^1 on the curve. So, if H_m has no other singularities, it must be a rational curve if $m = 2$, an elliptic curve if $m = 3$, and a hyperelliptic curve of genus $m-2$ if $m \geq 4$.

The homaloidal linear system defining the de Jonquières transformation J is generated by the curves D_1, D_2, D_3 whose equations are given by the right-hand sides in (7.44). The point $\mathfrak{o} = [1,0,0]$ is a base point of multiplicity $m-2$. Let us find other base points. Let $x = [\alpha,\beta,\gamma]$ be a base point different from $\mathfrak{o}$. Then either β or γ is not zero. Hence

$$\alpha b_{m-1}(\beta,\gamma) + b_m(\beta,\gamma) = \alpha a_{m-2}(\beta,\gamma) + a_{m-1}(\beta,\gamma) = 0.$$

If $\alpha \neq 0$ this happens if and only if

$$(b_{m-1}a_{m-1} - b_m a_{m-2})(\beta,\gamma) = 0.$$

If $\alpha = 0$, then the condition is $b_m(\beta,\gamma) = a_{m-1}(\beta,\gamma) = 0$, hence the point still satisfies the previous equation. Under some generality condition, this gives $2m-2$ base points $x_1,\dots,x_{2m-2}$. Obviously, the points x_i lie on H_m. They also lie on the curve

$$\Gamma = V(t_0 a_{m-2}(t_1,t_2) + a_{m-1}(t_1,t_2)). \quad (7.47)$$

This is a monoidal curve of degree $m-1$ with singular point $\mathfrak{o}$ of multiplicity $m-1$. It has the same tangent cone at $\mathfrak{o}$ as the curve H_m. Thus one expects to find $m(m-1) - (m-2)^2 - (m-2) = 2m-2$ points of intersection outside $\mathfrak{o}$.

Note that a general member D of the homaloidal linear system intersects the line $\overline{x_i\mathfrak{o}}$ with multiplicity $m-1$ at $\mathfrak{o}$ and multiplicity 1 at x_i. This implies that each line belongs to the P-locus of J. Also D intersects the curve Γ at $\mathfrak{o}$ with multiplicity $(m-1)^2$ and at the points x_i with multiplicity 1. Since $D \cdot \Gamma = m(m-1) = (m-1)^2 + 2m - 2$, this implies that Γ belongs to the P-locus two. The degree of the Jacobian is equal to $3(m-1) = m-1+2m-2$, thus there is nothing more in the P-locus.

Let us record what we have found so far.

Proposition 7.3.18 *Let J be the de Jonquières transformation given by (7.44). Assume that the binary form $b_{m-1}a_{m-1} - b_m a_{m-2}$ of degree $2m-2$ has no multiple roots. Then J has $2m-1$ proper fundamental points $\mathfrak{o}, x_1, \dots, x_{2m-2}$. The point $\mathfrak{o}$ is of multiplicity $m-2$, the remaining ones are of multiplicity 1.*

The characteristic vector is $(m, m-2, 1, \dots, 1)$. *The* P*-locus consists of a monoidal curve* Γ *given by Equation* (7.47) *and* $2m-2$ *lines* $\overline{x_i\mathfrak{o}}$.

If we drop the condition on the binary form $b_{m-1}a_{m-1} - b_m a_{m-2}$, some of the fundamental points become infinitely near.

Let us see when J satisfies $J = J^{-1}$. The affine equation shows that this happens if and only if the trace of the matrix $\begin{pmatrix} b'_{m-1} & b'_m \\ a'_{m-2} & a'_{m_1} \end{pmatrix}$ is equal to 0. Thus the condition is

$$a_{m-1}(t_1, t_2) + b_{m-1}(t_1, t_2) = 0. \tag{7.48}$$

In this case the hyperelliptic curve has the equation

$$t_0^2 a_{m-2}(t_1, t_2) + 2t_0 a_{m-1}(t_1, t_2) - b_m(t_1, t_2) = 0. \tag{7.49}$$

The curve Γ coincides with the first polar of H_m. The fundamental points are the ramification points of the projection of H_m from the point $\mathfrak{o}$. The transformation J is the de Jonquières involution described in Subsection 7.2.3. It is clear that the curve H_m is nonsingular if and only if we have $2m-2$ distinct fundamental points of multiplicity 1.

A space construction of a de Jonquières transformation is due to L. Cremona [143]. Consider a rational curve R of bidegree $(1, m-2)$ on a nonsingular quadric Q in $\mathbb{P}^3$. Let ℓ be a line on Q which intersects R at $m-2$ distinct points. For each point x in the space, there exists a unique line joining a point on ℓ and on R. In fact, the plane spanned by x and ℓ intersects R at a unique point r outside $R \cap \ell$ and the line $\overline{xr}$ intersects ℓ at a unique point s. Take two general planes Π and Π' and consider the following birational transformation $f : \Pi \dashrightarrow \Pi'$. Take a general point $p \in \Pi$, find the unique line joining a point $r \in R$ and a point $s \in \ell$. It intersects Π' at the point $f(p)$. For a general line ℓ in Π the union of lines $\overline{rs}, r \in R, s \in L$, that intersect ℓ is a ruled surface of degree m. Its intersection with Π' is a curve of degree m. This shows that the transformation f is of degree m. It has $2m-2$ simple base points. They are $m-1$ points in $\Pi' \cap R$ and $m-1$ points which are common to the line $\Pi \cap \Pi'$ and the $m-1$ lines joining the point $\ell \cap \Pi$ with the points in the intersection $\Pi \cap R$. Finally, the point $\ell \cap \Pi'$ is a base point of multiplicity $m-1$. Identifying Π and Π' by means of an isomorphism, we obtain a de Jonquières transformation.

7.4 Elementary transformations

7.4.1 Minimal rational ruled surfaces

First let us recall the definition of a minimal rational ruled surface $\mathbf{F}_n$. If $n = 0$ this is the surface $\mathbb{P}^1 \times \mathbb{P}^1$. If $n = 1$ it is isomorphic to the blow-up of one point in $\mathbb{P}^2$ with the ruling $\pi : \mathbf{F}_1 \to \mathbb{P}^1$ defined by the pencil of lines through the point. If $n > 1$, we consider the cone in $\mathbb{P}^{n+1}$ over a Veronese curve $\mathsf{V}_n^1 \subset \mathbb{P}^n$, i.e. we identify $\mathbb{P}^{n-1}$ with a hyperplane in $\mathbb{P}^n$ and consider the union of lines joining a fixed point p_0 not on the hyperplane with all points in V_n^1. The surface $\mathbf{F}_n$ is a minimal resolution of the vertex p_0 of the cone. The exceptional curve of the resolution is a smooth rational curve E_n with $E_n^2 = -n$. The projection from the vertex of the cone extends to a morphism $p : \mathbf{F}_n \to \mathbb{P}^1$ which defines a ruling. The curve E_n is its section, called the *exceptional section*. In the case $n = 1$, the exceptional curve E_1 of the blow-up $\mathbf{F}_1 \to \mathbb{P}^2$ is also a section of the corresponding ruling $p : \mathbf{F}_1 \to \mathbb{P}^1$. It is also called the exceptional section.

Recall from [282] some facts about vector and projective bundles that we will need next and later on. For any locally free sheaf $\mathcal{E}$ of rank $r + 1$ over a scheme S one defines the *vector bundle* $\mathbb{V}(\mathcal{E})$ as the scheme $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$. A local section $U \to \mathbb{V}(\mathcal{E})$ is defined by a homomorphism $\mathrm{Sym}(\mathcal{E}) \to \mathcal{O}(U)$ of $\mathcal{O}(U)$-algebras, and hence by a linear map $\mathcal{E}|U \to \mathcal{O}(U)$. Thus the sheaf of local sections of the vector bundle $\mathbb{V}(\mathcal{E})$ is isomorphic to the sheaf $\mathcal{E}^\vee$. The fiber $\mathbb{V}(\mathcal{E})_x$ over a point $x \in X$ is equal to $\mathrm{Spec}\ \mathrm{Sym}(\mathcal{E}(x)) = \mathcal{E}(x)^\vee$, where $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is the fiber of $\mathcal{E}$ at x considered as a vector space over the residue field $\kappa(x)$ of the point x.

The *projective bundle* associated with a vector bundle $\mathbb{V}(\mathcal{E})$ (or a locally free sheaf $\mathcal{E}$) is the scheme $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$. It comes with the natural morphism $p : \mathbb{P}(\mathcal{E}) \to S$. In the same notation as above,

$$\mathbb{P}(\mathcal{E})|U \cong \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_U^{r+1})) \cong \mathrm{Proj}(\mathcal{O}(U)[t_0, \dots, t_r]) \cong \mathbb{P}^r_U.$$

For any point $x \in X$, the fiber $\mathbb{P}(\mathcal{E})_x$ over x is equal to $\mathbb{P}(\mathcal{E}(x))$.

By definition of the projective spectrum, it comes with an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Its sections over $p^{-1}(U)$ are homogeneous elements of degree 1 in $\mathrm{Sym}(\mathcal{O}_U^{r+1})$. This gives, for any $k \geq 0$,

$$p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) \cong \mathrm{Sym}^k(\mathcal{E}).$$

Note that, for any invertible sheaf $\mathcal{L}$ over S, we have $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ as schemes; however, the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ has to be replaced with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes p^*\mathcal{L}$.

For any scheme $\pi : X \to S$ over S, a morphism of S-schemes $f : X \to \mathbb{P}(\mathcal{E})$ is defined by an invertible sheaf $\mathcal{L}$ over X and a surjection $\phi : \pi^*\mathcal{E} \to \mathcal{L}$. When we trivialize $\mathbb{P}(\mathcal{E})$ over U, the surjection ϕ defines $r + 1$ sections of

$\mathcal{L}|\pi^{-1}(U)$. This gives a local map $x \mapsto [s_0(x), \dots, s_r(x)]$ from $\pi^{-1}(U)$ to $p^{-1}(U) = \mathbb{P}^r_U$. These maps are glued together to define a global map. We have $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Example 7.4.1 Let us take $X = S$. Then an S-morphism $S \to \mathbb{P}(\mathcal{E})$ is a section $s : S \to \mathbb{P}(\mathcal{E})$. It is defined by an invertible sheaf $\mathcal{L}$ on S and a surjection $\phi : \mathcal{E} \to \mathcal{L}$. We have $\mathcal{L} = s^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Another special case is when we take $i : x = \mathrm{Spec}(\kappa(x)) \hookrightarrow S$ to be a closed point in S. Then an invertible sheaf on a point is the constant sheaf κ_x and $i^*\mathcal{E} = \mathcal{E}_x = \mathcal{E}/\mathfrak{m}_x\mathcal{E} = \mathcal{E}(x)$ is the fiber of the sheaf $\mathcal{E}$. The corresponding morphism $x \to \mathbb{P}(\mathcal{E})$ is defined by a surjection $\mathcal{E}(x) \to \kappa_x$, i.e. by a point in the projective space $\mathbb{P}(\mathcal{E}(x))$. This agrees with the description of fibres of a projective bundle from above.

Lemma 7.4.2 *Let $s : S \to \mathbb{P}(\mathcal{E})$ be a section, $\mathcal{L} = s^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and let $\mathcal{K} = \mathrm{Ker}(\mathcal{E} \to \mathcal{L})$. Let us identify S with $s(S)$. Then $\mathcal{K}^\vee \otimes \mathcal{L}$ is isomorphic to the normal sheaf of $s(S)$ in $\mathbb{P}(\mathcal{E})$.*

Proof We use exact sequence (7.9) to compute the normal sheaf. Recall that the sheaf $\Omega^1_{\mathbb{P}^n}$ of regular 1-forms on projective space can be defined by the exact sequence (the *Euler exact sequence*)

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

More generally, we have a similar exact sequence on any projective bundle $\mathbb{P}(\mathcal{E})$ over a scheme S:

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/S} \to p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0. \tag{7.50}$$

Here the homomorphism $p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is equal to the homomorphism $p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ after twisting by -1. Thus

$$\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1) \cong \mathrm{Ker}(p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)). \tag{7.51}$$

Applying s^* to both sides we get

$$\mathcal{K} \cong s^*\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1).$$

Since $\Omega^1_{s(S)/S} = \{0\}$, we get from (7.9)

$$s^*(\mathcal{N}^\vee_{s(S)/\mathbb{P}(\mathcal{E})}) \cong s^*\Omega^1_{\mathbb{P}(\mathcal{E})/S} \cong \mathcal{K} \otimes \mathcal{L}^{-1}.$$

Passing to the duals, we get the formula for the normal sheaf of the section. □

Let us apply this to minimal ruled surfaces $\mathbf{F}_n$. It is known that any locally free sheaf over $\mathbb{P}^1$ is isomorphic to the direct sum of invertible sheaves. Suppose $\mathcal{E}$ is of rank 2. Then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some integers a, b. Since

the projective bundle $\mathbb{P}(\mathcal{E})$ does not change if we tensor $\mathcal{E}$ with an invertible sheaf, we may assume that $a = 0$ and $b = -n \leq 0$ (this corresponds to the normalization taken in [282], Chapter V, §2, Proposition 2.8).

Proposition 7.4.3 *Let $\pi : X \to \mathbb{P}^1$ be a morphism of a nonsingular surface such that all fibres are isomorphic to $\mathbb{P}^1$. Suppose π has a section whose image E satisfies $E^2 = -n$ for some $n \geq 0$. Then $X \cong \mathbf{F}_n$.*

Proof Let $\mathfrak{f}$ be the divisor class of a fiber of π and let $\mathfrak{e}$ be the divisor class of the section E. For any divisor class D on X such that $D \cdot \mathfrak{f} = a$, we obtain $(D - a\mathfrak{e}) \cdot \mathfrak{f} = 0$. If D represents an irreducible curve C, this implies that $\pi(C)$ is a point, and hence C is a fibre. Writing every divisor as a linear combination of irreducible curves, we obtain that any divisor class is equal to $a\mathfrak{f} + b\mathfrak{e}$ for some integers a, b. Let us write $K_X = a\mathfrak{f} + b\mathfrak{e}$. By the adjunction formula, applied to a fiber and the section E, we get

$$-2 = (a\mathfrak{f} + b\mathfrak{e}) \cdot f, \quad -2 + n = (a\mathfrak{f} + b\mathfrak{e}) \cdot \mathfrak{e} = a - 2nb.$$

This gives

$$K_X = (-2 - n)\mathfrak{f} - 2\mathfrak{e}. \tag{7.52}$$

Assume $n > 0$. Consider the linear system $|n\mathfrak{f} + \mathfrak{e}|$. We have

$$(n\mathfrak{f} + \mathfrak{e})^2 = n, \ (n\mathfrak{f} + \mathfrak{e}) \cdot ((-2 - n)\mathfrak{f} - 2\mathfrak{e}) = -2 - n.$$

By Riemann-Roch, $\dim |n\mathfrak{f} + \mathfrak{e}| \geq n + 1$. The linear system $|n\mathfrak{f} + \mathfrak{e}|$ has no base points because it contains the linear system $|n\mathfrak{f}|$ with no base points. Thus it defines a regular map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Since $(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{e} = 0$, it blows down the section E to a point p. Since$(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{f} = a$, it maps fibres to lines passing through p. The degree of the image is $(n\mathfrak{f} + \mathfrak{e})^2 = n$. Thus the image of the map is a surface of degree n equal to the union of lines through a point. It must be a cone over the Veronese curve V_n^1 if $n > 1$ and $\mathbb{P}^2$ if $n = 1$. The map is its minimal resolution of singularities. This proves the assertion in this case.

Assume $n = 0$. We leave it to the reader to check that the linear system $|\mathfrak{f} + \mathfrak{e}|$ maps X isomorphically to a quadric surface in $\mathbb{P}^3$. □

Corollary 7.4.4

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \cong \mathbf{F}_n.$$

Proof The assertion is obvious if $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Assume $n > 0$. Consider the section of $\mathbb{P}(\mathcal{E})$ defined by the surjection

$$\phi : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-n), \tag{7.53}$$

corresponding to the projection to the second factor. Obviously, $\mathcal{N} = \mathrm{Ker}(\phi) \cong \mathcal{O}_{\mathbb{P}^1}$. Applying Lemma 7.4.2, we get

$$\mathcal{N}_{s(\mathbb{P}^1)/\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_{\mathbb{P}^1}(-n).$$

Now, if C is any curve on a surface X, its ideal sheaf is isomorphic to $\mathcal{O}_X(-C)$ and hence the conormal sheaf is isomorphic to $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)$. This easily implies that

$$\mathcal{N}_{C/X} = \mathcal{O}_X(C) \otimes \mathcal{O}_C. \tag{7.54}$$

In particular, we see that the degree of the invertible sheaf $\mathcal{N}_{C/X}$ on the curve C is equal to the self-intersection C^2.

Thus we obtain that the self-intersection of the section s defined by the surjection (7.53) is equal to $-n$. It remains for us for us to apply the previous Proposition. □

7.4.2 Elementary transformations

Let $\pi : \mathbf{F}_n \to \mathbb{P}^1$ be a ruling of $\mathbf{F}_n$ (the unique one if $n \neq 0$). Let $x \in \mathbf{F}_n$ and F_x be the fiber of the ruling containing x. If we blow up x, the proper transform $\bar{F}_x$ of F_x is an exceptional curve of the first kind. We can blow it down to obtain a nonsingular surface X. The projection π induces a morphism $\pi' : X \to \mathbb{P}^1$ with any fiber isomorphic to $\mathbb{P}^1$. Let S_0 be the exceptional section or any section with the self-intersection 0 if $n = 0$ (such a section is of course equal to a fiber of the second ruling of $\mathbf{F}_0$). Assume that $x \notin S_0$. The proper transform $\bar{S}_0$ of S_0 on the blow-up has the self-intersection equal to $-n$, and its image in X has the self-intersection equal to $-n+1$. Applying Proposition 7.4.3, we obtain that $X \cong \mathbf{F}_{n-1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_{n-1}.$$

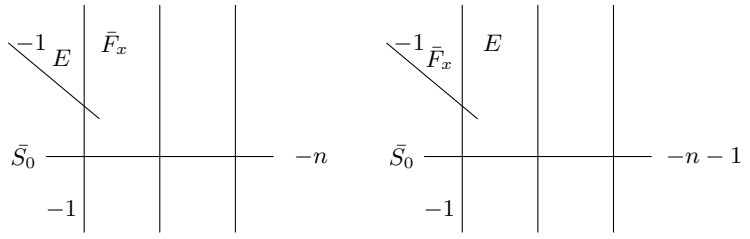


Figure 7.4 Elementary transformation

Here in Figure 7.4, on the left, we blow down $\bar{F}_x$ to obtain $\mathbf{F}_{n-1}$, and, on the right, we blow down $\bar{F}_x$ to obtain $\mathbf{F}_{n+1}$.

Assume that $x \in E_n$. Then the proper inverse transform of S_0 on the blow-up has self-intersection $-n-1$ and its image in X has the self-intersection equal to $-n-1$. Applying Proposition 7.4.3, we obtain that $X \cong \mathbf{F}_{n+1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_{n+1}.$$

A birational map elm_x is called an *elementary transformation*.

Remark 7.4.5 Let $\mathcal{E}$ be a locally free sheaf over a nonsingular curve B. As we explained in Example 7.4.1, a point $x \in \mathbb{P}(\mathcal{E})$ is defined by a surjection $\mathcal{E}(x) \to \kappa(x)$, where $\kappa(x)$ is considered as the structure sheaf of the closed point x. Composing this surjection with the natural surjection $\mathcal{E} \to \mathcal{E}(x)$, we get a surjective morphism of sheaves $\phi_x : \mathcal{E} \to \kappa(x)$. Its kernel $\mathrm{Ker}(\phi_x)$ is a subsheaf of $\mathcal{E}$ which has no torsion. Since the base is a regular 1-dimensional scheme, the sheaf $\mathcal{E}' = \mathrm{Ker}(\phi_x)$ is locally free. Thus we have defined an operation on locally free sheaves. It is also called an elementary transformation.

Consider the special case when $B = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. We have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \xrightarrow{\phi_x} \kappa_x \to 0.$$

The point x belongs to the exceptional section S_0 if and only if ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}(-n) \to \kappa_x$. Then $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)$ and $\mathbb{P}(\mathcal{E}') \cong \mathbf{F}_{n+1}$. The inclusion of sheaves $\mathcal{E}' \subset \mathcal{E}$ gives rise to a rational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathcal{P}(\mathcal{E}')$ which coincides with elm_x. If $x \notin S_0$, then ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}$, and we obtain $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In this case $\mathbb{P}(\mathcal{E}') \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n+1)) \cong \mathbf{F}_{n-1}$ and again, the inclusion $\mathcal{E}' \subset \mathcal{E}$ defines a rational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$ which coincides with elm_x. We refer for this sheaf-theoretical interpretation of elementary transformation to [281]. A more general definition applied to projective bundles over any algebraic variety can be found in [25], [568].

Let $x, y \in \mathbf{F}_n$. Assume that $x \in S_0$, $y \notin S_0$ and $\pi(x) \neq \pi(y)$. Then the composition

$$e_{x,y} = \mathrm{elm}_y \circ \mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_n$$

is a birational automorphism of $\mathbf{F}_n$. Here we identify the point y with its image in $\mathrm{elm}_x(\mathbf{F}_n)$. If $n = 0$, we have to fix one of the two structures of a projective bundle on $\mathbf{F}_0$. Similarly, we get a birational automorphism $e_{y,x} = \mathrm{elm}_y \circ \mathrm{elm}_x$ of $\mathbf{F}_n$. We can also extend this definition to the case when $y \succ_1 x$, where y does not correspond to the tangent direction defined by the fiber passing through x or the exceptional section (or any section with self-intersection 0). We blow up x, then y, and then blow down the proper transform of the fiber

through x and the proper inverse transform of the exceptional curve blown up from x.

7.4.3 Birational automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

Let X be a rational variety and let $\phi : X \dashrightarrow \mathbb{P}^n$ be a birational isomorphism. It defines a homomorphism of the groups of birational automorphisms

$$\mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(X), \quad f \mapsto \phi^{-1} \circ f \circ \phi$$

with the inverse

$$\mathrm{Bir}(X) \to \mathrm{Bir}(\mathbb{P}^n), \quad g \mapsto \phi \circ g \circ \phi^{-1}.$$

Here we realize this simple observation by taking $X = \mathbb{P}^1 \times \mathbb{P}^1$, identified with a nonsingular quadric Q in $\mathbb{P}^3$. We identify $\mathbb{P}^2$ with a plane in $\mathbb{P}^3$ and take $\phi : Q \dashrightarrow \mathbb{P}^2$ to be the projection map p_{x_0} from a point x_0. Let a, b be the images of the two lines on Q containing the point x_0. The inverse map ϕ^{-1} is given by the linear system $|2h - q_1 - q_2|$ of conics through the points q_1, q_2, and a choice of an appropriate basis in the linear system. Let

$$\Phi_{x_0} : \mathrm{Bir}(Q) \to \mathrm{Bir}(\mathbb{P}^2)$$

be the corresponding isomorphism of groups.

A birational automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by a linear system $|mh_1 + kh_2 - \eta|$, where h_1, h_2 are the divisor classes of fibres of the projection maps $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and η is a bubble cycle on Q. If we fix coordinates $(u_0, u_1), (v_0, v_1)$ on each factor of $\mathbb{P}^1 \times \mathbb{P}^1$, then a birational automorphism of the product is given by four bihomogeneous polynomials R_0, R_1, R'_0, R'_1 of bidegree (m, k):

$$([a_0, a_1], [b_0, b_1]) \mapsto ([R_1(a, b), R_2(a, b], [R'_0(a, b), R'_1(a, b)]).$$

Explicitly, let us use an isomorphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \to Q, \quad ([a_0, a_1], [b_0, b_1]) \mapsto [a_0b_0, a_0b_1, a_1b_0, a_1b_1],$$

where $Q = V(z_0z_3 - z_1z_2)$. Take $x_0 = [0, 0, 0, 1]$. The projection map p_{x_0} is given by $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$. The inverse map $p_{x_0}^{-1}$ can be given by the formulas

$$[t_0, t_1, t_2] \mapsto [t_0^2, t_0t_1, t_0t_2, t_1t_2].$$

It is not defined at the points $q_1 = [0, 1, 0]$ and $q_2 = [0, 0, 1]$.

If g is given by R_0, R_1, R'_0, R'_1, then $\Phi_{x_0}(g)$ is given by the formula

$$[z_0, z_1, z_2] \mapsto [R_0(a, b)R'_0(a, b), R_0(a, b)R'_1(a, b), R_1(a, b)R'_0(a, b)],$$

where $[z_0, z_1, z_2] = [a_0b_0, a_0b_1, a_1b_0]$ for some $[a_0, b_0], [b_0, b_1] \in \mathbb{P}^1$.

If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by the polynomials P_0, P_1, P_2, then $\Phi_{x_0}^{-1}(f)$ is given by the formula

$$[z_0, z_1, z_2, z_3] \mapsto [P_0(z')^2, P_0(z')P_1(z'), P_0(z')P_2(z'), P_1(z')P_2(z')], \tag{7.55}$$

where $P_i(z') = P_i(z_0, z_1, z_2)$.

Let $\mathrm{Aut}(Q) \subset \mathrm{Bir}(Q)$ be the subgroup of biregular automorphisms of Q. It contains a subgroup $\mathrm{Aut}(Q)^o$ of index 2 that leaves invariant each family of lines on Q. By acting on each factor of the product $\mathbb{P}^1 \times \mathbb{P}^1$, it becomes isomorphic to the product $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$.

Lemma 7.4.6 *Let $\sigma \in \mathrm{Aut}(Q)^o$. If $\sigma(x_0) \neq x_0$, then $\Phi_{x_0}(\sigma)$ is a quadratic transformation with fundamental points $a, b, p_{x_0}(\sigma^{-1}(x_0))$. If $\sigma(x_0) = x_0$, then $\Phi_{x_0}(\sigma)$ is a projective transformation.*

Proof If $x = \sigma(x_0) \neq x_0$, then the F-locus of $f = \Phi_{x_0}(\sigma)$ consists of three points q_1, q_2 and $p_{x_0}(x)$. It follows from (7.35), that it must be a quadratic transformation. If $\sigma(x_0) = x_0$, then the map f is not defined only at q_1 and q_2. The rational map $\phi : \mathbb{P}^2 \dashrightarrow Q$ can be resolved by blowing up the two points q_1, q_2 followed by blowing down the proper transform of the line $\overline{q_1q_2}$. It is clear that it does not have infinitely near fundamental points. Since any non-projective planar Cremona transformation has at least three fundamental points, we obtain that the map f extends to an automorphism of $\mathbb{P}^2$. □

Remark 7.4.7 The image $\Phi_{x_0}(\mathrm{Aut}(Q)$ consists of quadratic or projective transformations which leave invariant the linear system of conics through two points q_1, q_2. These are complex conics discussed in Subsection 2.2.3. Over reals, when we deal with real conics through the ideal points in the line at infinity, the group $\Phi_{x_0}(\mathrm{Aut}(Q))$ is known as the *Inversive group* in dimension 2 (see [394]).

The subgroup $\Phi_{x_0}(\mathrm{Aut}(Q))$ of $\mathrm{Cr}(2) = \mathrm{Bir}(\mathbb{P}^2)$ is an example of a linear algebraic subgroup of the Cremona group $\mathrm{Cr}(2)$. All such subgroups in $\mathrm{Cr}(2)$ were classified by F. Enriques [213]. In particular, he showed that any linear algebraic subgroup of rank 2 in $\mathrm{Cr}(2)$ is contained in a subgroup isomorphic to $\mathrm{Aut}(\mathbf{F}_n)$ for some n. There is a generalization of this result to the group $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n)$ (see [164]). Instead of minimal ruled surfaces one considers smooth toric varieties of dimension n.

Take two points x, y in Q which do not lie on a line and consider the birational transformation $e_{x,y} := \mathrm{elm}_x \circ \mathrm{elm}_y$ defined in the previous Subsection. Recall that to define $e_{x,y}$, we have to fix one of the two structures of a projective bundle on Q. We do not exclude the case when there is only one proper

point among x and y, say $y \succ x$. It is easy to see that the linear system defining the transformation $e_{x,y}$ is equal to $|2h_1 + h_2 - x - y|$, where h_1 is the class of a fiber of the projective bundle structure $\mathrm{pr} : Q \to \mathbb{P}^1$.

Proposition 7.4.8 *$\Phi_{x_0}(e_{x,y})$ is a product of quadratic transformations. Moreover, if $x_0 \in \{x, y\}$, then $\Phi_{x_0}(t_{x,y})$ is a quadratic transformation. Otherwise, $\Phi_{x_0}(t_{x,y})$ is the product of two quadratic transformations.*

Proof Let $\tau : X \to Q$ be the blow-up of the bubble cycle $x + y$. It factors into the composition of the blow-up $\tau_1 : Q_x \to Q$ of x and the blow-up $\tau_2 : Q' \to Q_x$ of y. Suppose $x_0 \in \{x, y\}$. Without loss of generality, we may assume that $x_0 = x$. The composition of rational maps $\pi = p_{x_0} \circ \tau : Q' \dashrightarrow \mathbb{P}^2$ is a regular map. Let $\alpha : X \to Q$ be the blowing-down of the proper transforms of the fiber ℓ_x (resp. ℓ_y) of $\mathrm{pr} : Q \to \mathbb{P}^1$ containing x (resp. y). The composition $\sigma = p_{x_0} \circ \alpha : Q' \to Q \dashrightarrow \mathbb{P}^2$ is also a regular map. The two morphisms $\pi, \sigma : X \to \mathbb{P}^2$ define a resolution of the birational map $\Phi_{x_0}(e_{x,y})$. It is immediate that this resolution coincides with a resolution of a quadratic transformation with fundamental points $q_1, q_2, p_{x_0}(y)$. Note that, if $y \succ x$, then $p_{x_0}(y) \succ q_2$, where the line ℓ_y is blown down to q_2 under the map $Q'_x \to \mathbb{P}^2$.

If $x_0 \neq x, y$, we compose $e_{x,y}$ with an automorphism g of Q such that $\sigma(x_0) = x$. Then

$$\Phi_{x_0}(e_{x,y} \circ g) = \Phi_{x_0}(e_{x_0,g^{-1}(y)}) = \Phi_{x_0}(e_{x,y}) \circ \Phi_{x_0}(g).$$

By Lemma 7.4.6, $\Phi_{x_0}(g)$ is a quadratic transformation. By the previous case, $\Phi_{x_0}(e_{x_0,\sigma^{-1}(y)})$ is a quadratic transformation. Also the inverse of a quadratic transformation is a quadratic transformation. Thus $\Phi_{x_0}(e_{x,y})$ is a product of two quadratic transformations.

□

Proposition 7.4.9 *Let $T : \mathbf{F}_n \dashrightarrow \mathbf{F}_m$ be a birational map. Assume that T commutes with the projections of the minimal ruled surfaces to $\mathbb{P}^1$. Then T is a composition of biregular maps and elementary transformations.*

Proof Let (X, π, σ) be a resolution of T. The morphism π (resp. σ) is the blow-up of an admissible ordered bubble cycle $\eta = (x_1, \ldots, x_N)$ (resp. $\xi = (y_1, \ldots, y_N)$). Let $p_1 : \mathbf{F}_n \to \mathbb{P}^1$ and $p_2 : \mathbf{F}_m \to \mathbb{P}^1$ be the structure morphisms of the projective bundles. The two composition $p_1 \circ \pi$ and $p_2 \circ \sigma$ coincide and define a map

$$\phi : X \to \mathbb{P}^1.$$

Let $a_1, \ldots, a_k$ be points in $\mathbb{P}^1$ such that $F_i = \phi^{-1}(a_i) = \pi^*(p_1^{-1}(a_i))$ is a

reducible curve. We have $\pi_*(F_i) = p_1^{-1}(a_i)$ and $\sigma_*(F_i) = p_2^{-1}(a_i)$. Let E_i be the unique component of R_i which is mapped surjectively to $p_1^{-1}(a_i)$ and E_i' be the unique component of F_i which is mapped surjectively to $p_2^{-1}(a_i)$. The preimages in X of the maximal points in η and ξ (with respect to the admissible order) are (-1)-curves $E_1, \ldots, E_k$ and $E_1', \ldots, E_{k'}'$. Let E be a (-1)-curve component of F_i that is different from $E_1, \ldots, E_k$ and $E_1', \ldots, E_{k'}'$. We can reorder the order of the blow-ups to assume that $\pi(E) = x_N$ and $\sigma(E) = y_N$. Let $\pi_N : X \to X_{N-1}$ be the blow-up of x_N and $\sigma_N : X \to Y_{N-1}$ be the blow-up of y_N. Since π_N and σ_N blow down the same curve, there exists an isomorphism $\phi : X_{N-1} \cong Y_{N-1}$. Thus, we can replace the resolution (X, π, σ) with

$$(X_{N-1}, \pi_1 \circ \ldots \circ \pi_{N-1}, \sigma_1 \circ \ldots \circ \sigma_{N-1} \circ \phi).$$

Continuing in this way, we may assume that x_N and y_N are the only maximal points of π and σ such that $p_1(x_N) = p_2(y_N) = a_i$. Let $E = \pi^{-1}(x_N)$ and $E' = \sigma^{-1}(y_N)$. Let $R \neq E'$ be a component of $\phi^{-1}(a_i)$ which intersects E. Let $x = \pi(R)$. Since $x_N \succ x$, and no other points is infinitely near x, we get $R^2 = -2$. Blowing down E, we get that the image of R has self-intersection -1. Continuing in this way, we get two possibilities:

(1)
$$F_i = E_i + E_i', \quad E_i^2 = E_i'^2 = -1, E_i \cdot E_i' = 1,$$

(2)
$$F_i = E_i + R_1 + \cdots + R_k + E_i', \quad E_i^2 = E_i'^2 = -1,$$

$$R_i^2 = -1, E_i \cdot R_1 = \ldots = R_i \cdot R_{i+1} = R_k \cdot E_i' = 1,$$

and all other intersections are equal to zero.

In the first case, $T = \mathrm{elm}_{x_N}$. In the second case, let $g : X \to X'$ be the blow-down of E_i, let $x = \pi(R_1 \cap E_i)$. Then $T = T' \circ \mathrm{elm}_x$, where T' satisfies the assumption of the proposition. Continuing in this way, we write T as the composition of elementary transformations. □

Let J be a de Jonquières transformation of degree m with fundamental points $\mathfrak{o}, x_1, \ldots, x_{2m-2}$. We use the notation from Subsection 7.3.6. Let $\pi : X \to \mathbb{P}^2$ be the blow-up of the base points. We factor π as the composition of the blow-up $\pi_1 : X_1 \to X_0 = \mathbb{P}^2$ of the point $\mathfrak{o}$ and the blow-ups $\pi_i : X_{i+1} \to X_i$ of the points x_i. Let $p : X_1 \to \mathbb{P}^1$ be the map given by the pencil of lines through the point $\mathfrak{o}$. The composition $\phi : X \to X_1 \to \mathbb{P}^1$ is a *conic bundle*. This means that its general fiber is isomorphic to $\mathbb{P}^1$ and it has

$2m-2$ singular fibres F_i over the points a_i corresponding to the lines $\ell_i = \overline{\mathfrak{o}x_i}$. Each singular fiber is equal to the union of two (-1)-curves $F_i' + F_i''$ intersecting transversally at one point x_i'. The curve F_i' is the proper transform of the line ℓ_i, and the curve F_i'' is the proper transform of the exceptional curve E_i of the blow-up $X_{i+1} \to X_i, i \geq 1$. The proper transform E of the exceptional curve of $X_1 \to X_0$ is a section of the conic bundle $\phi : X \to \mathbb{P}^1$. It intersects the components F_i'. The proper transform $\overline{\Gamma}$ of the curve Γ is another section. It intersects the components F_i''. Moreover, it intersects E at $2m-2$ points $z_1, \ldots, z_{2m-2}$ corresponding to the common branches of Γ and the proper transform H_m' of the hyperelliptic curve H_m at the point $\mathfrak{o}$. The curve H_m' is a 2-section of the conic bundle (i.e. the restriction of the map ϕ to H_m' is of degree 2).

Recall that the curve Γ and the lines ℓ_i form the P-locus of J. Let $\sigma : X \to \mathbb{P}^2$ be the blow-down of the curves $F_i', \ldots, F_{2m-2}'$ and $\overline{\Gamma}$. The morphisms $\pi, \sigma : X \to \mathbb{P}^2$ define a resolution of the transformation J. We may assume that σ is the composition of the blow-downs $X \to Y_{2m-3} \to \ldots \to Y_1 \to Y_0 = \mathbb{P}^2$, where $Y_1 \to Y_0$ is the blow-down of the image of $\overline{\Gamma}$ under the composition $X \to \ldots \to Y_1$, and $Y_2 \to Y_1$ is the blow-down of the image of F_1' in Y_2.

The surfaces X_1 and Y_1 are isomorphic to $\mathbf{F}_1$. The morphisms $X \to X_1$ and $X \to Y_1$ define a resolution of the birational map $T' : \mathbf{F}_1 \dashrightarrow \mathbf{F}_1$ equal to the composition of $2m-2$ elementary transformations

$$\mathbf{F}_1 \overset{\mathrm{elm}_{x_2'}}{\dashrightarrow} \mathbf{F}_0 \dashrightarrow \mathbf{F}_1 \dashrightarrow \ldots \dashrightarrow \mathbf{F}_0 \overset{\mathrm{elm}_{x_{2m-2}'}}{\dashrightarrow} \mathbf{F}_1.$$

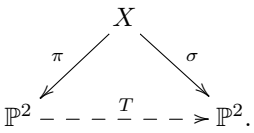
If we take x_0 to be the image of ℓ_1 under $\mathrm{elm}_{x_2'}$, and use it to define the isomorphism $\Phi_{x_0} : \mathrm{Bir}(\mathbf{F}_0) \to \mathrm{Bir}(\mathbb{P}^2)$, then we obtain that $T = \Phi_{x_0}(T')$, where T' is the composition of transformations $e_{x_i', x_{i+1}'} \in \mathrm{Bir}(\mathbf{F}_0)$, where $i = 3, 5, \ldots, 2m-3$. Applying Proposition 7.4.8, we obtain the following.

Theorem 7.4.10 *A de Jonquières transformation is equal to a composition of quadratic transformations.*

7.5 Noether's Factorization Theorem

7.5.1 Characteristic matrices

Consider a resolution (7.1) of a Cremona transformation T of degree d



Obviously, it gives a resolution of the inverse transformation T^{-1}. The roles of π and σ are interchanged. Let

$$\sigma : X = X_M \xrightarrow{\sigma_M} X_{M-1} \xrightarrow{\sigma_{M-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2 \tag{7.56}$$

be the factorization into a sequence of blow-ups similar to the one we had for π. It defines a bubble cycle ξ and the homaloidal net $|d'h - \xi|$ defining T^{-1} (it follows from Subsection 7.13 that $d' = d$). Let $\mathcal{E}'_1, \dots, \mathcal{E}'_M$ be the corresponding exceptional configurations. We will always take for X a minimal resolution. It must be isomorphic to the minimal resolution of the graph of ϕ.

Lemma 7.5.1 *Let $\mathcal{E}_1, \dots, \mathcal{E}_N$ be the exceptional configurations for π and $\mathcal{E}'_1, \dots, \mathcal{E}'_M$ be the exceptional configurations for σ. Then*

$$N = M.$$

Proof Let S be a nonsingular projective surface and $\pi : S' \to S$ be a blow-up of a point. Then the Picard group $\mathrm{Pic}(S')$ is generated by the preimage $\pi^*(\mathrm{Pic}(S))$ and the divisor class $[E]$ of the exceptional curve. Also we know that $[E]$ is orthogonal to any divisor class from $\pi^*(\mathrm{Pic}(S))$ and this implies that

$$\mathrm{Pic}(S') = \mathbb{Z}[E] \oplus \pi^*(\mathrm{Pic}(S)).$$

In particular, taking $S = \mathbb{P}^2$, we obtain, by induction that

$$\mathrm{Pic}(X) = \pi^*(\mathrm{Pic}(\mathbb{P}^2)) \bigoplus \bigoplus_{i=1}^{N} [\mathcal{E}_i].$$

This implies that $\mathrm{Pic}(X)$ is a free abelian group of rank $N + 1$. Replacing π with σ, we obtain that the rank is equal to $1 + M$. Thus $N = M$.

□

It could happen that all exceptional configurations of π are irreducible (i.e.

no infinitely points are used to define π) but some of the exceptional configurations of σ are reducible. This happens in the case of the transformation given in Exercise 7.2.

Definition 7.5.2 *An* ordered resolution *of a Cremona transformation is the diagram* (7.1) *together with an order of a sequence of the exceptional curves for σ and π (equivalently, a choice of an admissible order on the bubble cycles defining π and σ).*

Any ordered resolution of T defines two bases in $\mathrm{Pic}(X)$. The first basis is

$$\underline{e}' : e'_0 = \sigma^*(h),\ e'_1 = [\mathcal{E}'_1], \dots, e'_N = [\mathcal{E}'_N].$$

The second basis is

$$\underline{e} : e_0 = \pi^*(h),\ e_1 = [\mathcal{E}_1], \dots, e_N = [\mathcal{E}_N].$$

Here, as always, h denotes the class of a line in the plane.

We say that a resolution is *minimal* if $e'_j \neq e_i$ for any i, j. If $e'_j = e_i$, then the exceptional configurations $\mathcal{E}_i$ and $\mathcal{E}'_j$ are equal. We can change the admissible orders on the bubble cycles defining the maps π and σ to assume that $i = j = n - b$, where b is the number of irreducible components in $\mathcal{E}_i$, the exceptional divisor of $\pi_{N-i} : X \to X_i$ is equal to $\mathcal{E}_i$ and the exceptional divisor of $\sigma_{Ni} : X \to Y_i$ is equal to $\mathcal{E}'_i$. By the universal property of the blow-up, this implies that there exists an isomorphism $\phi : X_i \to Y_i$ such that $\phi \circ \pi_{Ni} = \sigma_{Nj}$. Thus, we can replace X with X_i and define a new resolution $\pi_{i0} : X_i \to \mathbb{P}^2, \sigma_{i0} \circ \phi : X_i \to \mathbb{P}^2$ of T. The old resolution factors through the new one.

From now on, we assume that we chose a minimal resolution of T. Write

$$e'_0 = de_0 - \sum_{i=1}^{N} m_i e_i, \quad e'_j = d_j e_0 - \sum_{i=1}^{N} m_{ij} e_i,\ j > 0.$$

By the minimality property, we may assume that $d, d_1, \dots, d_N > 0$. The matrix

$$A = \begin{pmatrix} d & d_1 & \dots & d_N \\ -m_1 & -m_{11} & \dots & -m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \dots & -m_{NN} \end{pmatrix} \tag{7.57}$$

is called the *characteristic matrix* of T with respect to an ordered resolution. It is the matrix of change of basis from $\underline{e}$ to $\underline{e}'$.

Here $(d; m_1, \dots, m_N)$ is the characteristic of T. In other columns the vectors $(d_j, m_{1j}, \dots, m_{Nj})$ describe the divisor classes of the exceptional configurations $\mathcal{E}'_j$ of σ. The image of $\mathcal{E}'_j$ in $\mathbb{P}^2$ is a curve in the linear system $|d_j h - \sum_{i=1}^N m_{ij} x_i|$. Its degree is equal to d_j. It may not be irreducible or reduced. Let E be a unique (-1)-component of the exceptional configuration $\mathcal{E}'_j$. It corresponds to a minimal point in the bubble cycle η' infinitely near x_j of order equal to the number of irreducible components of $\mathcal{E}'_j$ minus one. By the minimality assumption, the image $\pi(E)$ is an irreducible curve, and the image $\pi(\mathcal{E}_j)$ contains $\pi(E)$ with multiplicity equal to b_j.

The image of $\mathcal{E}_j$ under the map π is called a *total principal curve* of T. Its degree is equal to d_j. The reduced union of total principal curves is equal to the P-locus of T.

The characteristic matrix defines a homomorphism of free abelian groups

$$\phi_A : \mathbb{Z}^{1+N} \to \mathbb{Z}^{1+N}.$$

We equip $\mathbb{Z}^{1+N}$ with the standard hyperbolic inner product, where the norm v^2 of a vector $v = (a_0, a_1, \dots, a_N)$ is defined by

$$v^2 = a_0^2 - a_1^2 - \cdots - a_N^2.$$

The group $\mathbb{Z}^{1+N}$ equipped with this integral quadratic form is customary denoted by $I^{1,N}$. It is an example of a *quadratic lattice*, a free abelian group equipped with an integral valued quadratic form. We will discuss quadratic lattices in Chapter 8. Since both bases $\underline{e}$ and $\underline{e'}$ are orthonormal with respect to the inner product, we obtain that the characteristic matrix is orthogonal, i.e. belongs to the group $\mathrm{O}(I^{1,N}) \subset \mathrm{O}(1, N)$, where $O(1, N)$ is the real orthogonal group of the hyperbolic space $\mathbb{R}^{1,N}$ with the hyperbolic norm defined by the quadratic form $x_0^2 - x_1^2 - \cdots - x_n^2$.

Recall that the orthogonal group $\mathrm{O}(1, N)$ consists of $(N + 1) \times (N + 1)$ matrices M such that

$$M^{-1} = J_{N+1}\, {}^t M J_{N+1}, \tag{7.58}$$

where J_{N+1} is the diagonal matrix $\mathrm{diag}[1, -1, \dots, -1]$.

In particular, the characteristic matrix A^{-1} of T^{-1} satisfies

$$A^{-1} = J^t A J = \begin{pmatrix} d & m_1 & \dots & m_N \\ -d_1 & -m_{11} & \dots & -m_{N1} \\ \vdots & \vdots & \vdots & \vdots \\ -d_N & -m_{1N} & \dots & -m_{NN} \end{pmatrix}. \tag{7.59}$$

It follows that the vector $(d; d_1, \dots, d_N)$ is equal to the characteristic vector

of T^{-1}. In particular, we obtain again that the degree of T is equal to the degree of T^{-1}, the fact specific to dimension 2. Also, (7.35) implies that $d_1 + \cdots + d_N = d - 3$. This shows that the sum of the degrees of total principal curves of T is equal to the degree of the Jacobian J of the polynomials defining T. This explains the multiplicities of irreducible components of $V(J)$. They are larger than one when not all fundamental points are proper.

Let $f : X' \to X$ be a rational map of irreducible varieties. For any closed irreducible subvariety Z of X' with $X' \cap \mathrm{dom}(f) \neq \emptyset$, we denote by $f(Z)$ the closure of the image of $Z \cap \mathrm{dom}(f)$ under f.

Proposition 7.5.3 *Let $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a Cremona transformation with fundamental points $x_1, \ldots, x_N$ and fundamental points $y_1, \ldots, y_N$ of T^{-1}. Let A be the characteristic matrix of T. Let C be an irreducible curve on $\mathbb{P}^2$ of degree n which passes through the points y_i with multiplicities n_i. Let n' be the degree of $T(C)$ and let n'_i be the multiplicity of $T(C)$ at x_i. Then the vector $v = (n', -n'_1, \ldots, -n'_N)$ is equal to $A^{-1} \cdot v$, where $v = (n, -n_1, \ldots, -n_N)$.*

Proof Let (X, π, σ) be a minimal resolution of T. The divisor class of the proper inverse transform $\pi^{-1}(C)$ in X is equal to $v = ne_0 - \sum n_i e_i$. If we rewrite it in terms of the basis $(e'_0, e'_1, \ldots, e'_N)$ we obtain that it is equal to $v' = n'e_0 - \sum n'_i e_i$, where $v' = Av$. Now the image of $\pi^{-1}(C)$ under σ coincides with $\phi(C)$. By definition of the curves $\mathcal{E}_i$, the curve $\phi^{-1}(C)$ is a curve of degree n' passing through the fundamental points y_i of T^{-1} with multiplicities n'_i. □

Let C be a total principal curve of T and $ce_0 - \sum_{i=1}^N c_i e_i$ be the class of $\pi^{-1}(C)$. Let $v = (c, -c_1, \ldots, -c_N)$. Since $T(C)$ is a point, $A \cdot v = -e'_j$ for some j.

Example 7.5.4 The following matrix is a characteristic matrix of the standard quadratic transformation T_{st} or its degenerations $T_{\mathrm{st}}, T''_{\mathrm{st}}$.

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.60}$$

This follows from Example 7.1.9.

The following is a characteristic matrix of a de Jonquières transformation

$$A = \begin{pmatrix} m & m-1 & 1 & \dots & 1 \\ -m+1 & -m+2 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & 0 \\ -1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & 0 & \dots & 0 \\ -1 & -1 & 0 & \dots & -1 \end{pmatrix}. \tag{7.61}$$

Observe that the canonical class K_X is an element of $\mathrm{Pic}(X)$ which can be written in both bases as

$$K_X = -3e_0 + \sum_{i=1}^{N} e_i = -3e_0' + \sum_{i=1}^{n} e_i'.$$

This shows that the matrix A considered as an orthogonal transformation of $I^{1,N}$ leaves the vector

$$\mathbf{k}_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N = (-3, 1, \dots, 1)$$

invariant. Here, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{Z}^{1+N}$ with $(i+1)$-th coordinate equal to 1 and other coordinates equal to zero.

The matrix A defines an orthogonal transformation of the orthogonal complement $(\mathbb{Z}\mathbf{k}_N)^{\perp}$.

Lemma 7.5.5 *The following vectors form a basis of* $(\mathbb{Z}\mathbf{k}_N)^{\perp}$.

$$N \geq 3 : \alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \alpha_i = \mathbf{e}_{i-1} - \mathbf{e}_i,\ i = 2, \dots, N,$$
$$N = 2 : \alpha_1 = \mathbf{e}_0 - 3\mathbf{e}_1, \quad \alpha_2 = \mathbf{e}_1 - \mathbf{e}_2,$$
$$N = 1 : \alpha_1 = \mathbf{e}_0 - 3\mathbf{e}_1.$$

Proof Obviously, the vectors α_i are orthogonal to the vector $\mathbf{k}_N$. Suppose a vector $v = (a_0, a_1, \dots, a_N) \in (\mathbb{Z}\mathbf{k}_N)^{\perp}$. Thus $3a_0 + \sum_{i=1}^{N} a_i = 0$, hence $-a_N = 3a_0 + \sum_{i=1}^{N-1} a_i$. Assume $N \geq 3$. We can write

$$v = a_0(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) + (a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2) + (2a_0 + a_1 + a_2)(\mathbf{e}_2 - \mathbf{e}_3)$$
$$+ \sum_{i=3}^{N-1} (3a_0 + a_1 + \cdots + a_i)(\mathbf{e}_i - \mathbf{e}_{i+1}).$$

If $N = 2$, we write $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1) + (3a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2)$. If $N = 1$, $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1)$. □

It is easy to compute the matrix $Q_N = (a_{ij})$ of the restriction of the inner product to $(\mathbb{Z}\mathbf{k}_N)^\perp$ with respect to the basis (α_0, α_{N-1}). We have

$$(-8), \quad \text{if } N = 1, \quad \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}, \quad \text{if } N = 2.$$

If $N \geq 3$, we have

$$a_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1 \text{ and } i, j \geq 1, \\ 1 & \text{if } i = 0, j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

For $N \geq 3$ the matrix $A+2I_N$ is the incidence matrix of the graph from Figure 7.5 (the *Coxeter-Dynkin diagram* of type $T_{2,3,N-3}$).

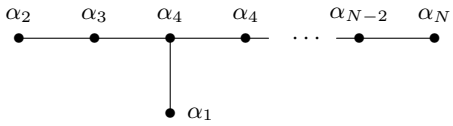


Figure 7.5 Coxeter-Dynkin diagram of type $T_{2,3,N-3}$

For $3 \leq N \leq 8$ this is the Coxeter-Dynkin diagram of the root system of the semi-simple Lie algebra $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$ of type $A_2 + A_1$ if $N = 3$, of $\mathfrak{sl}_5$ of type $\mathfrak{A}_4$ if $N = 4$, of $\mathfrak{so}_{10}$ of type D_5 if $N = 5$ and of the exceptional simple Lie algebra of type E_N if $N = 6, 7, 8$.

We have

$$\mathbf{k}_N^2 = 9 - N.$$

This shows that the matrix Q_N is negative definite if $N < 9$, semi-negative definite with 1-dimensional null-space for $N = 9$, and of signature $(1, N-1)$ for $N \geq 10$. By a direct computation one checks that its determinant is equal to $N - 9$.

Proposition 7.5.6 *Assume $N \leq 8$. There are only finitely many possible characteristic matrices. In particular, there are only finitely many possible characteristics of a homaloidal net with ≤ 8 base points.*

Proof Let

$$G = \{M \in \mathrm{GL}(N) : {}^tMQ_NM = Q_N\}.$$

Since Q_N is negative definite for $N \leq 8$, the group G is isomorphic to the

orthogonal group $\mathrm{O}(N)$. The latter group is a compact Lie group. A characteristic matrix belongs to the subgroup $\mathrm{O}(Q_N) = G \cap \mathrm{GL}(N, \mathbb{Z})$. Since the latter is discrete, it must be finite. □

There are further properties of characteristic matrices for which we refer to [2] for the modern proofs. The most important of these is the following *Clebsch Theorem*.

Theorem 7.5.7 *Let A be the characteristic matrix. There exists a bijection $\beta : \mathbb{N} \to \mathbb{N}$ such that for any set I of columns with $d_i = n, i \in I$, there exists a set of rows J with $\#I = \#J$ such that $\mu_j = \beta(a), j \in J$.*

Note that subtracting two columns (or rows) with the same first entry, and taking the inner product square, we easily get that they differ only at two entries by ± 1. This implies a certain symmetry of the matrix if one reorders the columns and rows according to Clebsch's Theorem. We refer for the details to [2].

7.5.2 The Weyl groups

Let $\mathbf{E}_N = (\mathbb{Z}\mathbf{k}_N)^\perp \cong \mathbb{Z}^N$ equipped with the quadratic form obtained by the restriction of the inner product in $I^{1,N}$. Assume $N \geq 3$. For any vector $\alpha \in \mathbf{E}_N$ with $\alpha^2 = -2$, we define the following element in $\mathrm{O}(E_N)$:

$$r_\alpha : v \mapsto v + (v, \alpha)\alpha.$$

It is called a *reflection* with respect to α. It leaves the orthogonal complement to α pointwisely fixed, and maps α to $-\alpha$.

Definition 7.5.8 *The subgroup $W(\mathbf{E}_N)$ of $\mathrm{O}(\mathbf{E}_N)$ generated by reflections r_{α_i} is called the* Weyl group *of $\mathbf{E}_N$.*

The following Proposition is stated without proof. It follows from the theory of groups generated by reflections (see, for example, [187], 4.3).

Proposition 7.5.9 *The Weyl group $W(\mathbf{E}_N)$ is of infinite index in $\mathrm{O}(\mathbf{E}_N)$ for $N > 10$. For $N \leq 10$,*

$$\mathrm{O}(\mathbf{E}_N) = W(\mathbf{E}_N) \rtimes (\tau),$$

where $\tau^2 = 1$ and $\tau = 1$ if $N = 7, 8$, $\tau = -1$ if $N = 9, 10$ and τ is induced by the symmetry of the Coxeter-Dynkin diagram for $N = 4, 5, 6$.

Note that any reflection can be extended to an orthogonal transformation of the lattice $I^{1,N}$ (use the same formula). The subgroup generated by reflections $r_{\alpha_i}, i \neq 1$, acts as the permutation group $\mathfrak{S}_N$ of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$.

Lemma 7.5.10 *(Noether's inequality) Let* $v = (d, m_1, \ldots, m_N)$. *Assume* $d > 0, m_1 \geq \ldots \geq m_N > 0$, *and*

(i) $\sum_{i=1}^{n} m_i^2 = d^2 + a$;
(ii) $\sum_{i=1}^{N} m_i = 3d - 2 + a$,

where $a \in \{-1, 0, 1\}$. *Then*

$$m_1 + m_2 + m_3 \geq d.$$

Proof We have

$$m_1^2 + \cdots + m_N^2 = d^2 - 1, \quad m_1 + \cdots + m_N = 3d - 3.$$

Multiplying equality (ii) by m_3 and subtracting it from equality (i), we obtain

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 + a - 3m_3(d - \tfrac{2-a}{3}).$$

We can rewrite the previous equality in the form

$$(d - \tfrac{2-a}{3})(m_1 + m_2 + m_3 - d - \tfrac{2-a}{3}) = (m_1 - m_3)(d - \tfrac{2-a}{3} - m_1) +$$

$$(m_2 - m_3)(d - \tfrac{2-a}{3} - m_2) + \sum_{i \geq 4} m_i(m_3 - m_i) + a + (\tfrac{2-a}{3})^2.$$

Note that $\frac{2-a}{3} < 1 \leq d$ unless $a = -1$ when $\frac{2-a}{3} = 1$. In any case, (i) and (ii) give that $d - \frac{2-a}{3} - m_i > 0$. Thus all summands in the right-hand side are positive. In the left-hand side, the factor $d - \frac{2-a}{3}$ is positive unless $d = 1, a = -1$. In the latter case, all $m_i = 0$ contradicting our assumption that $m_N > 0$. Thus we obtain $m_1 + m_2 + m_3 > d + \frac{2-a}{3}$. Since $\frac{2-a}{3} = -\frac{1}{3}$ if it is not positive, this implies $m_1 + m_2 + m_3 > d$. □

Corollary 7.5.11

$$m_1 > d/3.$$

We can apply Noether's Lemma to the case when $v = (d, m_1, \ldots, m_N)$ is the characteristic vector of a homaloidal net or when $d\mathbf{e}_0 - \sum m_i \mathbf{e}_i$ is the class of an exceptional configuration.

Definition 7.5.12 *Let* $v = d\mathbf{e}_0 - \sum_{i=1}^{N} m_i \mathbf{e}_i \in I^{1,N}$. *We say that* v *is of* homaloidal type *(resp.*conic bundle type*,* exceptional type*) if it satisfies conditions (i) and (ii) from the above with* $a = -1$ *(resp.* $a = 0$*, resp.* $a = 1$*). We say that* v *is of* proper *homaloidal (exceptional type) if there exists a Cremona transformation whose characteristic matrix has* v *as the first (resp. second column).*

Lemma 7.5.13 *Let $v = d\mathbf{e}_0 - \sum_{i=1}^n m_i\mathbf{e}_i$ belong to the $W(\mathbb{E}_N)$-orbit of $\mathbf{e}_1$. Then $d \geq 0$. Let $\eta = \sum_{i=1}^N x_i$ be a bubble cycle and $\alpha_\eta : I^{1,N} \to \mathrm{Pic}(Y_\eta)$ be an isomorphism of lattices defined by choosing some admissible order of η. Then $\alpha_\eta(v)$ is an effective divisor.*

Proof The assertion is true for $v = \mathbf{e}_1$. In fact, $\alpha_\eta(v)$ is the divisor class of the first exceptional configuration $\mathcal{E}_1$. Let $w = s_k \circ \cdots \circ s_1 \in W(\mathbb{E}_N)$ be written as the product of simple reflections with minimal possible k. One can show that k is uniquely defined by w. It is called the *length* of w. Let $v = w(\mathbf{e}_1) = (d', m'_1, \ldots, m'_N)$. We prove the assertion by using induction on the length of w. The assertion is obvious if $k = 1$ since $v' = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$ or differs from v by a permutation of the m_i's. Suppose the assertion is true for all w of length $\leq k$. Let w has length $k+1$. Without loss of generality, we may assume that s_{k+1} is the reflection with respect to some root $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. Then $d' = 2d - m_1 - m_2 - m_3 < 0$ implies $4d^2 < (m_1 + m_2 + m_3)^2 \leq 3(m_1^2 + m_2^2 + m_3^2)$, hence $d^2 - m_1^2 - m_2^2 - m_3^2 < -\frac{d^2}{3}$. If $d \geq 2$, this contradicts condition (i) of the exceptional type. If $d = 1$, we check the assertion directly by listing all exceptional types.

To prove the second assertion, we use the Riemann-Roch Theorem applied to the divisor class $\alpha_\eta(v)$. We have $\alpha_\eta(v)^2 = -1, \alpha_\eta(v) \cdot K_{Y_\eta} = -1$, hence $h^0(\alpha_\eta(v)) + h^0(K_{Y_\eta} - \alpha_\eta(v)) \geq 1$. Assume $h^0(K_{Y_\eta} - \alpha_\eta(v)) > 0$. Intersecting $K_Y - \alpha_\eta(v)$ with $e_0 = \alpha_\eta(\mathbf{e}_0)$, we obtain a negative number. However, the divisor class e_0 is nef on Y_η. This shows that $h^0(\alpha_\eta(v)) > 0$ and we are done. □

Lemma 7.5.14 *Let v be a proper homaloidal type. Then it belongs to the $W(\mathbb{E}_N)$-orbit of the vector $\mathbf{e}_0$.*

Proof Let $v = d\mathbf{e}_0 - \sum_{i=1}^N m_i\mathbf{e}_i$ be a proper homaloidal type and η be the corresponding homaloidal bubble cycle. Let $w \in W(\mathbb{E}_N)$ and $\mathbf{v}' = w(v) = d'\mathbf{e}_0 - \sum_{i=1}^N m'_i\mathbf{e}_i$. We have $m'_i = \mathbf{e}_i \cdot v' = w^{-1}(\mathbf{e}_i) \cdot \mathbf{v}$. Since $w^{-1}(\mathbf{e}_i)$ represents an effective divisor on Y_η and v is the characteristic vector of the corresponding homaloidal net, we obtain $w^{-1}(\mathbf{e}_i) \cdot v \geq 0$, hence $m_i \geq 0$.

Obviously, $m_i \geq 0$. We may assume that $v \neq \mathbf{e}_0$, i.e. the homaloidal net has at least three base points. Applying the Noether inequality, we find m_i, m_j, m_k such that $m_i + m_j + m_k > d$. We choose the maximal possible such m_i, m_j, m_k. After reordering, we may assume that $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_N$. Note that this preserves the properness of the homaloidal type since the new order on η is still admissible. Applying the reflection s with respect to the vector $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, we obtain a new homaloidal type $v' = d'\mathbf{e}_0 - \sum_{i=1}^N m'_i\mathbf{e}_i$ with $d' = 2d - m_1 - m_2 - m_3 < d$. As we saw

above, each $m_i \geq 0$. So, we can apply Noether's inequality again until we get $w \in W(\mathbb{E}_N)$ such that the number of nonzero coefficients m_i' of $\mathbf{v}' = w(v)$ is at most 2 (i.e. we cannot apply Noether's inequality anymore). A straightforward computation shows that such vector must be equal to $\mathbf{e}_0$. □

Remark 7.5.15 Observe that the characteristic matrix of a quadratic transformation with fundamental points x_1, x_2, x_3 is the matrix of the reflection s_{α_0} with respect to the vector $\alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. So, the previous Proposition seems to prove that by applying a sequence of quadratic transformation we obtain a Cremona transformation with characteristic vector $(1, 0, \ldots, 0)$. It must be a projective transformation. In other words, any Cremona transformation is the composition of quadratic and projective transformations. This is the content of Noether's Factorization Theorem, which we will prove later in this section. The original proof by Noether was along these lines, where he wrongly presumed that one can always perform a standard quadratic transformation with fundamental points equal to the highest multiplicities, say m_1, m_2, m_3. The problem here is that the three points x_1, x_2, x_3 may not represent the fundamental points of a standard Cremona transformation when one of the following cases happens for the three fundamental points x_1, x_2, x_3 of highest multiplicities:

(i) $x_2 \succ x_1, x_3 \succ x_1$;
(ii) the base ideal in an affine neighborhood of x_1 is equal to (u^2, v^3) (cuspidal case).

Theorem 7.5.16 *Let A be a characteristic matrix of a homaloidal net. Then A belongs to the Weyl group $W(\mathbf{E}_N)$.*

Proof Let $A_1 = (d, -m_1, \ldots, -m_N)$ be the first column of A. Applying the previous lemma, we obtain $w \in W(\mathbb{E}_N)$, identified with a $(N+1) \times (N+1)$-matrix, such that the $w \cdot A_1 = \mathbf{e}_0$. Thus the matrix $A' = w \cdot A$ has the first column equal to the vector $(1, 0, \ldots, 0)$. Since A' is an orthogonal matrix (with respect to the hyperbolic inner product), it must be the block matrix of the unit matrix I_1 of size 1 and an orthogonal matrix O of size $n-1$. Since O has integer entries, it is equal to the product of a permutation matrix P and the diagonal matrix with ± 1 at the diagonal. Since $A \cdot \mathbf{k}_N = \mathbf{k}_N$ and ${}^t w \cdot \mathbf{k}_N = \mathbf{k}_N$, this easily implies that O is the identity matrix I_N. Thus $w \cdot A = I_{N+1}$ and $A \in W(\mathbb{E}_N)$. □

Proposition 7.5.17 *Every vector v in the $W(\mathbb{E}_N)$-orbit of $\mathbf{e}_0$ is a proper homaloidal type.*

Proof Let $v = w(\mathbf{e}_0)$ for some $w \in W(\mathbb{E}_N)$. Write w as the composition of

simple reflections $s_k \circ \cdots \circ s_1$. Choose an open subset U of $(\mathbb{P}^2)^N$ such that an ordered set of points $(x_1, \ldots, x_N) \in U$ satisfies the following conditions:

(i) $x_i \neq x_j$ for $i \neq j$;

(ii) if $s_1 = s_{\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k}$, then x_i, x_j, x_k are not collinear;

(iii) let T be the involutive quadratic transformation with the fundamental points x_i, x_j, x_k and let $(y_1, \ldots, y_N)$ be the set of points with $y_i = x_i, y_j = x_j, y_k = x_k$ and $y_h = T(x_h)$ for $h \neq i, j, k$. Then $(y_1, \ldots, y_N)$ satisfies conditions (i) and (ii) for s_1 is replaced with s_2. Next do it again by taking s_3 and so on.

It is easy to see that in this way U is a non-empty Zariski open subset of $(\mathbb{P}^2)^N$ such that $w(\mathbf{e}_0)$ represents the characteristic vector of a homaloidal net. □

Corollary 7.5.18 *Every vector v in the $W(\mathbb{E}_N)$-orbit of $\mathbf{e}_1$ can be realized as a proper exceptional type.*

Proof Let $v = w(\mathbf{e}_1)$ for some $w \in W(\mathbb{E}_N)$. Then let η be a bubble cycle realizing the homaloidal type $w(\mathbf{e}_0)$ and T be the corresponding Cremona transformation with characteristic matrix A. Then v is its second column, and hence corresponds to the first exceptional configuration $\mathcal{E}'_1$ for ϕ^{-1}. □

7.5.3 Noether-Fano inequality

First we generalize Corollary 7.5.11 to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $T : S \dashrightarrow S'$ be a birational map of surfaces. Let $\pi : X \to S, \sigma : X \to S'$ be its resolution. Let $\mathcal{H}'$ be a linear system on X' without base points. For any $H' \in \mathcal{H}', H \in \mathcal{H}$,

$$\sigma^*(H') \sim \pi^*(H) - \sum_i m_i \mathcal{E}_i$$

where $\mathcal{E}_i$ are the exceptional configurations of the map π. Since $\mathcal{H}'$ has no base points, $\sigma^*(\mathcal{H}')$ has no base points. Thus any divisor $\sigma^*(H')$ intersects non-negatively any curve on X. In particular,

$$\sigma^*(H') \cdot \mathcal{E}_i = -m_i \mathcal{E}_i^2 = m_i \geq 0. \tag{7.62}$$

This can be interpreted by saying that $T^{-1}(H')$ belongs to the linear system $|H - \eta|$, where $\eta = \sum m_i x_i$ is the bubble cycle on S defined by π.

Theorem 7.5.19 *(Noether-Fano inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that $|H + mK_S| \neq \emptyset$ there exists i such that*

$$m_i > m.$$

Moreover, we may assume that x_i is a proper point in S.

Proof We know that $K_X = \pi^*(K_S) + \sum_i \mathcal{E}_i$. Thus we have the equality in $\mathrm{Pic}(X)$

$$\sigma^*(H') + mK_X = (\pi^*(H + mK_S)) + \sum (m - m_i)\mathcal{E}_i.$$

Applying σ_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption cannot be effective. Since $|\pi^*(H + mK_S)| \neq \emptyset$, applying σ_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i (m - m_i)\mathcal{E}_i$. If all $m - m_i$ are non-negative, it is also an effective divisor, and we get a contradiction. Thus there exists i such that $m - m_i < 0$.

The last assertion follows from the fact that $m_i \geq m_j$ if $x_j \succ x_i$. □

Example 7.5.20 Assume $S = S' = \mathbb{P}^2$, $[H] = dh$ and $[H'] = h$. We have $|H + K_{S'}| = |-2h| = \emptyset$. Thus we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d - 3a)h| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.5.11.

Example 7.5.21 Let $S = \mathbf{F}_n$ and $S' = \mathbf{F}_r$ be the minimal rational ruled surfaces. Let $\mathcal{H}' = |\mathfrak{f}'|$, where $\mathfrak{f}'$ is the divisor class of a fiber of the fixed projective bundle structure on S'. The linear system $|\mathfrak{f}'|$ is a pencil without base points. So we can write $\sigma^*(\mathcal{H}') = |\pi^*(a\mathfrak{f} + b\mathfrak{e}) - \eta|$ for some bubble cycle, where $\mathfrak{f}, \mathfrak{e}$ are the divisor classes of a fiber and the exceptional section on S. Here (X, π, σ) is a resolution of T. Thus $\mathcal{H} \subset |a\mathfrak{f} + b\mathfrak{e}|$.

By (**??**),

$$K_S = -(2 + n)\mathfrak{f} - 2\mathfrak{e}, K_{S'} = -(2 + r)\mathfrak{f}' - 2\mathfrak{e}'. \tag{7.63}$$

Thus $|H' + K_{S'}| = |(-1 - n)\mathfrak{f} - 2\mathfrak{e}| = \emptyset$. We take $m_0 = 1$. We have

$$|a\mathfrak{f} + b\mathfrak{e} + mK_S| = |(a - m(2 + n))\mathfrak{f} + (b - 2m)\mathfrak{e}|.$$

Assume that

$$1 < b \leq \frac{2a}{2 + n}.$$

If $m = [b/2]$, then $m \geq m_0$ and both coefficients $a - m(2 + n)$ and $b - 2m$ are non-negative. Thus we can apply Theorem 7.5.19 to find an index i such that $m_i > m \geq b/2$.

In the special case, when $n = 0$, i.e. $S = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality $b \leq a$ implies that there exists i such that $m_i > b/2$.

Similar argument can be also applied to the case $S = \mathbb{P}^2, S' = \mathbb{F}_r$. In this case, $\mathcal{H} = |ah|$ and $|h + mK_S| = |(a - 3m)h|$. Thus, we can take $m = [a/3]$ and find i such that $m_i > a/3$.

7.5.4 Noether's Factorization Theorem

We shall prove the following.

Theorem 7.5.22 *The group* $\mathrm{Bir}(\mathbf{F}_0)$ *is generated by biregular automorphisms and a birational automorphism* $e_{x,y}$ *for some pair of points* x, y.

Applying Proposition 7.4.8, we obtain the following Noether's Factorization Theorem.

Corollary 7.5.23 *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by projective automorphisms and quadratic transformations.*

Now let us prove Theorem 7.5.22. Let $T : \mathbf{F}_n \dashrightarrow \mathbf{F}_m$ be a birational map. Let

$$\mathrm{Pic}(\mathbf{F}_n) = \mathbb{Z}\mathfrak{f} + \mathbb{Z}\mathfrak{e}, \quad \mathrm{Pic}(\mathbf{F}_m) = \mathbb{Z}\mathfrak{f}' + \mathbb{Z}\mathfrak{e}',$$

where we use the notation from the previous Subsection. We have two bases in $\mathrm{Pic}(X)$

$$\underline{e} : \pi^*(\mathfrak{f}),\ \pi^*(\mathfrak{e}),\ e_i = [\mathcal{E}_i],\ i = 1, \dots, N,$$

$$\underline{e'} : \pi^*(\mathfrak{f}'),\ \pi^*(\mathfrak{e}'),\ e'_i = [\mathcal{E}'_i],\ i = 1, \dots, N.$$

For simplicity of notation, let us identify $\mathfrak{f}, \mathfrak{e}, \mathfrak{f}', \mathfrak{e}'$ with their inverse transforms in $\mathrm{Pic}(X)$. Similar to the case of birational maps of projective plane, we can use an ordered resolution (X, π, σ) of T to define its characteristic matrix A.

Lemma 7.5.24 *Let* T *be a quadratic transformation with two (resp. one) proper base points. Then* T *is equal to the composition of two (resp. four or less) quadratic transformations with proper base points.*

Proof Composing the transformation T with a projective transformation, we may assume that T is either T'_{st} or $T = T''_{\mathrm{st}}$ (see Example 7.1.9). In the first case, we compose T with the quadratic transformation T' with fundamental points $[1, 0, 0], [0, 1, 0], [1, 0, 1]$ given by the formula:

$$[t'_0, t'_1, t'_2] = [t_1 t_2, t_1(t_0 - t_2), t_2(t_0 - t_2)].$$

The composition $T' \circ T'_{\mathrm{st}}$ is given by the formula

$$[t'_0, t'_1, t'_2] = [t_0^2 t_1 t_2, t_0 t_2^2 (t_2 - t_0), t_0 t_1 t_2 (t_2 - t_0)] = [t_0 t_1, t_2(t_2 - t_0), t_1(t_2 - t_0)].$$

It is a quadratic transformation with three fundamental points $[0, 1, 0], [1, 0, 0]$, and $[1, 0, 1]$.

In the second case, we let T' be the quadratic transformation

$$[t'_0, t'_1, t'_2] = [t_0 t_1, t_1 t_2, t_2^2]$$

with two proper fundamental points $[1,0,0]$, $[0,1,0]$. The composition $T' \circ T''_{\text{st}}$ is given by

$$[t'_0, t'_1, t'_2] = [t_1^2(t_2^2 - t_0t_1), t_1^2t_2^2, t_1^3t_2] = [t_2^2 - t_0t_1, t_1t_2, t_2^2].$$

It is a quadratic transformation with two proper base points. By the above, T' and $T' \circ T$ are equal to the composition of two quadratic transformations with three proper points. Thus T is a composition of four, or less, quadratic transformations with three proper base points. □

Lemma 7.5.25 *Let $T : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ be a birational automorphism equal to a composition of elementary transformations. Then T is equal to a composition of biregular automorphisms of $\mathbf{F}_0$ and a transformation $e_{x,y}$ for a fixed pair of points x, y, where y is not infinitely near x.*

Proof It follows from Proposition 7.4.8 and the previous Lemma that $e_{x,y}$, where $y \succ_1 x$ can be written as a composition of two transformations of type $e_{x',y'}$ with no infinitely near points. Now notice that the transformations $e_{x,y}$ and $e_{x',y'}$ for different pairs of points differ by an automorphism of $\mathbf{F}_0$ which sends x to x' and y to y'. Suppose we have a composition T of elementary transformations

$$\mathbf{F}_0 \overset{\text{elm}_{x_1}}{\dashrightarrow} \mathbf{F}_1 \overset{\text{elm}_{x_2}}{\dashrightarrow} \dots \overset{\text{elm}_{x_{k-1}}}{\dashrightarrow} \mathbf{F}_1 \overset{\text{elm}_{x_k}}{\dashrightarrow} \mathbf{F}_0.$$

If no $\mathbf{F}_0$ occurs among the surfaces $\mathbf{F}_n$ here, then T is a composition of even number k of elementary transformations preserving the projections to $\mathbb{P}^1$. It is clear that not all points x_i are images of points in $\mathbf{F}_0$ lying on the same exceptional section as x_1. Let x_i be such a point (maybe infinitely near x_1). Then we compose T with e_{x_i,x_1} to obtain a birational map $T' : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ which is a composition of $k-2$ elementary transformations. Continuing in this way we write T as a composition of transformations $e_{x',y'}$.

If $\mathbf{F}_1 \overset{\text{elm}_{x_{i-1}}}{\dashrightarrow} \mathbf{F}_0 \overset{\text{elm}_{x_i}}{\dashrightarrow} \mathbf{F}_1$ occurs, then elm_{x_i} may be defined with respect to another projection to $\mathbb{P}^1$. Then we write this as a composition of the automorphism τ of $\mathbb{P}^1 \times \mathbb{P}^1$ which switches the factors and the elementary transformation with respect to the first projection. Then we repeat this if such $(\mathbf{F}_0, \text{elm}_{x_j})$ occurs again. □

Let $T : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ be a birational transformation. Assume the image of $|\mathfrak{f}|$ is equal to $|a\mathfrak{f} + b\mathfrak{e} - \sum m_x x|$. Applying the automorphism τ, if needed, we may assume that $b \leq a$. Thus, by using Example 7.5.21, we can find a point x with $m_x > b/2$. Composing T with elm_x, we obtain that the image of $|\mathfrak{f}|$ in $\mathbf{F}_1$ is the linear system $|a'\mathfrak{f}' + b\mathfrak{e}' - m_{x'}x' - \sum_{y \neq x'} m_y y|$, where $m_{x'} = b - m_x < m_x$. Continuing in this way, we get a map $T' : \mathbf{F}_0 \dashrightarrow \mathbf{F}_q$ such that the image of

$|\mathfrak{f}|$ is the linear system $|a'\mathfrak{f}' + b\mathfrak{e}' - \sum m_x x|$, where all $m_x \leq b/2$. If $b = 1$, we get all $m_i = 0$. Thus T' is everywhere defined and hence $q = 0$. The assertion of the Theorem is verified.

Assume $b \geq 2$. Since all $m_i \leq b/2$, we must have, by Example 7.5.21,

$$b > \frac{2a'}{2+q}.$$

Since the linear system $|a'f' + bs'|$ has no fixed components, we get

$$(a'\mathfrak{f}' + b\mathfrak{e}') \cdot \mathfrak{e}' = a' - bq \geq 0.$$

Thus $q \leq a'/b < (2+q)/2$, and hence $q \leq 1$. If $q = 0$, we get $b > a'$. Applying τ, we will decrease b and will start our algorithm again until we either arrive at the case $b = 1$, and we are done, or arrive at the case $q = 1$, and $b > 2a'/3$ and all $m_{x'} \leq b/2$.

Let $\pi : \mathbf{F}_1 \to \mathbb{P}^2$ be the blowing down the exceptional section to a point q. Then the image of a fiber $|\mathfrak{f}|$ on $\mathbf{F}_1$ under π is equal to $|h - q|$. Hence the image of our linear system in $\mathbb{P}^2$ is equal to $|a'h - (a'-b)q - \sum_{p \neq q} m'_p p|$. Obviously, we may assume that $a' \geq b$, hence the coefficient at q is non-negative. Since $b > 2a'/3$, we get $a' - b < a'/3$. By Example 7.5.21, there exists a point $p \neq q$ such that $m'_p > a'/3$. Let $\pi(x) = p$ and $\mathcal{E}_1$ be the exceptional curve corresponding to x and s be the exceptional section in $\mathbf{F}_1$. If $x \in S$, the divisor class $\mathfrak{e} - e_1$ is effective and is represented by the proper inverse transform of s in the blow-up of x. Then

$$(a'\mathfrak{f} + b\mathfrak{e} - m'_x e_1 - \sum_{i>1} m'_i e_i) \cdot (\mathfrak{e} - e_1) \leq a' - b - m'_x < 0.$$

This is impossible because the linear system $|a'\mathfrak{f} + b\mathfrak{e} - m_x x - \sum_{y \neq x} y|$ on $\mathbf{F}_1$has no fixed part. Thus x does not lie on the exceptional section. If we apply elm_x, we arrive at $\mathbf{F}_0$ and may assume that the new coefficient at f' is equal to $a' - m'_x$. Since $m'_x > a'/3$ and $a' < 3b/2$, we see that $a' - m'_x < b$. Now we apply the switch automorphism τ to decrease b. Continuing in this way, we obtain that T is equal to a product of elementary transformations and automorphisms of $\mathbf{F}_0$. We finish the proof of Theorem 7.5.22 by applying Lemma 7.5.25.

Applying Lemma 7.4.6, Proposition 7.4.8 and Lemma 7.5.25, we obtain the following.

Corollary 7.5.26 *The group* $\mathrm{Cr}(2)$ *of Cremona transformations of* $\mathbb{P}^2$ *is generated by projective automorphisms and the standard Cremona transformation* T_{st}.

Remark 7.5.27 It is known that for $n > 2$, the *Cremona groups* $\mathrm{Cr}(n) := \mathrm{Bir}(\mathbb{P}^n)$ cannot be generated by the subgroup of projective transformations and a countable set of other transformations. For $n = 3$, this is a classical result of Hilda Hudson [307]. A modern, and different, proof for $n \geq 3$ can be found in [428].

Exercises

7.1 Show that Cremona transformations with reduced 0-dimensional base scheme exist only in dimension 2.

7.2 Find all possible Cremona transformations of $\mathbb{P}^3$ with base scheme equal to the union of skew lines. Describe their P-loci.

7.3 Prove that the base scheme of a Cremona transformation is not a complete intersection of hypersurfaces.

7.4 Let Z be the union of four 4 skew lines in $\mathbb{P}^3$ and two lines intersecting them. Show that the linear system of cubic surfaces through Z defines a Cremona transformation. Find its P-locus, as well as the base scheme and the P-locus of the inverse Cremona transformation.

7.5 A Cremona transformation T of $\mathbb{P}^n$ is called *regularizable* if there exists a rational variety X, a birational morphism $\phi : X \to \mathbb{P}^n$, and an automorphism g of X such that $T = \phi \circ g \circ \phi^{-1}$. Show that any T of finite order in $\mathrm{Cr}(n)$ is regularizable. On the other hand, a general quadratic transformation is not regularizable.

7.6 Consider a minimal resolution X of the standard quadratic transformation T_{st} with three proper base points. Show that T_{st} lifts to an automorphism σ of X. Show that σ has four fixed points and the orbit space $X/(\sigma)$ is isomorphic to the cubic surface with four nodes given by the equation $t_0t_1t_2 + t_0t_1t_3 + t_1t_2t_3 + t_0t_2t_3 = 0$.

7.7 Consider the rational map defined by

$$[t_0, t_1, t_2] \mapsto [t_1t_2(t_0 - t_2)(t_0 - 2t_1), t_0t_2(t_1 - t_2)(t_0 - 2t_1), t_0t_1(t_1 - t_2)(t_0 - t_2)].$$

Show that it is a Cremona transformation and find the Enriques diagram of the corresponding bubble cycle.

7.8 Let C be a plane curve of degree d with a singular point p. Let $\pi : X \to \mathbb{P}^2$ be a sequence of blow-ups which resolves the singularity. Define the bubble cycle $\eta(C, p) = \sum m_i x_i$ as follows: $x_1 = p$ and $m_1 = \mathrm{mult}_p C$, $x_2, \ldots, x_k$ are infinitely near points to p of order 1 such that the proper transform C' of C under the blow-up at p contains these points, $m_i = \mathrm{mult}_{x_i} C'$, $i = 2, \ldots, k$, and so on.

(i) Show that the arithmetic genus of the proper transform of C in X is equal to $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_i m_i(m_i - 1)$.

(ii) Describe the Enriques diagram of $\eta(C, p)$, where $C = V(t_0^{b-a}t_1^a + t_2^b), p = [1, 0, 0]$, and $a \leq b$ are positive integers.

7.9 Show that two hyperelliptic plane curves H_m and H'_m of degree m and genus $m-2$ are birationally isomorphic if and only if there exists a de Jonquières transformation which transforms one curve to another.

7.10 Consider a set of five points $\mathfrak{o}, x_1, x_2, x_3, x_4$ such that the last three points are collinear. Consider a de Joinquières transformation of degreee 3 with fundamental point $\mathfrak{o}$ of multiplicity 2 and simple fundamental points at the points x_i. Show that one of five total principal curves is reducible.

7.11 Let H_{g+2} be a hyperelliptic curve given by Equation (7.46). Consider the linear system of hyperelliptic curves $H_{q+2} = V(t_2^2 g_q(t_0, t_1) + 2t_2 g_{q+1}(t_0, t_1) + g_{q+2}(t_0, t_1))$ such that $f_g g_{q+2} - 2f_{g+1}g_{q+1} + f_{g+2}g_q = 0$. Show that

(i) the curves H_{q+2} exist if $q \geq (g-2)/2$;
(ii) the branch points of H_{g+2} belong to H_{q+2} and vice versa;
(iii) the curve H_{q+2} is invariant with respect to the de Jonquières involution IH_{g+2} defined by the curve H_{g+2} and the curve H_{g+2} is invariant with respect to the de Jonquières involution IH_{q+2} defined by the curve H_{q+2};
(iv) the involutions IH_{g+2} and IH_{q+2} commute with each other;
(v) the fixed locus of the composition $H_{g+2} \circ H_{q+2}$ is given by the equation

$$f_{g+q+3} = \det \begin{pmatrix} f_g & f_{g+1} & f_{g+2} \\ g_q & g_{q+1} & g_{q+2} \\ 1 & -t_2 & t_2^2 \end{pmatrix} = 0;$$

(vi) the de Jonquières transformations that leave the curve H_{g+2} invariant form a group. It contains an abelian subgroup of index 2 that consists of transformations which leave H_{g+2} pointwisely fixed.

7.12 Consider the linear system $L_{a,b} = |af + bs|$ on $\mathbf{F}_n$, where s is the divisor class of the exceptional section, and f is the divisor class of a fibre. Assume $a, b \geq 0$. Show that

(i) $L_{a,b}$ has no fixed part if and only if $a \geq nb$;
(ii) $L_{a,b}$ has no base points if and only if $a \geq nb$;
(iii) Assume $b = 1$ and $a \geq n$. Show that the linear system $L_{a,1}$ maps $\mathbf{F}_n$ in $\mathbb{P}^{2a-n+1}$ onto a surface $X_{a,n}$ of degree $2a - n$;
(iv) show that the surface $X_{a,n}$ is isomorphic to the union of lines $\overline{v_a(x)v_{a-n}(x)}$, where $v_a : \mathbb{P}^1 \to \mathbb{P}^a, v_{2a-n} : \mathbb{P}^1 \to \mathbb{P}^{a-n}$ are the Veronese maps, and $\mathbb{P}^a$ and $\mathbb{P}^{a-n}$ are identified with two disjoint projective subspaces of $\mathbb{P}^{2a-n+1}$.

7.13 Find the automorphism group of the surface $\mathbf{F}_n$.

7.14 Compute characteristic matrices of symmetric Cremona transformations of degree 5,8 and 17.

7.15 Let C be an irreducible plane curve of degree $d > 1$ passing through the points $x_1, \ldots, x_n$ with multiplicities $m_1 \geq \ldots \geq m_n$. Assume that its proper inverse transform under the blowing up the points $x_1, \ldots, x_n$ is a smooth rational curve $\bar{C}$ with $\bar{C}^2 = -1$. Show that $m_1 + m_2 + m_3 > d$.

7.16 Let $(m, m_1, \ldots, m_n)$ be the characteristic vector of a Cremona transformation. Show that the number of fundamental points with $m_i > m/3$ is less than 9.

7.17 Compute the characteristic matrix of the composition $T \circ T'$ of a de Jonquières transformation T with fundamental points $\mathfrak{o}, x_1, x_2, \ldots, x_{2d-2}$ and a quadratic transformation T' with fundamental points $\mathfrak{o}, x_1, x_2$.

7.18 Let $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ be an automorphism of the affine plane given by a formula $(x, y) \to (x + P(y), y)$, where P is a polynomial of degree d in one variable. Consider

σ as a Cremona transformation. Compute its characteristic matrix. In the case $d = 3$ write as a composition of projective transformations and quadratic transformations.

7.19 Show that every Cremona transformation is a composition of the following maps ("links"):

(i) the switch involution $\tau : \mathbf{F}_0 \to \mathbf{F}_0$;
(ii) the blow-up $\sigma : \mathbf{F}_1 \to \mathbb{P}^2$;
(iii) the inverse $\sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbf{F}_1$;
(iv) an elementary transformation $\mathrm{elm}_x : \mathbf{F}_q \dashrightarrow \mathbf{F}_{q\pm 1}$.

7.20 Show that any planar Cremona transformation is a composition of de Jonquières transformations and projective automorphisms.

7.21 Let $x_0 = [0,1] \times [1,0] \in \mathbb{P}^1 \times \mathbb{P}^1$, $y_0 = \tau(t_0)$, where $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the switch of the factors. Show that e_{y_0,x_0} is given by the formula $[u_0,u_1] \times [v_0,v_1] \mapsto [u_0,u_1] \times [u_0v_1, u_2v_0]$. Check that the composition $T = \tau \circ e_{y_0,x_0}$ satisfies $T^3 = \mathrm{id}$.

7.22 Let $\mathcal{P}$ be a linear pencil of plane curves whose general member is a curve of geometric genus 1 and $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be a rational map it defines.

(i) Show that there exist birational morphisms $\pi : X \to \mathbb{P}^2, \phi : X \to \mathbb{P}^1$ with $f = \phi \circ \pi^{-1}$ such that $\phi : X \to \mathbb{P}^1$ is a relatively minimal rational elliptic surface.
(ii) Use the formula for the canonical class of an elliptic surface to show that the divisor class of a fiber is equal to $-mK_X$ for some positive integer m.
(iii) Show that there exists a birational morphism $\sigma : X \to \mathbb{P}^2$ such that the image of the elliptic fibration is an *Halphen pencil* of index m, i.e. a linear pencil of curves of degree $3m$ with nine m-multiple base points (including infinitely near).
(iv) Conclude by deducing *Bertini's Theorem*, which states that any linear pencil of plane elliptic curves can be reduced by a plane Cremona transformation to an Halphen pencil.

7.23 Find all possible characteristic vectors of planar Cremona transformations with $N \leq 8$ base points.

Historical Notes

A comprehensive history of the theory of Cremona transformations can be found in several sources [128], [307], and [535]. Here we give only a brief sketch.

The general study of plane Cremona transformations was first initiated by L. Cremona in his two papers [142] and [143] published in 1863 and 1864. However, examples of birational transformations have been known since the antiquity, for example, the inversion transformation. The example of a quadratic transformation, which we presented in Example 7.3.17 goes back to Poncelet [452], although the first idea of a general quadratic transformation must be credited to C. MacLaurin [374]. It was generally believed that all birational transformations must be quadratic and much work was done in developing

the general theory of quadratic transformations. The first transformation of arbitrary degree was constructed in 1859 by E. de Jonquières in [161], the de Jonquières transformations. His memoir remained unpublished until 1885 although an abstract was published in 1864 [160]. In his first memoir [142] Cremona gives a construction of a general de Jonquières transformation without reference to de Jonquières. We reproduced his construction in Section 7.3.6. Cremona gives the credit to de Jonquières in his second paper. Symmetric transformations of order 5 were first studied by R. Sturm [550], of order 8 by C. Geiser [238], and of order 17 much later by E. Bertini [40]. In his second paper Cremona lays the foundation of the general theory of plane birational transformations. He introduces the notion of fundamental points and principal curves, establishes the equalities (7.35), proves that the numbers of fundamental points of the transformation and its inverse coincide, proves that principal curves are rational and computes all possible characteristic vectors up to degree 10. The notion of a homaloidal linear system was introduced by Cremona later, first for space transformations in [146] and then for plane transformations in [147]. The word "homaloid" means flat and was used by J. Sylvester to mean a linear subspace of a projective space. More generally, it was applied by A. Cayley to rational curves and surfaces. Cremona also introduced the net of isologues and proved that the number of fixed points of a general transformation of degree d is equal to $d+2$. In the special case of de Jonquière transformations this was also done by de Jonquière in [161]. The notion of isologue curves belongs to him as well as the formula for the number of fixed points.

Many special Cremona transformations in $\mathbb{P}^3$ are discussed in Hudson's book [307]. In her words, the most interesting space transformation is the bilinear cubo-cubic transformation with base curve of genus 3 and degree 6. It was first obtained by L. Magnus in 1837 [376]. In modern times bilinear transformations, under the name determinantal transformations, were studied by I. Pan [430], [429] and by G. Gonzales-Sprinberg [251].

The first major result in the theory of plane Cremona transformations after Cremona's work was Noether's Theorem. The statement of the Theorem was guessed by W. Clifford in 1869 [115]. The original proof of M. Noether in [415] based on Noether's inequality contained a gap which we explained in Remark 7.5.15. Independently, J. Rosanes found the same proof and made the same mistake [477]. In [416] Noether tried to correct his mistake, taking into account the presence of infinitely near fundamental points of highest multiplicities where one cannot apply a quadratic transformation. He took into account the case of infinitely near points with different tangent direction but overlooked the cuspidal case. The result was accepted for thirty years, until in 1901, C. Segre pointed out that the cuspidal case was overlooked [516]. In the

same year, G. Castelnuovo [67] gave a complete proof along the same lines as used in this chapter. In 1916, J. Alexander [4] raised objections to Castelnuovo's proof and gives a proof without using de Jonquières transformations [4]. This seems to be a still accepted proof. It is reproduced, for example, in [3].

The characteristic matrices of Cremona transformation were used by S. Kantor [327] and later by P. Du Val [195]. The latter clearly understood the connection to reflection groups. The description of proper homaloidal and exceptional types as orbits of the Weyl groups were essentially known to H. Hudson. There are numerous modern treatments; these started from M. Nagata [408] and culminated in the monograph of M. Alberich-Carramiñana [2]. A modern account of Clebsch's Theorem and its history can be also found there. Theorem 7.5.16 is usually attributed to Nagata, although it was known to S. Kantor and A. Coble.

The original proof of Bertini's Theorem on elliptic pencils discussed in Exercise 7.20 can be found in [40]. The Halphen pencils were studied by G. Halphen in [276]. A modern proof of Bertini's Theorem can be found in [174]. A survey of results about reducing other linear system of plane curves by planar Cremona transformation to linear systems of curves of lower degree can be found in [535] and in [250]. The formalism of bubble spaces originated from the classical notion of infinitely near points first introduced by Yu. Manin [377].

The theory of decomposition of Cremona transformation via composition of elementary birational isomorphisms between minimal ruled surfaces has a vast generalization to higher dimensions under the name *Sarkisov program* (see [131]).

We intentionally omitted the discussion of finite subgroups of the Cremona group $\mathrm{Cr}(2)$; the modern account of this classification and the history can be found in [188].

8

Del Pezzo surfaces

8.1 First properties

8.1.1 Surfaces of degree d in $\mathbb{P}^d$

Recall that a subvariety $X \subset \mathbb{P}^n$ is called *nondegenerate* if it is not contained in a proper linear subspace. All varieties we consider here are assumed to be reduced. Let $d = \deg(X)$. We have the following well-known (i.e., can be found in modern text-books, e.g. [267], [278]) result.

Theorem 8.1.1 *Let X be an irreducible nondegenerate subvariety of $\mathbb{P}^n$ of dimension k and degree d. Then $d \geq n - k + 1$, and the equality holds only in one of the following cases:*

(i) X is a quadric hypersurface;
(ii) X is a Veronese surface V_2^4 in $\mathbb{P}^5$;
(iii) X is a cone over a Veronese surface V_2^4 in $\mathbb{P}^5$;
(iv) X is a rational normal scroll.

Recall that a *rational normal scroll* is defined as follows. Choose k disjoint linear subspaces $L_1, \ldots, L_k$ in $\mathbb{P}^n$ which together span the space. Let $a_i = \dim L_i$. We have $\sum_{i=1}^k a_i = n-k+1$. Consider Veronese maps $sfv_{a_i} : \mathbb{P}^1 \to L_i$ and define $S_{a_1,\ldots,a_k;n}$ to be the union of linear subspaces spanned by the points $v_{a_1}(x), \ldots, v_{a_k}(x)$, where $x \in \mathbb{P}^1$. It is clear that $\dim S_{a_1,\ldots,a_k;n} = k$ and it is easy to see that $\deg S_{a_1,\ldots,a_k;n} = a_1 + \cdots + a_k$ and $\dim S_{a_1,\ldots,a_k;n} = k$. In this notation, it is assumed that $a_1 \leq a_2 \leq \ldots \leq a_k$.

A rational normal scroll $S_{a_1,a_2,n}$ of dimension 2 with $a_1 = a, a_2 = n-1-a$ will be redenoted by $S_{a,n}$. Its degree is $n-1$ and it lies in $\mathbb{P}^n$. For example, $S_{1,3}$ is a nonsingular quadric in $\mathbb{P}^3$ and $S_{0,3}$ is an irreducible quadric cone.

Corollary 8.1.2 *Let S be an irreducible nondegenerate surface of degree d*

in $\mathbb{P}^n$. Then $d \geq n-1$ and the equality holds only in one of the following cases:

(i) X is a nonsingular quadric in $\mathbb{P}^3$;
(ii) X is a quadric cone in $\mathbb{P}^3$;
(iii) X is a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;
(iv) X is a rational normal scroll $S_{a,n} \subset \mathbb{P}^n$.

The del Pezzo surfaces come next. Let X be an irreducible nondegenerate surface of degree d in $\mathbb{P}^d$. A general hyperplane section H of X is an irreducible curve of degree d. Let $p_a = h^1(X, \mathcal{O}_X)$ denote its arithmetic genus. There are two possibilities: $p_a = 0$ or $p_a = 1$. In fact, projecting to $\mathbb{P}^3$ from a general set of $d-3$ nonsingular points, we get an irreducible curve H' of degree 4 in $\mathbb{P}^3$. Taking nine general points in H', we find an irreducible quadric surface Q containing H'. If Q is singular, then its singular point lies outside H'. We assume that Q is nonsingular, the other case is considered similarly. Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be the divisor classes of the two rulings generating $\mathrm{Pic}(Q)$. Then $H' \in |a\mathfrak{f}_1 + b\mathfrak{f}_2|$ with $a, b \geq 0$ and $a + b = \deg H' = 4$. This gives $(a, b) = (3, 1), (1, 3)$, or $(2, 2)$. In the first two cases $p_a(H') = 0$, in the third case $p_a(H') = 1$.

Proposition 8.1.3 *An irreducible nondegenerate surface X of degree d in $\mathbb{P}^d$ with hyperplane sections of arithmetic genus equal to 0 is isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

Proof Obviously, X is a rational surface. Assume that X is embedded in $\mathbb{P}^d$ by a complete linear system, otherwise it is a projection from a surface of the same degree in $\mathbb{P}^{N+1}$. A birational map $f : \mathbb{P}^2 \dashrightarrow X$ is given by a linear system $|mh - \eta|$ for some bubble cycle $\eta = \sum m_i x_i$. By Proposition 7.3.2, we have

$$d = \deg X = m^2 - \sum_{i=1}^{N} m_i^2,$$

$$r = \dim |mh - \eta| \geq \tfrac{1}{2}(m(m+3) - \sum_{i=1}^{n} m_i(m_i + 1)).$$

Since hyperplane sections of X are curves of arithmetic genus 0, we get

$$(m-1)(m-2) = \sum_{i=1}^{N} m_i(m_i - 1).$$

Combining all this together, we easily get

$$r \geq d + 1.$$

Since X is nondegenerate, we must get the equality $r = d + 1$. Thus X is a surface of degree d in $\mathbb{P}^{d+1}$, and we get a contradiction. □

Recall that an irreducible reduced curve of arithmetic genus $p_a = 0$ is a nonsingular rational curve. It follows from the Proposition that every surface X embedded in $\mathbb{P}^n$ by a complete linear system with rational hyperplane sections has degree $n + 1$. By Corollary 8.1.2, it must be either a scroll or a Veronese surface. For example, if we take $m = 4, N = 3, m_1 = m_2 = m_3 = 2$, we obtain a surface of degree 4 in $\mathbb{P}^5$. It is a Veronese surface in disguise. Indeed, if we compose the map with a quadratic transformation with fundamental points at x_1, x_2, x_3, we obtain the result that the image is given by the linear system of conics in the plane, so the image is a Veronese surface. On the hand, if we take $m = 3, N = 1, m_1 = 2$, we get a surface X of degree 5 in $\mathbb{P}^6$. The family of lines through the point x_1 is mapped to a ruling of lines on X, so X is a scroll.

Proposition 8.1.4 *Suppose X is a scroll of degree d in $\mathbb{P}^d, d > 3$, that is not a cone. Then X is a projection of a scroll of degree d in $\mathbb{P}^{d+1}$.*

Proof Projecting a scroll from a point on the surface we get a surface of degree d' in $\mathbb{P}^{d-1}$ satisfying

$$d = kd' + 1, \tag{8.1}$$

where k is the degree of the rational map defined by the projection. Since the image of the projection is a nondegenerate surface, we obtain $d' \geq d - 2$, the only solution is $k = 1$ and $d' = d - 1$. Continuing in this way, we arrive at a cubic surface in $\mathbb{P}^3$. By Proposition 8.1.6, it is a cone, hence it is a rational surface. We will see later, in Chapter 10, that a rational scroll is a projection of a normal rational scroll $S_{q,n}$ of degree $n - 1$ in $\mathbb{P}^n$. □

The classical definition of a del Pezzo surface is the following.

Definition 8.1.5 *A* del Pezzo surface *is a nondegenerate irreducible surface of degree d in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

According to a classical definition (see [526], 4.5.2), a subvariety X is called *normal subvariety* if it is not a projection of a subvariety of the same degree.

Recall that a closed nondegenerate subvariety X of degree d in $\mathbb{P}^n$ is called *linearly normal* if the restriction map

$$r : H^0(\mathbb{P}^n, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(1)) \tag{8.2}$$

is bijective.

The relation between the two definitions is the following one.

Proposition 8.1.6 *Suppose X is a normal nondegenerate subvariety in $\mathbb{P}^n$. Then X is linearly normal. Conversely, if X is linearly normal and normal (i.e. coincides with its normalization), then it is a normal subvariety.*

Proof It is clear that X is nondegenerate if and only if r is injective. If it is not surjective, then linear system $|\mathcal{O}_X(1)|$ embeds X in $\mathbb{P}^m$ with $m > n$ with the image X' of the same degree, and X is a projection of X.

Conversely, suppose the restriction map r is surjective and X is a projection of X' of the same degree. The center of the projection does not belong to X', so the projection is a regular map $p : X' \to X$. We have $p^*\mathcal{O}_X(1) \cong \mathcal{O}_{X'}(1)$. By the projection formula $p_*p^*\mathcal{O}_{X'}(1) \cong \mathcal{O}_X(1) \otimes \pi_*\mathcal{O}_{X'}$. Since X is normal, $p_*\mathcal{O}_{X'} \cong \mathcal{O}_X$ (see [282]). Thus the canonical homomorphism

$$H^0(X, \mathcal{O}_X(1)) \to H^0(X', \mathcal{O}_{X'}(1)) \cong H^0(X, p_*p^*\mathcal{O}_X(1)) \tag{8.3}$$

is bijective. Since r is bijective,

$$\dim H^0(X, \mathcal{O}_X(1)) = \dim H^0(X', \mathcal{O}_{X'}(1)) = n + 1.$$

Since X' is nondegenerate, $\dim H^0(X', \mathcal{O}_{X'}(1)) \geq n + 2$. This contradiction proves the assertion. □

Let $S_d \subset \mathbb{P}^d$ be a del Pezzo surface. Assume $d \geq 4$. As in the proof of Proposition 8.1.4, we project S_d from a general subset of $d - 3$ nonsingular points to obtain a cubic surface S_3 in $\mathbb{P}^3$. Suppose S_3 is a cone over a cubic curve with vertex x_0. A general plane section of S_3 is the union of three concurrent lines. Its preimage in S_4 is the union of four lines passing through the preimage x_0' of x_0. This means that the point x_0' is a singular point of multiplicity 4 equal to the degree of S_4. Clearly, it must be a cone. Proceeding in this way back to S_d, we obtain that S_d is a cone, a possibility which we have excluded. Next assume that S_3 is not a normal surface. We will see later that it must be a scroll. A general hyperplane section of S_4 passing through the center of the projection $S_4 \dashrightarrow S_3$ is a curve of degree 4 and arithmetic genus 1. Its image in S_3 is a curve of degree 3 and arithmetic genus 1. So, it is not a line. The preimage of a general line on S_3 must be a line on S_4. Thus S_4 is a scroll. Going back to S_d, we obtain that S_d is a scroll. This has been also excluded.

Thus, we obtain that a general projection of S_d from a set of $d-3$ nonsingular points is a normal cubic surface.

Let us derive immediate corollaries of this.

Proposition 8.1.7 *The degree d of a del Pezzo surface S_d is less than or equal to 9.*

Proof We follow the original argument of del Pezzo. Let $S_d \dashrightarrow S_{d-1}$ be the projection from a general point $p_1 \in S_d$. It extends to a regular map $S'_d \to S_{d-1}$, where S'_d is the blow-up of p_1. The image of the exceptional curve E_1 of the blow-up is a line ℓ_1 in S'_d. Let $S_{d-1} \to S_{d-2}$ be the projection from a general point in S_{d-1}. We may assume that the projection map $S_d \dashrightarrow S_{d-1}$ is an isomorphism over p_2 and that p_2 does not lie on ℓ_1. Continuing in this way, we arrive at a normal cubic surface S_3, and the images of lines ℓ_1, and so on, will be a set of disjoint lines on S_3. We will see later that a normal cubic surface does not have more than six skew lines. This shows that $d \leq 9$. □

Proposition 8.1.8 *A del Pezzo surface S_d is a normal surface (i.e. coincides with its normalization in the field of rational functions).*

Proof We follow the same projection procedure as in the previous proof. The assertion is true for $d = 3$. The map $S'_4 \to S_3$ is birational map onto a normal surface. Since we may assume that the center p of the projection $S_4 \dashrightarrow S_3$ does not lie on a line, the map is finite and of degree 1. Since S_3 is normal, it must be an isomorphism. In fact, the local ring A of a point $x \in S'_4$ is integral over the local ring A' point of its image x' and both rings have the same fraction field Q. Thus the integral closure of A in Q is contained in the integral closure of A' equal to A'. This shows that A coincides with A'. Thus we see that S_4 is a normal surface. Continuing in this way, we get that $S_5, \ldots, S_d$ are normal surfaces. □

8.1.2 Rational double points

Here we recall without proofs some facts about rational double points (RDP) singularities which we will often use later. The proofs can be found in many sources [16], [458], [441].

Recall that we say that a variety X has *rational singularities* if there exists a resolution of singularities $\pi : Y \to X$ such that $R^i\pi_*\mathcal{O}_Y = 0, i > 0$. One can show that, if there exists one resolution with this property, any resolution of singularities satisfies this property. Also, one can give a local definition of a rational singularity $x \in X$ by requiring that the stalk $(R^i\pi_*\mathcal{O}_Y)_x$ vanishes for $i > 0$. Note that a nonsingular points is, by definition, a rational singularity.

We will be interested in rational singularities of normal algebraic surfaces. Let $\pi : Y \to X$ be a resolution of singularities. We can always choose it to be minimal in the sense that it does not factor nontrivially through another resolution of singularities. This is equivalent to that the fibres of π do not contain (-1)-curves. A minimal resolution always exists and is unique, up to isomorphism. A curve in the fiber $\pi^{-1}(x)$ is called an *exceptional curve*.

Let $Z = \sum n_i E_i$, where $n_i \geq 0$ and E_i are irreducible components of $\pi^{-1}(x)$, called *exceptional components*. We say that Z is a *fundamental cycle* if $Z \cdot E_i \leq 0$ for all E_i and Z is minimal (in terms of order on the set of effective divisors) with this property. A fundamental cycle always exists unique.

Proposition 8.1.9 *The following properties are equivalent:*

(i) x is a rational singularity;

(ii) the canonical maps $\pi^* : H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y)$ are bijective;

(iii) for every curve (not necessarily reduced) Z supported in $\pi^{-1}(x)$, one has $H^1(Z, \mathcal{O}_Z) = 0$;

(iv) for every curve Z supported in $\pi^{-1}(x)$, $p_a(Z) := 1 + \frac{1}{2} Z \cdot (Z + K_Y) \leq 0$.

Recall that the *multiplicity* of a point x on a variety X is the multiplicity of the maximal ideal $\mathfrak{m}_{X,x}$ defined in any text-book in Commutative Algebra. If X is a hypersurface, then the multiplicity is the degree of the first nonzero homogeneous part in the Taylor expansion of the affine equation of X at the point x.

If x is a rational surface singularity, then $-Z^2$ is equal to its multiplicity, and $-Z^2 + 1$ is equal to the embedding dimension of x (the dimension of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$) [16], Corollary 6. It follows that a rational double point is locally isomorphic to a hypersurface singularity, and hence is a Gorenstein singularity. The converse is also true, a rational Gorenstein singularity has multiplicity 2.

Suppose now that x is a rational double point of a normal surface X. Then each exceptional component E satisfies $H^1(E, \mathcal{O}_E) = 0$. This implies that $E \cong \mathbb{P}^1$. Since the resolution is minimal, $E^2 \leq -2$. By the adjunction formula, $E^2 + E \cdot K_Y = -2$ implies $E \cdot K_Y \geq 0$. Let $Z = \sum n_i E_i$ be a fundamental cycle. Then, by (iii) from above,

$$0 = 2 + Z^2 \leq -Z \cdot K_Y = -\sum n_i (E_i \cdot K_Y).$$

This gives $E_i \cdot K_Y = 0$ for every E_i. By the adjunction formula, $E_i^2 = -2$.

Let K_X be a canonical divisor on X. This is a Weyl divisor, the closure of a canonical divisor on the open subset of nonsingular points. Let $\pi^*(K_X)$ be its

preimage on Y. We can write

$$K_Y = \pi^*(K_X) + \Delta,$$

where Δ is a divisor supported in $\pi^{-1}(x)$. Suppose x is a Gorenstein singularity. This means that ω_X is locally free at x, i.e. one can choose a representative of K_X which is a Cartier divisor in an open neighborhood of x. Thus, we can choose a representative of $\pi^*(K_X)$ which is disjoint from $\pi^{-1}(x)$. For any exceptional component E_i, we have

$$0 = K_Y \cdot E_i = E_i \cdot \pi^*(K_X) + E_i \cdot \Delta = E_i \cdot \Delta.$$

It is known that the *intersection matrix* $(E_i \cdot E_j)$ of exceptional components is negative definite [403]. This implies that $\Delta = 0$.

To sum up, we have the following.

Proposition 8.1.10 *Let $\pi : Y \to X$ be a minimal resolution of a rational double point x on a normal surface X. Then each exceptional component of π is a (-2)-curve and $K_Y = \pi^*(K_X)$.*

8.1.3 A blow-up model of a del Pezzo surface

Let us show that a del Pezzo surface satisfies the following properties that we will take for a more general definition of a del Pezzo surface.

Theorem 8.1.11 *Let S be a del Pezzo surface of degree d in $\mathbb{P}^d$. Then all its singularities are rational double points and ω_S^{-1} is an ample invertible sheaf.*

Proof The assertion is true if $d = 3$. It follows from the proof of Proposition 8.1.8 that S is isomorphic to the blow-up of a cubic surface at $d - 3$ nonsingular points. Thus the singularities of S are isomorphic to singularities of a cubic surface which are RDP. In particular, the canonical sheaf ω_S of S is an invertible sheaf.

Let C be a general hyperplane section. It defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(1) \to \mathcal{O}_C(1) \to 0.$$

Tensoring by ω_S, and applying the adjunction formula for C, we obtain an exact sequence

$$0 \to \omega_S \to \omega_S(1) \to \omega_C \to 0.$$

Applying Serre's duality and Proposition 8.1.9, we obtain

$$H^1(S, \omega_S) \cong H^1(S, \mathcal{O}_S) = 0.$$

Since C is an elliptic curve, $\omega_C \cong \mathcal{O}_C$. The exact sequence implies that

$H^0(S, \omega_S(1)) \neq 0$. Let D be an effective divisor defined by a nonzero section of $\omega_S(1)$. By the adjunction formula, its restriction to a general hyperplane section is zero. Thus D is zero. This shows that $\omega_S(1) \cong \mathcal{O}_S$, hence $\omega_S \cong \mathcal{O}_S(-1)$. In particular, $\omega_S^{-1} \cong \mathcal{O}_S(1)$ is ample (in fact, very ample). □

Definition 8.1.12 *A normal algebraic surface S is called a* del Pezzo surface *if its canonical sheaf ω_S is invertible, ω_S^{-1} is ample and all singularities are rational double points.*

By the previous Theorem and by Propositions 8.1.8, a del Pezzo surface of degree d in $\mathbb{P}^d$ is a del Pezzo surface in this new definition. Note that one takes a more general definition of a del Pezzo surface without assuming the normality property (see [459]). However, we will not pursue this.

Let $\pi : X \to S$ be a minimal resolution of singularities of a del Pezzo surface. Our goal is to show that X is a rational surface isomorphic either to a minimal rational surface $\mathbf{F}_0$, or $\mathbf{F}_2$, or is obtained from $\mathbb{P}^2$ by blowing up a bubble cycle of length ≤ 8.

Lemma 8.1.13 *Any irreducible reduced curve C on X with negative self-intersection is either a (-1)-curve or (-2)-curve.*

Proof By the adjunction

$$C^2 + C \cdot K_S = \deg \omega_C = 2 \dim H^1(C, \mathcal{O}_C) - 2 \geq -2.$$

By Proposition 8.1.10, the assertion is true if C is an exceptional curve of the resolution of singularities $\pi : X \to S$. Suppose $\pi(C) = C'$ is a curve. Since $-K_S$ is ample, there exists some $m > 0$ such that $|-mK_S|$ defines an isomorphism of S onto a surface S' in $\mathbb{P}^n$. Thus $|-mK_X|$ defines a morphism $X \to S'$ which is an isomorphism outside the exceptional divisor of π. Taking a general section in $\mathbb{P}^n$, we obtain that $-mK_X \cdot C > 0$. By the adjunction formula, the only possibility is $C^2 = -1$, and $H^1(C, \mathcal{O}_C) = 0$. □

Recall that a divisor class D on a nonsingular surface X is called *nef* if $D \cdot C \geq 0$ for any curve C on X. It is called *big* if $D^2 > 0$. It follows from the proof of the previous Lemma that $-K_X$ is nef and big.

Lemma 8.1.14 *Let X be a minimal resolution of a del Pezzo surface S. Then*

$$H^i(X, \mathcal{O}_X) = 0, \ i \neq 0.$$

Proof Since S has rational double points, by Proposition 8.1.10, the sheaf ω_S is an invertible ample sheaf and

$$\omega_X \cong \pi^*(\omega_S). \tag{8.4}$$

Since, $\omega_S \cong \mathcal{O}_S(-A)$ for some ample divisor A, we have $\omega_X \cong \mathcal{O}_X(-A')$, where $A' = \pi^*(A)$ is nef and big. We write $0 = K_X + A$ and apply Ramanujam's Vanishing Theorem ([453], [355], vol. I, Theorem 4.3.1): for any nef and big divisor D on a nonsingular projective variety X

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0,\ i > 0.$$

□

Theorem 8.1.15 *Let X be a minimal resolution of a del Pezzo surface. Then, either $X \cong \mathbf{F}_0$, or $X \cong \mathbf{F}_2$, or X is obtained from $\mathbb{P}^2$ by blowing up $N \leq 8$ points in the bubble space.*

Proof Let $f : X \to X'$ be a morphism onto a minimal model of X. Since $-K_X$ is nef and big, $K_{X'} = f_*(K_X)$ is not nef but big. It follows from the classification of algebraic surfaces that X' is a minimal ruled surface. Assume X' is not a rational surface. By Lemma 8.1.14, $H^1(X', \mathcal{O}_{X'}) = 0$. If $p : X' \to B$ is a ruling of X', we must have $B \cong \mathbb{P}^1$ (use that the projection $p : X' \to B$ satisfies $p_*\mathcal{O}_{X'} \cong \mathcal{O}_B$ and this defines a canonical injective map $H^1(B, \mathcal{O}_B) \to H^1(X', \mathcal{O}_{X'})$). Thus $X' = \mathbf{F}_n$ or $\mathbb{P}^2$. Assume $X' = \mathbf{F}_n$. If $n > 2$, the proper transform in X of the exceptional section of X' has self-intersection $-r \leq -n < -2$. This contradicts Lemma 8.1.13. Thus $n \leq 2$. If $n = 1$, then composing the map $X' = \mathbf{F}_1 \to \mathbb{P}^2$, we obtain a birational morphism $X \to X' \to \mathbb{P}^2$, so the assertion is verified.

Assume $n = 2$, and the birational morphism $f : X \to X' = \mathbf{F}_2$ is not an isomorphism. Then it is an isomorphism over the exceptional section (otherwise we get a curve on X with self-intersection < -2). Thus, it factors through a birational morphism $f : X \to Y \to \mathbf{F}_2$, where Y is the blow-up of a point $y \in \mathbf{F}_2$ not on the exceptional section. Let $Y \to Y'$ be the blow-down morphism of the proper transform of a fiber of the ruling of $\mathbf{F}_2$ passing through the point y. Then Y' is isomorphic to $\mathbf{F}_1$, and the composition $X \to X' \to Y \to Y' \to \mathbb{P}^2$ is a birational morphism to $\mathbb{P}^2$.

Assume $n = 0$ and $f : X \to \mathbf{F}_2$ is not an isomorphism. Again, we factor f as the composition $X \to Y \to \mathbf{F}_0$, where $Y \to \mathbf{F}_0$ is the blow-up of a point $y \in \mathbf{F}_0$. Blowing down the proper transforms of the lines through y, we get a morphism $Y \to \mathbb{P}^2$ and the composition $X \to Y \to \mathbb{P}^2$.

The last assertion follows from the known behavior of the canonical class under a blow-up. If $\pi : S \to \mathbb{P}^2$ is a birational morphism which is a composition of N blow-ups, then

$$K_X^2 = K_{\mathbb{P}^2}^2 - N = 9 - N. \tag{8.5}$$

Since $K_X^2 > 0$, we obtain $N < 9$.

□

Definition 8.1.16 *The number $d = K_X^2$ is called the* degree *of a del Pezzo surface.*

It is easy to see that it does not depend on a minimal resolution of S. Note that this definition agrees with the definition of the degree of a del Pezzo surface $S \subset \mathbb{P}^d$ in its classical definition. Indeed, let H be a hyperplane section of S, the intersection theory of Cartier divisors show that

$$d = H^2 = \pi^*(H)^2 = p^*(-K_S)^2 = (-K_X)^2 = K_X^2.$$

Suppose S is a nonsingular del Pezzo surface. Since $K_{\mathbf{F}_2}$ is not ample, we obtain the following.

Corollary 8.1.17 *Assume that S is a nonsingular del Pezzo surface. Then $S \cong \mathbf{F}_0$ or is obtained by blowing-up of a bubble cycle in $\mathbb{P}^2$ of ≤ 8 points.*

Definition 8.1.18 *A* weak del Pezzo surface *is a nonsingular surface S with $-K_S$ nef and big.*

So, we see that a minimal resolution of a singular del Pezzo surface is a weak del Pezzo surface. The proof of Theorem 8.1.15 shows that a weak del Pezzo surface is isomorphic to $\mathbf{F}_0, \mathbf{F}_2$ or to the blow-up of a bubble cycle on $\mathbb{P}^2$ that consists of ≤ 8 points.

Remark 8.1.19 Recall that a Fano variety is a nonsingular projective variety X with $-K_X$ ample. A quasi-Fano variety is a nonsingular variety with $-K_X$ big and nef. Thus a nonsingular del Pezzo surface is a Fano variety of dimension 2, and a weak del Pezzo surface is a quasi-Fano variety of dimension 2.

Definition 8.1.20 *A* blowing down structure *on a weak del Pezzo surface S is a composition of birational morphisms*

$$\pi : S = S_N \xrightarrow{\pi_N} S_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2,$$

where each $\pi : S_i \to S_{i-1}$ is the blow-up a point x_i in the bubble space of $\mathbb{P}^2$.

Recall from Section 7.5.1 that a blowing-down structure of a weak del Pezzo surface defines a basis $(e_0, e_1, \dots, e_N)$ in $\mathrm{Pic}(S)$, where e_0 is the class of the full preimage of a line and e_i is the class of the exceptional configurations $\mathcal{E}_i$ defined by the point x_i. We call it a *geometric basis*. As we explained in the previous Chapter, a blowing-down structure defines an isomorphism of free abelian groups

$$\phi : \mathbb{Z}^{N+1} \to \mathrm{Pic}(S) \quad \text{such that } \phi(k_N) = K_S,$$

where $k_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N$. The class e_0 is the full preimage of the class h of a line in the plane, and the classes e_i are the divisor classes of the exceptional configurations $\mathcal{E}_i$. We call such an isomorphism a *geometric marking*.

Definition 8.1.21 *A pair (S, ϕ), where S is a weak del Pezzo surface and ϕ is a marking (resp. geometric marking) $\mathbb{Z}^{N+1} \to \mathrm{Pic}(S)$ is called a* marked *(resp.* geometrically marked*) weak del Pezzo surface.*

The bubble cycle η appearing in a blowing-up model of a weak del Pezzo surface must satisfy some restrictive conditions. Let us find them.

Lemma 8.1.22 *Let X be a nonsingular projective surface with $H^1(X, \mathcal{O}_X) = 0$. Let C be an irreducible curve on X such that $|-K_X - C| \neq \emptyset$ and $C \notin |-K_X|$. Then $C \cong \mathbb{P}^1$.*

Proof We have $-K_X \sim C + D$ for some nonzero effective divisor D, and hence $K_X + C \sim -D \not\sim 0$. This shows that $|K_X + C| = \emptyset$. By Riemann-Roch,

$$0 = h^0(\mathcal{O}_X(K_X + C)) = \tfrac{1}{2}((K_X + C)^2 - (K_X + C) \cdot K_X) + 1$$

$$-h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \geq 1 + \tfrac{1}{2}(C^2 + K_X \cdot C) = h^1(\mathcal{O}_C).$$

Thus $H^1(C, \mathcal{O}_C) = 0$, and, as we noted earlier, this implies that $C \cong \mathbb{P}^1$. □

Proposition 8.1.23 *Let S be a weak del Pezzo surface.*

(i) Let $f : S \to \bar{S}$ be a blowing down of a (-1)-curve E. Then $\bar{S}$ is a weak del Pezzo surface.

(ii) Let $\pi : S' \to S$ be the blowing-up with center at a point x not lying on any (-2)-curve. Assume $K_S^2 > 1$. Then S' is a weak del Pezzo surface.

Proof (i) We have $K_S = f^*(K_{\bar{S}}) + E$, and hence, for any curve C on $\bar{S}$, we have

$$K_{\bar{S}} \cdot C = f^*(K_{\bar{S}}) \cdot f^*(C) = (K_S - E) \cdot f^*(C) = K_S \cdot f^*(C) \leq 0.$$

Also $K_{\bar{S}}^2 = K_S^2 + 1 > 0$. Thus $\bar{S}$ is a weak del Pezzo surface.

(ii) Since $K_S^2 > 1$, we have $K_{S'}^2 = K_S^2 - 1 > 0$. By Riemann-Roch,

$$\dim |-K_{S'}| \geq \tfrac{1}{2}((-K_{S'})^2 - (-K_{S'} \cdot K_{S'})) = K_{S'}^2 \geq 0.$$

Thus $|-K_{S'}| \neq \emptyset$, and hence, any irreducible curve C with $-K_{S'} \cdot C < 0$ must be a proper component of some divisor from $|-K_{S'}|$ (it cannot be linearly equivalent to $-K_{S'}$ because $(-K_{S'})^2 > 0$). Let $E = \pi^{-1}(x)$. We

have $-K_{S'} \cdot E = 1 > 0$. So we may assume that $C \neq E$. Let $\bar{C} = f(C)$. We have

$$-K_{S'} \cdot C = \pi^*(-K_S) \cdot C - E \cdot C = -K_S \cdot \bar{C} - \text{mult}_x(\bar{C}).$$

Since $f_*(K_{S'}) = K_S$ and $C \neq E$, the curve $\bar{C}$ is a proper irreducible component of some divisor from $|-K_S|$. By Lemma 8.1.22, $\bar{C} \cong \mathbb{P}^1$. Thus $\text{mult}_x \bar{C} \leq 1$ and hence $0 > -K_{S'} \cdot C \geq -K_S \cdot \bar{C} - 1$. This gives $-K_S \cdot \bar{C} = 0$ and $x \in \bar{C}$ and hence $\bar{C}$ is a (-2)-curve. Since x does not lie on any (-2)-curve we get a contradiction. □

Corollary 8.1.24 *Let $\eta = \sum_{i=1}^r x_i$ be a bubble cycle on $\mathbb{P}^2$ and S_η be its blow-up. Then S_η is a weak del Pezzo surface if and only if*

(i) $r \leq 8$;
(ii) the Enriques diagram of η is the disjoint union of chains;
(iii) $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of four points;
(iv) $|\mathcal{O}_{\mathbb{P}^2}(2) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of seven points.

Proof The necessity of condition (i) is clear. We know that S does not contain curves with self-intersection < -2. In particular, any exceptional cycle $\mathcal{E}_i$ of the birational morphism $\pi : S \to \mathbb{P}^2$ contains only smooth rational curves E with $E^2 = -1$ or -2. This easily implies that the bubble points corresponding to each exceptional configuration $\mathcal{E}_i$ represent a totally ordered chain. This checks condition (ii).

Suppose (iii) does not hold. Let D be an effective divisor from the linear system $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'|$. We can change the admissible order on η to assume that $\eta' = x_1 + x_2 + x_3 + x_4$. Then the divisor class of the proper transform of D in Y_η is equal to $e_0 - e_1 - e_2 - e_3 - e_4 - \sum_{i \geq 4} m_i e_i$. Its self-intersection is obviously ≤ -3.

Suppose (iv) does not hold. Let $D \in |\mathcal{O}_{\mathbb{P}^2}(2) - \eta'|$. Arguing as above, we find that the divisor class of the proper transform of D is equal to $2e_0 - \sum_{i=1}^7 e_i - \sum_{i \geq 7} m_i e_i$. Its self-intersection is again ≤ -3.

Let us prove the sufficiency. Let $\mathcal{E}_N = \pi_N^{-1}(x_N)$ be the last exceptional configuration of the blow-down $Y_\eta \to \mathbb{P}^2$. It is an irreducible (-1)-curve. Obviously, $\eta' = \eta - x_N$ satisfies conditions (i)-(iv). By induction, we may assume that $S' = S_{\eta'}$ is a weak del Pezzo surface. Applying Proposition 8.1.23, we have to show that x_N does not lie on any (-2)-curve on S'. Condition (ii) implies that it does not lie on any irreducible component of the exceptional configurations $\mathcal{E}_i, i \neq N$. We will show in the next section that any (-2)-curve on a week del Pezzo surface S' of degree ≤ 7 is either blown down to a point under the canonical map $S_{\eta'} \to \mathbb{P}^2$ or equal to the proper inverse transform

of a line through three points, or a conic through five points. If x_N lies on the proper inverse transform of such a line (resp. a conic), then condition (iii) (resp. (iv)) is not satisfied. This proves the assertion. □

A set of bubble points satisfying conditions (i)-(iv) above is called a set of points in *almost general position*.

We say that the points are in *general position* if the following hold:

(i) all points are proper points;
(ii) no three points are on a line;
(iii) no six points on a conic;
(iv) no cubic passes through the points with one of the point being a singular point.

Proposition 8.1.25 *The blow-up of $N \leq 8$ points in $\mathbb{P}^2$ is a del Pezzo surface if and only if the points are in general position.*

8.2 The $\mathbf{E}_N$-lattice

8.2.1 Quadratic lattices

A (quadratic) *lattice* is a free abelian group $M \cong \mathbb{Z}^r$ equipped with a symmetric bilinear form $M \times M \to \mathbb{Z}$. A relevant example of a lattice is the second cohomology group modulo torsion of a compact smooth 4-manifold (e.g. a nonsingular projective surface) with respect to the cup-product. Another relevant example is the Picard group modulo numerical equivalence of a nonsingular projective surface equipped with the intersection pairing.

The values of the symmetric bilinear form will be often denoted by (x, y) or $x \cdot y$. We write $x^2 = (x, x)$. The map $x \mapsto x^2$ is an integer valued quadratic form on M. Conversely, such a quadratic form $q : M \to \mathbb{Z}$ defines a symmetric bilinear form by the formula $(x, y) = q(x + y) - q(x) - q(y)$. Note that $x^2 = 2q(x)$.

Let $M^\vee = \mathrm{Hom}_Z(M, \mathbb{Z})$ and

$$\iota_M : M \to M^\vee, \quad \iota_M(x)(y) = x \cdot y.$$

We say that M is *nondegenerate* if the homomorphism ι_M is injective. In this case the group

$$\mathrm{Disc}(M) = M^\vee / \iota_M(M)$$

is a finite abelian group. It is called the *discriminant group* of M. If we choose a basis to represent the symmetric bilinear form by a matrix A, then the order

of $\mathrm{Disc}(M)$ is equal to $|\det(A)|$. The number $\mathrm{disc}(M) = \det(A)$ is called the *discriminant* of M. A different choice of a basis changes A to ${}^tCAC$ for some $C \in \mathrm{GL}(n, \mathbb{Z})$, so it does not change $\det(A)$. A lattice is called *unimodular* if $|\mathrm{disc}(M)| = 1$.

Tensoring M with reals, we get a real symmetric bilinear form on $M_{\mathbb{R}} \cong \mathbb{R}^r$. We can identify M with an abelian subgroup of the inner product space $\mathbb{R}^r$ generated by a basis in $\mathbb{R}^r$. The Sylvester signature (t_+, t_-, t_0) of the inner product space $M_{\mathbb{R}}$ is called the *signature* of M. We write (t_+, t_-) if $t_0 = 0$. For example, the signature of $H^2(X, \mathbb{Z})/\mathrm{Torsion} \cong \mathbb{Z}^{b_2}$ for a non-singular projective surface X is equal to $(2p_g + 1, b_2 - 2p_g - 1)$, where $p_g = \dim H^0(X, \mathcal{O}_X(K_X))$. This follows from the Hodge Theory (see [267]). The signature on the lattice of divisor classes modulo numerical equivalence $\mathrm{Num}(X) = \mathrm{Pic}(X)/_{\equiv} \cong \mathbb{Z}^{\rho}$ is equal to $(1, \rho - 1)$ (this is called the *Hodge Index Theorem*, see [282], Chapter 5, Theorem 1.9).

Let $N \subset M$ be a subgroup of M. The restriction of the bilinear form to N defines a structure of a lattice on N. We say that N together with this form is a *sublattice* of M. We say that N is of *finite index* m if M/N is a finite group of order m. Let

$$N^{\perp} = \{x \in M : x \cdot y = 0, \forall y \in N\}.$$

Note that $N \subset (N^{\perp})^{\perp}$ and the equality takes place if and only if N is a *primitive sublattice* (i.e. M/N is torsion-free).

We will need the following Lemmas.

Lemma 8.2.1 *Let M be a nondegenerate lattice and let N be its nondegenerate sublattice of finite index m. Then*

$$|\mathrm{disc}(N)| = m^2 |\mathrm{disc}(M)|.$$

Proof Since N is of finite index in M, the restriction homomorphism $M^{\vee} \to N^{\vee}$ is injective. We will identify $M^{\vee}$ with its image in $N^{\vee}$. We will also identify M with its image $\iota_M(M)$ in $M^{\vee}$. Consider the chain of subgroups

$$N \subset M \subset M^{\vee} \subset N^{\vee}.$$

Choose a basis in M, a basis in N, and the dual bases in $M^{\vee}$ and $N^{\vee}$. The inclusion homomorphism $N \to M$ is given by a matrix A and the inclusion $N^{\vee} \to M^{\vee}$ is given by its transpose ${}^tA$. The order m of the quotient M/N is equal to $|\det(A)|$. The order of $N^{\vee}/M^{\vee}$ is equal to $|\det({}^tA)|$. They are equal. Now the chain of lattices from the above has the first and the last quotient of order equal to m, and the middle quotient is of order $|\mathrm{disc}(M)|$. The total quotient $N^{\vee}/N$ is of order $|\mathrm{disc}(N)|$. The assertion follows. □

Lemma 8.2.2 *Let M be a unimodular lattice and N be its nondegenerate primitive sublattice. Then*

$$|\mathrm{disc}(N^\perp)| = |\mathrm{disc}(N)|.$$

Proof Consider the restriction homomorphism $r : M \to N^\vee$, where we identify M with $M^\vee$ by means of ι_M. Its kernel is equal to $N^\perp$. Composing r with the projection $N^\vee/\iota_N(N)$ we obtain an injective homomorphism

$$M/(N + N^\perp) \to N^\vee/\iota_N(N).$$

Notice that $N^\perp \cap N = \{0\}$ because N is a nondegenerate sublattice. Thus $N^\perp + N = N^\perp \oplus N$ is of finite index i in M. Also the sum is orthogonal, so that the matrix representing the symmetric bilinear form on $N \oplus N^\perp$ can be chosen to be a block matrix. We denote the orthogonal direct sum of two lattices M_1 and M_2 by $M_1 \oplus M_2$. This shows that $\mathrm{disc}(N \perp N^\perp) = \mathrm{disc}(N)\mathrm{disc}(N^\perp)$. Applying Lemma 8.2.1, we get

$$\#(M/N \perp N^\perp) = \sqrt{|\mathrm{disc}(N^\perp)||\mathrm{disc}(N)|} \leq \#(N^\vee/N) = |\mathrm{disc}(N)|.$$

This gives $|\mathrm{disc}(N^\perp)| \leq |\mathrm{disc}(N)|$. Since $N = (N^\perp)^\perp$, exchanging the roles of N and $N^\perp$, we get the opposite inequality. □

Lemma 8.2.3 *Let N be a nondegenerate sublattice of a unimodular lattice M. Then*

$$\iota_M(N^\perp) = \mathrm{Ann}(N) := \mathrm{Ker}(r : M^\vee \to N^\vee) \cong (M/N)^\vee.$$

Proof Under the isomorphism $\iota_M : M \to M^\vee$ the image of $N^\perp$ is equal to $\mathrm{Ann}(N)$. Since the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is left exact, applying it to the exact sequence

$$0 \to N \to M \to M/N \to 0,$$

we obtain an isomorphism $\mathrm{Ann}(N) \cong (M/N)^\vee$. □

A morphism of lattices $\sigma : M \to N$ is a homomorphism of abelian groups preserving the bilinear forms. If M is a nondegenerate lattice, then σ is necessarily injective. We say in this case that σ is an *embedding* of lattices. An embedding is called *primitive* if its image is a primitive sublattice. An invertible morphism of lattices is called an *isometry*. The group of isometries of a lattice M to itself is denoted by $\mathrm{O}(M)$ and is called the *orthogonal group* of M.

Let $M_{\mathbb{Q}} := M \otimes \mathbb{Q} \cong \mathbb{Q}^n$ with the symmetric bilinear form of M extended to a symmetric $\mathbb{Q}$-valued bilinear form on $M_{\mathbb{Q}}$. The group $M^\vee$ can be identified with the subgroup of $M_{\mathbb{Q}}$ consisting of vectors v such that $(v, m) \in \mathbb{Z}$ for any

$m \in M$. Suppose that M is nondegenerate lattice. The finite group $\mathrm{Disc}(M)$ can be equipped with a quadratic form defined by

$$q(\bar{x}) = (x, x) \mod \mathbb{Z},$$

where $\bar{x}$ denotes a coset $x + \iota_M(M)$. If M is an *even lattice*, i.e. $m^2 \in 2\mathbb{Z}$ for all $m \in M$, then we take values modulo $2\mathbb{Z}$. The group of automorphisms of $\mathrm{Disc}(M)$ leaving the quadratic form invariant is denoted by $\mathrm{O}(\mathrm{Disc}(M))$.

The proof of the next Lemma can be found in [414].

Lemma 8.2.4 *Let $M \subset N$ be a sublattice of finite index. Then the inclusion $M \subset N \subset N^\vee \subset M^\vee$ defines the subgroup N/M in $\mathrm{Disc}(M) = M^\vee/M$ such that the restriction of the quadratic form of $\mathrm{Disc}(M)$ to it is equal to zero. Conversely, any such subgroup defines a lattice N containing M as a sublattice of finite index.*

The group $\mathrm{O}(M)$ acts naturally on the dual group $M^\vee$ preserving its bilinear form and leaving the subgroup $\iota_M(M)$ invariant. This defines a homomorphism of groups

$$\alpha_M : \mathrm{O}(M) \to \mathrm{O}(\mathrm{Disc}(M)).$$

Lemma 8.2.5 *Let N be a primitive sublattice in a nondegenerate lattice M. Then an isometry $\sigma \in \mathrm{O}(N)$ extends to an isometry of M acting identically on $N^\perp$ if and only if $\sigma \in \mathrm{Ker}(\alpha_N)$.*

8.2.2 The $\mathbf{E}_N$-lattice

Let $I^{1,N} = \mathbb{Z}^{N+1}$ equipped with the symmetric bilinear form defined by the diagonal matrix $\mathrm{diag}(1, -1, \ldots, -1)$ with respect to the standard basis

$$\mathbf{e}_0 = (1, 0, \ldots, 0),\ \mathbf{e}_1 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_N = (0, \ldots, 0, 1)$$

of $\mathbb{Z}^{N+1}$. Any basis defining the same matrix will be called an *orthonormal basis*. The lattice $I^{1,N}$ is a unimodular lattice of signature $(1, N)$.

Consider the special vector in $I^{1,N}$ defined by

$$\mathbf{k}_N = (-3, 1, \ldots, 1) = -3\mathbf{e}_0 + \sum_{i=1}^{N} \mathbf{e}_i. \tag{8.6}$$

We define the $\mathbf{E}_N$*-lattice* as a sublattice of $I^{1,N}$ given by

$$\mathbf{E}_N = (\mathbb{Z}\mathbf{k}_N)^\perp.$$

Since $\mathbf{k}_N^2 = 9 - N$, it follows from Lemma 8.2.2, that $\mathbf{E}_N$ is a negative

definite lattice for $N \leq 8$. Its discriminant group is a cyclic group of order $9-N$. Its quadratic form is given by the value on its generator equal to $-\frac{1}{9-N} \mod \mathbb{Z}$ (or $2\mathbb{Z}$ if N is odd).

Lemma 8.2.6 *Assume $N \geq 3$. The following vectors form a basis of $\mathbf{E}_N$*

$$\boldsymbol{\alpha}_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3,\ \boldsymbol{\alpha}_i = \mathbf{e}_{i-1} - \mathbf{e}_i,\ i = 2, \ldots, N.$$

The matrix of the symmetric bilinear form of $\mathbf{E}_N$ with respect to this basis is equal to

$$C_N = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & -2 & 1 \end{pmatrix}. \tag{8.7}$$

Proof By inspection, each $\boldsymbol{\alpha}_i$ is orthogonal to $\mathbf{k}_N$. Suppose $(a_0, a_1, \ldots, a_N)$ is orthogonal to $\mathbf{k}_N$. Then

$$3a_0 + a_1 + \cdots + a_N = 0. \tag{8.8}$$

We can write this vector as follows

$$(a_0, a_1, \ldots, a_N) = a_0\boldsymbol{\alpha}_1 + (a_0 + a_1)\boldsymbol{\alpha}_2 + (2a_0 + a_1 + a_2)\boldsymbol{\alpha}_3$$

$$+(3a_0 + a_1 + a_2 + a_3)\boldsymbol{\alpha}_4 + \cdots + (3a_0 + a_1 + \cdots + a_{N-1})\boldsymbol{\alpha}_N.$$

We use here that (8.8) implies that the last coefficient is equal to $-a_N$. We leave the computation of the matrix to the reader. □

One can express the matrix C_N by means of the incidence matrix A_N of the following graph with N vertices.

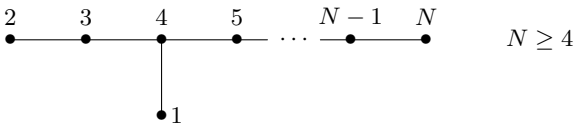


Figure 8.1 Coxeter-Dynkin diagram of type E_N

We have $C_N = -2I_N + A_N$.

8.2.3 Roots

A vector $\alpha \in \mathbf{E}_N$ is called a *root* if $\alpha^2 = -2$. A vector $(d, m_1, \dots, m_N) \in I^{1,N}$ is a root if and only if

$$d^2 - \sum_{i=1}^{N} m_i^2 = -2, \quad 3d - \sum_{i=1}^{N} m_i = 0. \tag{8.9}$$

Using the inequality $(\sum_{i=1}^{N} m_i)^2 \leq N \sum_{i=1}^{N} m_i^2$, it is easy to find all solutions.

Proposition 8.2.7 *Let $N \leq 8$ and*

$$\boldsymbol{\alpha}_{ij} = \mathbf{e}_i - \mathbf{e}_j, 1 \leq i < j \leq N,$$
$$\boldsymbol{\alpha}_{ijk} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k,\ 1 \leq i < j < k \leq N.$$

Any root in $\mathbf{E}_N$ is equal to $\pm\alpha$, where α is one of the following vectors:

N= 3 : $\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{123}$. Their number is 8.
N= 4 :$\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{ijk}$. Their number is 20.
N= 5 : $\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{ijk}$. Their number is 40.
N= 6 :$\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6$. Their number is 72.
N= 7 : $\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_7 - \mathbf{e}_i$. Their number is 126.
N= 8 :$\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 - \mathbf{e}_i - \mathbf{e}_j, 3\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 - \mathbf{e}_i$. Their number is 240.

For $N \geq 9$, the number of roots is infinite. From now on we assume

$$3 \leq N \leq 8.$$

An ordered set B of roots $\{\beta_1, \dots, \beta_r\}$ is called a *root basis* if they are linearly independent over $\mathbb{Q}$ and

$$\beta_i \cdot \beta_j \geq 0.$$

A root basis is called *irreducible* if it is not equal to the union of non-empty subsets B_1 and B_2 such that $\beta_i \cdot \beta_j = 0$ if $\beta_i \in B_1$ and $\beta_j \in B_2$. The symmetric $r \times t$-matrix $C = (a_{ij})$, where $a_{ij} = \beta_i \cdot \beta_j$ is called the *Cartan matrix* of the root basis.

Definition 8.2.8 *A* Cartan matrix *is a symmetric integer matrix (a_{ij}) with $a_{ii} = -2$ and $a_{ij} \geq 0$, or such a matrix multiplied by -1.*

We will deal only with Cartan matrices C with $a_{ii} = -2$. The matrix $C+2I$, where I is the identity matrix of the size equal to the size of C, can be taken as the incidence matrix of a non-oriented graph Γ_C with an ordered set of vertices

in which we put the number $a_{ij} - 2$ at the edge corresponding to vertices i and j if this number is positive. The graph is called the *Coxeter-Dynkin diagram* of C. The Cartan matrix C_N for $N = 6, 7, 8$ has the corresponding graph pictured in Figure 8.2.

Cartan matrix is called *irreducible* if the graph Γ_C is connected.

If C is negative definite irreducible Cartan matrix, then its Coxeter-Dynkin diagram is one of the types indicated in Figure 8.2 (see [52]). A lattice with quadratic form defined by a negative (positive) definite Cartan matrix is called a *root lattice*. Thus the lattice $\mathbf{E}_N, N \leq 8$, is an example of a root lattice.

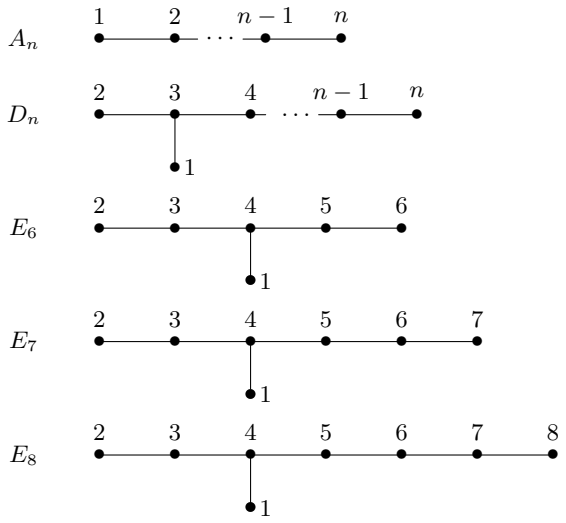


Figure 8.2 Coxeter-Dynkin diagrams of types A,D, E

For $3 \leq n \leq 5$, we will use $\mathbf{E}_n$ to denote the Coxeter-Dynkin diagrams of types $A_2 + A_1 (N = 3)$, $A_4 (N = 4)$ and $D_5 (N = 5)$.

Example 8.2.9 We know that exceptional components E_i of a minimal resolution of a RDP are (-2)-curves. We have already used the fact that the intersection matrix $(E_i \cdot E_j)$ is negative definite. This implies that the intersection matrix is a Cartan matrix.

Proposition 8.2.10 *The Cartan matrix C of an irreducible root basis in $\mathbf{E}_N$ is equal to an irreducible Cartan matrix of type A_r, D_r, E_r with $r \leq N$.*

Definition 8.2.11 *A* canonical root basis *in $\mathbf{E}_N$ is a root basis with Cartan matrix* (8.7) *and the Coxeter-Dynkin diagram from Figure 8.1.*

An example of a canonical root basis is the basis $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$.

Theorem 8.2.12 *Any canonical root basis is obtained from a unique orthonormal basis* $(v_0, v_1, \dots, v_n)$ *in* $I^{1,N}$ *such that* $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$ *by the formula*

$$\beta_1 = v_0 - v_1 - v_2 - v_3,\ \beta_i = v_{i-1} - v_i,\ i = 2, \dots, N. \tag{8.10}$$

Proof Given a canonical root basis $(\beta_1, \dots, \beta_N)$ we solve for v_i in the system of equations (8.10). We have

$$v_i = v_N + \sum_{i=2}^{N} \beta_i,\ i = 1, \dots, N-1,$$

$$v_0 = \beta_1 + v_1 + v_2 + v_3 = \beta_1 + 3v_N + 3\sum_{i=4}^{N} \beta_i + 2\beta_3 + \beta_2,$$

$$-\mathbf{k}_N = 3v_0 - v_1 - \cdots - v_N = 9v_N + 9\sum_{i=4}^{N} \beta_i + 6\beta_3 + 3\beta_2$$

$$-(v_N + \sum_{i=2}^{N} \beta_i) - (v_N + \sum_{i=3}^{N} \beta_i) - \cdots - (v_N + \beta_N) - v_N.$$

This gives

$$v_N = -\frac{1}{9-N}(\mathbf{k}_N + 3\beta_1 + 2\beta_2 + 4\beta_3 + \sum_{i=3}^{N}(9-i)\beta_{i+1}).$$

Taking the inner product of both sides with β_i, we find $(v_N, \beta_i) = 0, i = 1, \dots, N-1$, and $(v_N, \beta_N) = 1$. Thus all v_i belong to $(\mathbf{k}_N \perp \mathbf{E}_N)^\vee$. The discriminant group of this lattice is isomorphic to $(\mathbb{Z}/(9-N)\mathbb{Z})$ and the only isotropic subgroup of order $9-N$ is the diagonal subgroup. This shows that $\mathbf{E}_N^\vee$ is the only sublattice of $(\mathbf{k}_N \perp \mathbf{E}_N)^\vee$ of index $9-N$, hence $v_i \in \mathbf{E}_N^\vee$ for all i. It is immediately checked that $(v_0, v_1, \dots, v_N)$ is an orthonormal basis and $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$. □

Corollary 8.2.13 *Let* $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$ *be the stabilizer subgroup of* $\mathbf{k}_N$. *Then* $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$ *acts simply transitively on the set of canonical root bases in* $\mathbf{E}_N$.

Each canonical root basis $\underline{\beta} = (\beta_1, \dots, \beta_N)$ defines a partition of the set of roots $\mathcal{R}$

$$\mathcal{R} = \mathcal{R}_+ \coprod \mathcal{R}_-,$$

where $\mathcal{R}_+$ is the set of non-negative linear combinations of β_i. The roots from

$\mathcal{R}_+$ ($\mathcal{R}_-$) are called *positive* (*negative*) roots with respect to the root basis $\underline{\beta}$. It is clear that $\mathcal{R}_- = \{-\alpha : \alpha \in \mathcal{R}_+\}$.

For any canonical root basis $\underline{\beta}$, the subset

$$C_{\underline{\beta}} = \{x \in I^{1,N} \otimes \mathbb{R} : (x, \beta_i) \geq 0\}$$

is called a *Weyl chamber* with respect to $\underline{\beta}$. A subset of a Weyl chamber that consists of vectors such that $(v, \beta_i) = 0$ for some subset $I \subset \{1, \ldots, N\}$ is called a *face*. A face corresponding to the empty set is equal to the interior of the Weyl chamber. The face corresponding to the subset $\{1, \ldots, N\}$ is spanned by the vector $\mathbf{k}_N$.

For any root α, let

$$r_\alpha : I^{1,N} \to I^{1,N}, \quad v \mapsto v + (v, \alpha)\alpha.$$

It is immediately checked that $r_\alpha \in \mathrm{O}(I^{1,N})_{\mathbf{k}_N}$, $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(v) = v$ if $(v, \alpha) = 0$. The isometry r_α is called the *reflection* in the root α. By linearity, r_α acts as an orthogonal transformation of the real inner product space $\mathbb{R}^{1,N} := I^{1,N} \otimes \mathbb{R}$.

The following is a basic fact from the theory of finite reflection groups. We refer for the proof to numerous text-books on this subject (e.g. [52], [326]).

Theorem 8.2.14 *Let C be a Weyl chamber defined by a canonical root basis $\underline{\beta}$. Let $W(\mathbf{E}_N)$ be the subgroup of $\mathrm{O}(\mathbf{E}_N)$ generated by reflections r_{β_i}. For any $x \in \mathbb{R}^{1,N}$ there exists $w \in W(\mathbf{E}_N)$ such that $w(x) \in C$. If $x, w(x) \in C$, then $x = w(x)$ and x belongs to a face of C. The union of Weyl chambers is equal to $\mathbb{R}^{1,N}$. Two Weyl chambers intersect only along a common face.*

Corollary 8.2.15 *The group $W(\mathbf{E}_N)$ acts simply transitively on canonical root bases, and Weyl chambers. It coincides with the group $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$.*

The first assertion follows from the Theorem. The second assertion follows from Corollary 8.2.13 since $W(\mathbf{E}_N)$ is a subgroup of $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$.

Corollary 8.2.16

$$\mathrm{O}(\mathbf{E}_N) = W(\mathbf{E}_N) \times \langle \tau \rangle,$$

where τ is an isometry of $\mathbf{E}_N$, which is realized by a permutation of roots in a canonical basis leaving invariant the Coxeter-Dynkin diagram. We have $\tau = 1$ for $N = 7, 8$ and $\tau^2 = 1$ for $N \neq 7, 8$.

Proof By Lemma 8.2.5, the image of the restriction homomorphism

$$\mathrm{O}(I^{1,N})_{\mathbf{k}_N} \to \mathrm{O}(\mathbf{E}_N)$$

is equal to the kernel of the homomorphism $\alpha : \mathrm{O}(\mathbf{E}_N) \to \mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$. It

is easy to compute $\mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$ and find that it is isomorphic to $\mathbb{Z}/\tau\mathbb{Z}$. Also it can be checked that α is surjective and the image of the symmetry of the Coxeter-Dynkin diagram is the generator of $\mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$. It remains for us to apply the previous Corollary. □

The definition of the group $W(\mathbf{E}_N)$ does not depend on the choice of a canonical basis and hence coincides with the definition of Weyl groups $W(\mathbf{E}_N)$ from Chapter 7. Note that Corollary 8.2.15 also implies that $W(\mathbf{E}_N)$ is generated by reflections r_α for all roots α in $\mathbf{E}_N$. This is true for $N \leq 10$ and is not true for $N \geq 11$.

Proposition 8.2.17 *If $N \geq 4$, the group $W(\mathbf{E}_N)$ acts transitively on the set of roots.*

Proof Let $(\beta_1, \dots, \beta_N)$ be a canonical basis from (8.10). Observe that the subgroup of $W(\mathbf{E}_N)$ generated by the reflections with respect to the roots $\beta_2, \dots, \beta_N$ is isomorphic to the permutation group $\mathfrak{S}_N$. It acts on the set $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ by permuting its elements and leaves $\mathbf{e}_0$ invariant. This implies that $\mathfrak{S}_N$ acts on the roots $\alpha_{ij}, \alpha_{ijk}$, via its action on the set of subsets of $\{1, \dots, N\}$ of cardinality 2 and 3. Thus it acts transitively on the set of roots α_{ij} and on the set of roots $\boldsymbol{\alpha}_{ijk}$. Similarly, we see that it acts transitively on the set of roots $2\mathbf{e}_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_6}$ and $-\mathbf{k}_8 - \mathbf{e}_i$ if $N = 8$. Also applying r_α to α we get $-\alpha$. Now the assertion follows from the following computation

$$\begin{aligned} r_{\beta_1}(-\mathbf{k}_8 - \mathbf{e}_8) &= 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_4 - \cdots - \mathbf{e}_8, \\ r_{\beta_1}(2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6) &= \boldsymbol{\alpha}_{456}, \\ r_{\beta_1}(\boldsymbol{\alpha}_{124}) &= \boldsymbol{\alpha}_{34}. \end{aligned}$$

□

A sublattice R of $\mathbf{E}_N$ isomorphic to a root lattice is called a *root sublattice*. By definition, it has a root basis $(\beta_1, \dots, \beta_r)$ such that the matrix $(\beta_i \cdot \beta_j)$ is a Cartan matrix. Each such sublattice is isomorphic to the orthogonal sum of root lattices with irreducible Cartan matrices.

The types of root sublattices in the lattice $\mathbf{E}_N$ can be classified in terms of their root bases by the following procedure due to A. Borel and J. de Siebenthal [48] and, independently by E. Dynkin [198].

Let D be the Coxeter-Dynkin diagram. Consider the extended diagram by adding one more vertex which is connected to other edges as shown on the following *extended Coxeter-Dynkin diagrams*. Consider the following set of elementary operations over the diagrams D and their disconnected sums $D_1 +$

$\cdots + D_k$. Extend one of the components D_i to get the extended diagram. Consider its subdiagram obtained by deleting subset of vertices. Now all possible root bases are obtained by applying recursively the elementary operations to the initial Coxeter-Dynkin diagram of type $\mathbf{E}_N$ and all its descendants.

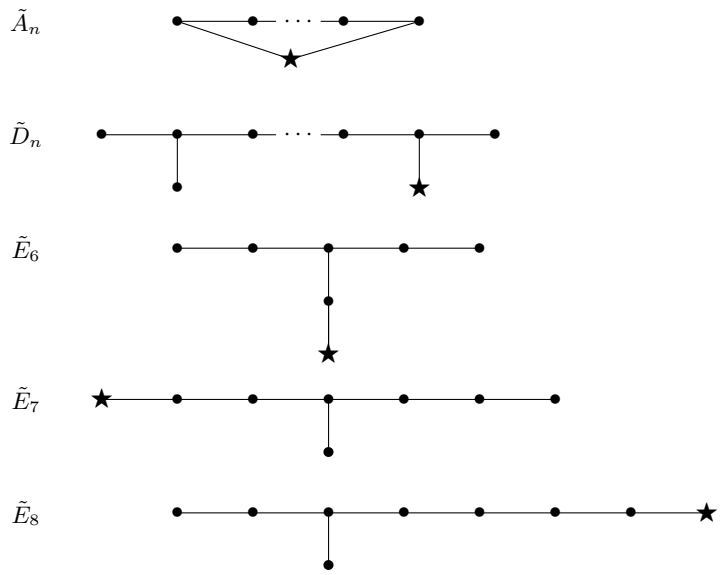


Figure 8.3 Extended Coxeter-Dynkin diagrams of types $\tilde{A}, \tilde{D}, \tilde{E}$

8.2.4 Fundamental weights

Let $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ be a canonical root basis (8.10) in $\mathbf{E}_N$. Consider its dual basis $(\beta_1^*, \ldots, \beta_N^*)$ in $\mathbf{E}_N^\vee \otimes \mathbb{Q}$. Its elements are called *fundamental weights*. We use the expressions for β_i from Theorem 8.2.12. Let us identify $\mathbf{E}_N^\vee$ with $(\mathbf{k}_N^\perp)^\vee = I^{1,N}/\mathbb{Z}\mathbf{k}_N$. Then we can take for representatives of β_j^* the following vectors from $I^{1,N}$:

$$\begin{aligned}
\beta_1^* &= v_0,\\
\beta_2^* &= v_0 - v_1,\\
\beta_3^* &= 2v_0 - v_1 - v_2,\\
\beta_i^* &= v_i + \cdots + v_N,\ i = 4, \ldots, N.
\end{aligned}$$

Definition 8.2.18 *A vector in* $I^{1,N}$ *is called an* exceptional vector *if it belongs to the* $W(\mathbf{E}_N)$*-orbit of* β_N^*.

Proposition 8.2.19 *A vector $v \in I^{1,N}$ is exceptional if and only if $\mathbf{k}_N \cdot v = -1$ and $v^2 = -1$. The set of exceptional vectors is the following*

$$N = 3, 4 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j;$$
$$N = 5 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_5;$$
$$N = 6 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6 + e_i;$$
$$N = 7 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_7 + \mathbf{e}_i + \mathbf{e}_j; -\mathbf{k}_7 - \mathbf{e}_i;$$
$$N = 8 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 + \mathbf{e}_i + \mathbf{e}_j + e_k; -\mathbf{k}_8 + \mathbf{e}_i - \mathbf{e}_j;$$
$$-\mathbf{k}_8 + \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k, -\mathbf{k}_8 + 2e_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_6}, -2\mathbf{k}_8 - e_i.$$

The number of exceptional vectors is given by Table 8.1.

N	3	4	5	6	7	8
#	6	10	16	27	56	240

Table 8.1 *Number of exceptional vectors*

Proof Similarly to the case of roots, we solve the equations

$$d^2 - \sum_{i=1}^{N} m_i^2 = -1,\ 3d - \sum_{i=1}^{N} m_i = 1.$$

First we immediately get the inequality $(3d-1)^2 \leq N(d^2+1)$ which gives $0 \leq d \leq 4$. If $d = 0$, the condition $\sum m_i^2 = d^2 + 1$ and $k_N \cdot v = -1$ gives the vectors $\mathbf{e}_i$. If $d = 1$, this gives the vectors $\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$, and so on. Now we use the idea of Noether's inequality from Chapter 7 to show that all these vectors $(d, m_1, \ldots, m_N)$ belong to the same orbit of $W(\mathbf{E}_N)$. We apply permutations from $\mathfrak{S}_N$ to assume $m_1 \geq m_2 \geq m_3$, then use the reflection $r_{\alpha_{123}}$ to decrease d. □

Corollary 8.2.20 *The orders of the Weyl groups $W(\mathbf{E}_N)$ are given by Table 8.2.*

N	3	4	5	6	7	8
$\#W(\mathbf{E}_N)$	12	5!	$2^4 \cdot 5!$	$2^3 \cdot 3^2 \cdot 6!$	$2^6 \cdot 3^2 \cdot 7!$	$2^7 \cdot 3^3 \cdot 5 \cdot 8!$

Table 8.2 *Orders of the Weyl groups*

Proof Observe that the orthogonal complement of $\mathbf{e}_N$ in $I^{1,N}$ is isomorphic to I^{N-1}. Since $\mathbf{e}_N^2 = -1$, by Lemma 8.2.5, the stabilizer subgroup of $\mathbf{e}_N$ in

$\mathrm{O}(I^{1,N})$ is equal to $\mathrm{O}(I^{1,N-1})$. This implies that the stabilizer subgroup of $\mathbf{e}_N$ in $W(\mathbf{E}_N)$ is equal to $W(\mathbf{E}_{N-1})$. Obviously, $W(\mathbf{E}_3) \cong \mathfrak{S}_3 \times \mathfrak{S}_2$ and $W(\mathbf{E}_4) \cong \mathfrak{S}_5$. Thus

$$\begin{aligned}
\#W(\mathbf{E}_5) &= 16 \cdot \#W(\mathbf{E}_4) = 2^4 \cdot 5!,\\
\#W(\mathbf{E}_6) &= 27 \cdot \#W(\mathbf{E}_5) = 2^3 \cdot 3^2 \cdot 6!,\\
\#W(\mathbf{E}_7) &= 56 \cdot \#W(\mathbf{E}_6) = 2^6 \cdot 3^2 \cdot 7!,\\
\#W(\mathbf{E}_8) &= 240 \cdot \#W(\mathbf{E}_7) = 2^7 \cdot 3^3 \cdot 5 \cdot 8!.
\end{aligned}$$

□

Proposition 8.2.21 *Let $N \leq 8$. For any two different exceptional vectors $v, w \in \mathbf{E}_N$, such that $v + w + 2\mathbf{k}_8 \neq 0$,*

$$(v, w) \in \{0, 1, 2\}.$$

Proof This can be seen directly from the list, however we prefer to give a proof independent of the classification. Obviously, we may assume that $n = 8$. It is immediately seen that all vectors $\mathbf{e}_i$ are exceptional. Because $(v, \mathbf{k}_8) = (w, \mathbf{k}_8)$, we have $v - w \in \mathbf{E}_8$. Because $\mathbf{E}_8$ is a negative definite even lattice we have $(v-w, v-w) = -2-2(v, w) \leq -2$. This gives $(v, w) \geq 0$. Assume $(v, w) > 2$. Let $h = 2\mathbf{k}_8 + v + w$. We have $(v+w)^2 = -2+2(v, w) \geq 4$ and $h^2 = 4-8+(v+w)^2 \geq 0,\ h\cdot\mathbf{k}_8 = 0$. Thus $I^{1,8}$ contains two non-proportional orthogonal nonzero vectors h and $\mathbf{k}_8$ with non-negative norm square. Since the signature of $I^{1,N}$ is equal to $(1, N)$, we get a contradiction.

□

8.2.5 Gosset polytopes

Consider the real vector space $\mathbb{R}^{N,1} = \mathbb{R}^{N+1}$ with the inner product $\langle , \rangle$ defined by the quadratic form on $I^{1,N}$ multiplied by -1. All exceptional vectors lie in the affine space $V_N = \{x \in \mathbb{R}^{N,1} : (\mathbf{k}_N, x) = 1\}$ and belong to the unit sphere S^N. Let Σ_N be the convex hull of the exceptional vectors. For any two vectors $w, w' \in \mathsf{S}^N$, the vector $w - w'$ belongs to the even quadratic lattice $\mathbf{E}_N$, hence $2 \leq \langle w - w', w - w'\rangle = 2 - 2\langle w, w'\rangle$. This shows that the minimal distance $\langle w - w', w - w'\rangle^{1/2}$ between two vertices is equal to $\sqrt{2}$ and occurs only when the vectors w and w' are orthogonal. This implies that the edges of Σ_N correspond to pairs of orthogonal exceptional vectors. The difference of such vectors is a root $\alpha = w - w'$ such that $\langle \alpha, w\rangle = 1$. The reflections $s_\alpha : x \mapsto x - \langle x, \alpha\rangle\alpha$ sends w to w'. Thus the reflection hyperplane $H_\alpha = \{x \in V_N : \langle x, \alpha\rangle = 0\}$ intersects the edge at the midpoint. It permutes

two adjacent vertices. The Weyl group $W(\mathbf{E}_N)$ acts on Σ_N with the set of vertices forming one orbit. The edges coming out of a fixed vertex correspond to exceptional vectors orthogonal to the vertex. For example, if we take the vertex corresponding to the vector $\mathbf{e}_N$, then the edges correspond to exceptional vectors for the root system $\mathbf{E}_{N-1}$. Thus the vertex figure at each vertex (i.e. the convex hull of midpoints of edges coming from the vertex) is isomorphic to Σ_{N-1}. A convex polytope with isomorphic vertex figures is called a *semi-regular polytope* (a *regular polytope* satisfies the additional property that all facets are isomorphic).

The polytopes Σ_N are Gosset polytopes discovered by T. Gosset in 1900 [257]. Following Gosset they are denoted by $(N-4)_{21}$. We refer to [135], p. 202, for their following facts about their combinatorics. Each polytope Σ_N has two $W(\mathbf{E}_N)$-orbits on the set of facets. One of them is represented by the convex hull of exceptional vectors $\mathbf{e}_1, \dots, \mathbf{e}_N$ orthogonal to the vector e_0. It is a $(N-1)$-simplex α_{N-1}. The second one is represented by the convex hull of exceptional vectors orthogonal to $\mathbf{e}_0 - \mathbf{e}_1$. It is a cross-polytope β_{N-1} (a cross-polytope β_i is the bi-pyramide over β_{i-1} with β_2 being a square). The number of facets is equal to the index of the stabilizer group of $\mathbf{e}_0$ or $\mathbf{e}_0 - \mathbf{e}_1$ in the Weyl group. The rest of faces are obtained by induction on N. The number of k-faces in Σ_N is given in Table 8.3 (see [135], 11.8).

k/N	3	4	5	6	7	8
0	6	10	16	27	56	240
1	$3\alpha + 6\alpha$	30	80	216	756	6720
2	$2\alpha + 3\beta$	$10\alpha + 20\alpha$	160	720	4032	60480
3		$5\alpha + 5\beta$	$40\alpha + 80\alpha$	1080	10080	241920
4			$16\alpha + 10\beta$	$432\alpha +$ 216α	12096	483840
5				$72\alpha_5 +$ 27β	$2016\alpha +$ 4032α	483840
6					$576\alpha +$ 126β	$69120\alpha +$ 138240α
7						$17280\alpha +$ $+2160\beta$

Table 8.3 *Gosset polytopes*

The Weyl group $W(\mathbf{E}_N)$ acts transitively on the set of k-faces when $k \leq N-2$. Othwerwise there are two orbits, their cardinality can be found in the table. The dual (reciprocal) polytopes are not semi-regular anymore since the group of symmetries has two orbits on the set of vertices. One is represented by the vector $\mathbf{e}_0$ and another by $\mathbf{e}_0 - \mathbf{e}_1$.

8.2.6 (-1)-curves on del Pezzo surfaces

Let $\phi : I^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a weak del Pezzo surface S. The intersection form on $\mathrm{Pic}(S)$ equips it with a structure of a lattice. Since ϕ sends an orthonormal basis of $I^{1,N}$ to an orthonormal basis of $\mathrm{Pic}(S)$, the isomorphism ϕ is an isomorphism of lattices. The image $K_S^{\perp}$ of $\mathbf{E}_N$ is isomorphic to the lattice $\mathbf{E}_N$.

The image of an exceptional vector is the divisor class E such that $E^2 = E \cdot K_S = -1$. By Riemann-Roch, E is an effective divisor class. Write it as a sum of irreducible components $E = R_1 + \cdots + R_k$. Intersecting with K_S, we obtain that there exists a unique component, say R_1 such that $R_1 \cdot K_S = -1$. For all other components we have $R_i \cdot K_S = 0$. It follows from the adjunction formula that any such component is a (-2)-curve. So, if S is a nonsingular del Pezzo surface, the image of any exceptional divisor is a (-1)-curve on S, and we have a bijection between the set of exceptional vectors in $\mathbf{E}_N$ and (-1)-curves on S. If S is a weak del Pezzo surface, we use the following.

Lemma 8.2.22 *Let D be a divisor class with $D^2 = D \cdot K_S = -1$. Then $D = E + R$, where R is a non-negative sum of (-2)-curves, and E is either a (-1)-curve or $K_S^2 = 1$ and $E \in |-K_S|$ and $E \cdot R = 0, R^2 = -2$. Moreover D is a (-1)-curve if and only if for each (-2)-curve R_i on S we have $D \cdot R_i \geq 0$.*

Proof Fix a geometric basis $e_0, e_1, \ldots, e_N$ in $\mathrm{Pic}(S)$. We know that $e_0^2 = 1, e_0 \cdot K_S = -3$. Thus $((D \cdot e_0)K_S + 3D) \cdot e_0 = 0$ and hence

$$\big((D \cdot e_0)K_S + 3D\big)^2 = -6D \cdot e_0 - 9 + (D \cdot e_0)^2 K_S^2 < 0.$$

Thus $-6D \cdot e_0 - 9 < 0$ and hence $D \cdot e_0 > -9/6 > -2$. This shows that $(K_S - D) \cdot e_0 = -3 - D \cdot e_0 < 0$, and since e_0 is nef, we obtain that $|K_S - D| = \emptyset$. Applying Riemann-Roch, we get $\dim |D| \geq 0$. Write an effective representative of D as a sum of irreducible components and use that $D \cdot (-K_S) = 1$. Since $-K_S$ is nef, there is only one component E entering with coefficient 1 and satisfying $E \cdot K_S = -1$, all other components are (-2)-curves. If $D \sim E$, then $D^2 = E^2 = -1$ and E is a (-1)-curve. Let $\pi : S' \to S$ be a birational morphism of a weak del Pezzo surface of degree 1 (obtained by blowing up $8 - k$ points on S in general position not lying on E). We identify E with its preimage in S'. Then $(E + K_{S'}) \cdot K_{S'} = -1 + 1 = 0$, hence, by Hodge Index Theorem, either $S' = S$ and $E \in |-K_S|$, or

$$(E + K_{S'})^2 = E^2 + 2E \cdot K_{S'} + K_{S'}^2 = E^2 - 1 < 0.$$

Since $E \cdot K_S = -1$, E^2 is odd. Thus, the only possibility is $E^2 = -1$. If

$E \in |-K_S|$, we have $E \cdot R_i = 0$ for any (-2)-curve R_i, hence $E \cdot R = 0, R^2 = -2$.

Assume $R \neq 0$. Since $-1 = E^2 + 2E \cdot R + R^2$ and $E^2 \leq 1$, $R^2 \leq -2$, we get $E \cdot R \geq 0$, where the equality take place only if $E^2 = 1$. In both cases we get

$$-1 = (E+R)^2 = (E+R) \cdot R + (E+R) \cdot E \geq (E+R) \cdot R.$$

Thus if $D \neq E$, we get $D \cdot R_i < 0$ for some irreducible component of R. This proves the assertion. □

The number of (-1)-curves on a nonsingular del Pezzo surface is given in Table 8.1. It is also can be found in Table 8.3. It is the number of vertices of the Gosset polytope. Other faces give additional information about the combinatorics of the set of (-1)-curves. For example, the number of k-faces of type α is equal to the number of sets of $k+1$ non-intersecting (-1)-curves.

We can also see the geometric realization of the fundamental weights:

$$w_1 = \mathbf{e}_0,\ w_2 = \mathbf{e}_0 - \mathbf{e}_1,\ w_3 = 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2,\ w_i = \mathbf{e}_1 + \cdots + \mathbf{e}_N,\ i = 4, \ldots, N.$$

The image of w_1 under a geometric marking represents the divisor class e_0. The image of w_2 represents $e_0 - e_1$. The image of w_3 is $2e_0 - e_1 - e_2$. Finally, the images of the remaining fundamental weights represent the classes of the sums of disjoint (-1)-curves.

Recall the usual attributes of the minimal model program. Let $\mathrm{Eff}(S)$ be the *effective cone* of a smooth projective surface S, i.e. the open subcone in $\mathrm{Pic}(S) \otimes \mathbb{R}$ spanned by effective divisor classes. Let $\overline{\mathrm{Eff}}(S)$ be its closure. The Cone Theorem [343] states that

$$\overline{\mathrm{Eff}}(S) = \overline{\mathrm{Eff}}(S)_{K_S \geq 0} + \sum_i \mathbb{R}[C_i],$$

where $\overline{\mathrm{Eff}}(S)_{K_S \geq 0} = \{x \in \overline{\mathrm{Eff}}(S) : x \cdot K_S \geq 0\}$ and $[C_i]$ are *extremal rays* spanned by classes of smooth rational curves C_i such that $-C_i \cdot K_X \leq 3$.

Recall that a subcone τ of a cone K is extremal if there exists a linear function ϕ such that $\phi(K) \geq 0$ and $\phi^{-1}(0) \cap K = \tau$. In the case when K is a polyhedral cone, an extremal subcone is a face of K.

Theorem 8.2.23 *Let S be a nonsingular del Pezzo surface of degree d. Then*

$$\overline{\mathrm{Eff}}(S) = \sum_{i=1}^{k} \mathbb{R}[C_i],$$

where the set of curves C_i is equal to the set of (-1)-curves if $d \neq 8, 9$. If $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $k = 2$, and the $[C_i]$'s are the

classes of the two rulings on S. If $d = 8$ and $S \cong \mathbf{F}_1$, then $k = 2$ and $[C_1]$ is the class of the exceptional section, and $[C_2]$ is the class of a fibre. If $d = 9$, then $k = 1$ and $[C_1]$ is the class of a line.

Proof Since S is a del Pezzo surface, $\overline{\text{Eff}}(S)_{K_S \geq 0} = \{0\}$, so it suffices to find the extremal rays. It is clear that $E \cdot K_S = -1$ implies that any (-1)-curve generates an extremal ray. Choose a geometric marking on S to identify $\text{Pic}(S)$ with $I^{1,N}$. Let C be a smooth rational curve such that $c = -C \cdot K_S \leq 3$. By the adjunction formula, $C^2 = -2 + c$. If $c = 1$, C is a (-1)-curve. If $c = 2$, applying Lemma 7.5.10, we follow the proof of Proposition 8.2.19 to obtain that all vectors with $v \in I^{1,N}$ satisfying $v \cdot \mathbf{k}_N = -2$ and $(v, v) = 0$ belong to the same orbit of $W(\mathbf{E}_N)$. Thus, if $d < 8$, we may assume that $v = e_0 - e_1$, but then $v = (e_0 - e_1 - e_2) + e_2$ is equal to the sum of two exceptional vectors, hence $[C]$ is not extremal. If $c = 3$, then $C^2 = 1, C \cdot K_S = -3$. Again, we can apply Noether's inequality and the proof of Lemma 7.5.14 to obtain that all such vectors belong to the same orbit. Take $v = e_0$ and write $e_0 = (e_0 - e_1 + e_2) + e_1 + e_2$ to obtain that $[C]$ is not extremal if $d < 8$. We leave the cases $d = 8, 9$ to the reader. □

Corollary 8.2.24 *Assume $d < 8$. Let $\phi : I^{1,N} \to \text{Pic}(S)$ be a geometric marking of a nonsingular del Pezzo surface. Then $\phi^{-1}(\overline{\text{Eff}}(S))$ is equal to the Gosset polytope.*

Recall from [343] that any extremal face F of $\overline{\text{Eff}}(S)$ defines a contraction morphism $\phi_F : S \to Z$. The two types of extremal faces of a Gosset polytope define two types of contraction morphisms: α_k-type and β_k-type. The contraction of the α_k-type blows down the set of disjoint (-1)-curves that are the vertices of the set. The contraction of the β_k-type defines a conic bundle structure on S. It is a morphism onto $\mathbb{P}^1$ with general fiber isomorphic to $\mathbb{P}^1$ and singular fibres equal to the union of two (-1)-curves intersecting transversally at one point. Thus the number of facets of type β of the Gosset polytope is equal to the number of conic bundle structures on S.

Another attribute of the minimal model program is the *nef cone* $\text{Nef}(S)$ in $\text{Pic}(S) \otimes \mathbb{R}$ spanned by divisor classes D such that $D \cdot C \geq 0$ for any effective divisor class C. The nef cone is the dual of $\overline{\text{Eff}}(S)$. Under a geometric marking it becomes isomorphic to the dual of the Gosset polytope. It has two types of vertices represented by the normal vectors to facets. One type is represented by the Weyl group orbit of the vector e_0 and another by the vector $e_0 - e_1$.

8.2.7 Effective roots

Let $\phi : I^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a weak del Pezzo surface of degree $d = 9 - N$. The image of a root $\alpha \in \mathbf{E}_N$ is a divisor class D such that $D^2 = -2$ and $D \cdot K_S = 0$. We say that α is an *effective root* if $\phi(\alpha)$ is an effective divisor class. An effective root representing a (-2)-curve will be called a *nodal root*. Let $\sum_{i\in I} n_i R_i$ be its effective representative. Since $-K_S$ is nef, we obtain that $R_i \cdot K_S = 0$. Since $K_S^2 > 0$, we also get $R_i^2 < 0$. Together with the adjunction formula this implies that each R_i is a (-2)-curve. Since a (-2)-curve does not move, we will identify it with its divisor class.

Proposition 8.2.25 *Let S be a weak del Pezzo surface of degree $d = 9 - N$. The number r of (-2)-curves on S is less than or equal to N. The sublattice $\mathcal{N}_S$ of* $\mathrm{Pic}(S)$ *generated by (-2)-curves is a root lattice of rank r.*

Proof Since each nodal curve is contained in $K_S^\perp$ and $R_i \cdot R_j \geq 0$ for $i \neq j$, it suffices to prove that the set of (-2)-curves is linearly independent over $\mathbb{Q}$. Suppose that this is not true. Then we can find two disjoint sets of curves $R_i, i \in I$, and $R_j, j \in J$, such that

$$\sum_{i\in I} n_i R_i \sim \sum_{j\in J} m_j R_j,$$

where n_i, m_j are some non-negative rational numbers. Taking the intersection of both sides with R_i we obtain that

$$R_i \cdot \sum_{i\in I} n_i R_i = R_i \cdot \sum_{j\in J} m_j R_j \geq 0.$$

This implies that

$$(\sum_{i\in I} n_i R_i)^2 = \sum_{i\in I} n_i R_i \cdot \big(\sum_{i\in I} n_i R_i\big) \geq 0.$$

Since $(\mathbb{Z}K_S)^\perp$ is negative definite, this could happen only if $\sum_{i\in I} n_i R_i \sim 0$. Since all coefficients are non-negative, this happens only if all $n_i = 0$. For the same reason each m_i is equal to 0. □

Let $\eta = x_1 + \cdots + x_N$ be the bubble cycle defined by the blowing down structure $S = S_N \to S_{N-1} \to \ldots S_1 \to S_0 = \mathbb{P}^2$ defining the geometric marking. It is clear that $\phi(\alpha_{ij}) = e_i - e_j$ is effective if and only if $x_i \succ_{i-j} x_j$. It is a nodal root if and only if $i = j + 1$.

A root α_{ijk} is effective if and only if there exists a line whose proper transform on the surfaces $S_{i-1}, S_{j-1}, S_{k-1}$ pass through the x_i, x_j, x_k. It is a nodal root if and only if all roots $\alpha_{i',j',k'}$ with $x_{i'} \succ x_i, x_{j'} \succ x_j, x_{k'} \succ x_k$ are not effective.

The root $2\mathbf{e}_0-\mathbf{e}_{i_1}-\cdots-\mathbf{e}_{i_6}$ is nodal if and only if its image in $\mathrm{Pic}(S)$ is the divisor class of the proper transform of an irreducible conic passing through the points $x_{i_1},\ldots,x_{i_6}$.

The root $3\mathbf{e}_0-\mathbf{e}_1-\cdots-\mathbf{e}_8-\mathbf{e}_i$ is nodal if and only if its image in $\mathrm{Pic}(S)$ is the divisor class of the proper transform of an irreducible cubic with double points at x_i and passing through the rest of the points.

Definition 8.2.26 *A* Dynkin curve *is a reduced connected curve R on a projective nonsingular surface X such that its irreducible components R_i are -2-curves and the matrix $(R_i\cdot R_j)$ is a Cartan matrix. The type of a Dynkin curve is the type of the corresponding root system.*

Under a geometric marking a Dynkin curve on a weak del Pezzo surface S corresponds to an irreducible root base in the lattice $\mathbf{E}_N$. We use the Borel-de Siebenthal-Dynkin procedure to determine all possible root bases in $\mathbf{E}_N$.

Theorem 8.2.27 *Let R be a Dynkin curve on a projective nonsingular surface X. There is a birational morphism $f: X\to Y$, where Y is a normal surface satisfying the following properties:*

(i) $f(R)$ is a point;
(ii) the restriction of f to $X\setminus R$ is an isomorphism;
(iii) $f^*\omega_Y\cong\omega_X$.

Proof Let H be a very ample divisor on X. Since the intersection matrix of components of $R=\sum_{i=1}^n R_i$ has nonzero determinant, we can find rational numbers r_i such that

$$(\sum_{i=1}^n r_iR_i)\cdot R_j=-H\cdot R_j,\quad j=1,\ldots,n.$$

It is known and that the entries of the inverse of a Cartan matrix are non-positive. Thus all r_i's are non-negative numbers. Replacing H by some multiple mH, we may assume that all r_i are non-negative integers. Let $D=\sum r_iR_i$. Since $H+D$ is an effective divisor and $(H+D)\cdot R_i=0$ for each i, we have $\mathcal{O}_X(H+D)\otimes\mathcal{O}_{R_i}=\mathcal{O}_{R_i}$. Consider the standard exact sequence

$$0\to\mathcal{O}_X(H)\to\mathcal{O}_X(H+D)\to\mathcal{O}_D\to 0.$$

Replacing H by mH, we may assume, by Serre's Duality, that $h^1(\mathcal{O}_X(H))=0$ and $\mathcal{O}_X(H)$ is generated by global sections. Let $s_0,\ldots,s_{N-1}$ be sections of $\mathcal{O}_X(H)$ which define an embedding of X in $\mathbb{P}^{N-1}$. Consider them as sections of $\mathcal{O}_X(H+D)$. Let s_N be a section of $\mathcal{O}_X(H+D)$ which maps to $1\in H^0(X,\mathcal{O}_D)$. Consider the map $f': X\to\mathbb{P}^N$ defined by the sections

$(s_0, \dots, s_N)$. Then $f'(D) = [0, \dots, 0, 1]$ and $f'|X \subset D$ is an embedding. So we obtain a map $f : X \to \mathbb{P}^N$ satisfying properties (i) and (ii). Since X is normal, f' factors through a map $f : X \to Y$, where Y is normal. Let ω_Y be the canonical sheaf of Y (it is defined to be equal to the sheaf $j_*\omega_{Y \setminus f'(R)}$, where $j : Y \setminus f'(R) \to Y$ is the natural open embedding). We have

$$\omega_X = f^*\omega_Y \otimes \mathcal{O}_X(A)$$

for some divisor A. Since $K_X \cdot R_i = 0$ for each i, and $f^*\omega_Y \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$ we get $A \cdot R_i = 0$. Since the intersection matrix of R is negative definite we obtain $A = 0$. □

Applying the projection formula and property (iii), we obtain

$$\omega_Y \cong f_*\omega_X.$$

Since f is a resolution of singularities and Y is a normal surface, and hence Cohen-Macaulay, this property is equivalent to that Y has rational singularities [343], Lemma 5.12. For any canonical root basis $\beta_1, \dots, \beta_N$ in a root system of type $\mathbf{E}_N, N \leq 8$, there exists a positive root $\beta_{\max}$ satisfying the property $\beta_{\max} \cdot \beta_i \leq 0, i = 1, \dots, N$. For an irreducible root system, it is equal to the following vector

$$\begin{aligned}
&A_n : \beta_{\max} = \beta_1 + \cdots + \beta_n;\\
&D_n : \beta_{\max} = \beta_1 + \beta_2 + 2\beta_3 + \cdots + 2\beta_{n-1} + \beta_n;\\
&E_6 : \beta_{\max} = 2\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6;\\
&E_7 : \beta_{\max} = 2\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7;\\
&E_8 : \beta_{\max} = 3\beta_1 + 2\beta_2 + 4\beta_3 + 6\beta_4 + 5\beta_5 + 4\beta_6 + 3\beta_7 + 2\beta_8.
\end{aligned}$$

In the root sublattice defined by a Dynkin curve it represents the fundamental cycle Z. Since $\beta_{\max}^2 = -2$, we see that there the singular point $f(R)$ admits a fundamental cycle Z with $Z^2 = -2$. Thus $f(R)$ is a RDP. As we already observed in Example 8.2.9 the exceptional components of a RDP form a Dynkin curve.

An example of a RDP is the singularity of the orbit of the origin of the orbit space $V = \mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of SL(2). The orbit space is isomorphic to the affine spectrum of the algebra of invariant polynomials $A = \mathbb{C}[X, Y]^\Gamma$. It has been known since F. Klein that the algebra A is generated by three elements u, v, w with one single basic relation $F(u, v, w) = 0$. The origin $(0, 0, 0)$ of the surface $V(F) \subset \mathbb{C}^3$ is a RDP with the Dynkin diagram of type A_n, D_n, E_n dependent on Γ in the following way. A nontrivial cyclic group of order $n + 1$ corresponds to type A_n, a binary dihedral group of order

$4n, n \geq 2$, corresponds to type D_{n+2}, a binary tetrahedral group of order 24 corresponds to type E_6, a binary octahedron group of order 48 corresponds to type E_7, and binary icosahedral group of order 120 corresponds to type E_8. It is known that the local analytic isomorphism class of a RDP is determined by the Dynkin diagram (see [440]). This gives the following.

Theorem 8.2.28 *A RDP is locally analytically isomorphic to one of the following singularities*

$$\begin{aligned} &A_n : z^2 + x^2 + y^{n+1} = 0, \quad n \geq 1, \\ &D_n : z^2 + y(x^2 + y^{n-2}) = 0, \quad n \geq 4, \\ &E_6 : z^2 + x^3 + y^4 = 0, \\ &E_7 : z^2 + x^3 + xy^3 = 0, \\ &E_8 : z^2 + x^3 + y^5 = 0. \end{aligned} \tag{8.11}$$

The corresponding Dynkin curve is of respective type A_n, D_n, E_n.

Comparing this list with the list of simple singularities of plane curves from definition 4.2.16, we find that a surface singularity is a RDP if and only if it is locally analytically isomorphic to a singularity at the origin of the double cover of $\mathbb{C}^2$ branched along a curve $F(x, y)$ with simple singularity at the origin. The types match.

Remark 8.2.29 A RDP is often named an *ADE-singularity* for the reason clear from above. Also it is often called a *Du Val singularity* in honor of P. Du Val who was the first to characterize them by property (iii) from Theorem 8.2.2. They are also called *Klein singularities* for the reason explained in above.

8.2.8 Cremona isometries

Definition 8.2.30 *Let S be a weak del Pezzo surface. An orthogonal transformation σ of* $\mathrm{Pic}(S)$ *is called a* Cremona isometry *if $\sigma(K_S) = K_S$ and σ sends any effective class to an effective class. The group of Cremona isometries will be denoted by* $\mathrm{Cris}(S)$.

It follows from Corollary 8.2.15 that $\mathrm{Cris}(S)$ is a subgroup of $W(S)$.

Lemma 8.2.31 *Let*

$$C^n = \{D \in \mathrm{Pic}(S) : D \cdot R \geq 0 \text{ for any } (-2)\text{-curve } R\}.$$

For any $D \in \mathrm{Pic}(S)$ there exists $w \in W(S)^n$ such that $w(D) \in C^n$. If $D \in C^n$ and $w(D) \in C^n$ for some $w \in W(S)^n$, then $w(D) = D$. In other words, C^n is a fundamental domain for the action of $W(S)^n$ in $\mathrm{Pic}(S)$.

Proof The set of (-2)-curves form a root basis in the Picard lattice $\mathrm{Pic}(S)$ and $W(S)^n$ is its Weyl group. The set C^n is a chamber defined by the root basis. Now the assertion follows from the theory of finite reflection groups which we have already employed for a similar assertion in the case of a canonical root basis in $\mathbf{E}_N$. □

Proposition 8.2.32 *An isometry σ of $\mathrm{Pic}(S)$ is a Cremona isometry if and only if it preserves the canonical class and sends a (-2)-curve to a (-2)-curve.*

Proof Clearly, any Cremona isometry sends the class of an irreducible curve to the class of an irreducible curve. Since it also preserves the intersection form, it sends a (-2)-curve to a (-2)-curve.

Let us prove the converse. Let D be an effective class in $\mathrm{Pic}(S)$ with $D^2 \geq 0$. Then $-K_S \cdot D > 0$ and $(K_S - D) \cdot D < 0$. This gives $-K_S \cdot \sigma(D) > 0$, $\sigma(D)^2 \geq 0$. Since $(K_S - \sigma(D)) \cdot (-K_S) = -K_S^2 + \sigma(D) \cdot K_S < 0$, we have $|K_S - \sigma(D)| = \emptyset$. By Riemann-Roch, $|\sigma(D)| \neq \emptyset$.

So it remains to show that σ sends any (-1)-curve E to an effective divisor class. By the previous Lemma, for any (-2)-curve R, we have $0 < E \cdot R = \sigma(E) \cdot \sigma(R)$. Since $\sigma(R)$ is a (-2)-curve, and any (-2) curve is obtained in this way, we see that $\sigma(E) \in C^n$. Hence $\sigma(E)$ is a (-1)-curve. □

Corollary 8.2.33 *Let $\mathcal{R}$ be the set of effective roots of a marked del Pezzo surface (S, ϕ). Then the group of Cremona isometries $\mathrm{Cris}(S)$ is isomorphic to the subgroup of the Weyl group of $\mathbf{E}_N$ that leaves the subset $\mathcal{R}$ invariant.*

Let $W(S)^n$ be the subgroup of $W(S)$ generated by reflections with respect to (-2)-curves. It acts on a marking $\varphi : I^{1,N} \to \mathrm{Pic}(S)$ by composing on the left.

By Lemma 8.2.22, a divisor D with $D^2 = D \cdot K_S = -1$ belongs to C^n if and only if it is a (-1)-curve. This and the previous Lemma imply the following.

Proposition 8.2.34 *Let $\phi : W(S) \to W(\mathbb{E}_N)$ be an isomorphism of groups defined by a geometric marking on S. There is a natural bijection*

$$(-1)\text{-curves on } S \longleftrightarrow W(S)^n \backslash \phi^{-1}(Exc_N),$$

where Exc_N is the set of exceptional vectors in $I^{1,N}$.

Theorem 8.2.35 *For any marked weak del Pezzo surface (S, φ), there exists $w \in W(S)^n$ such that $(S, w \circ \varphi)$ is geometrically marked weak del Pezzo surface.*

Proof We use induction on $N = 9 - K_S^2$. Let $e_i = \phi(\mathbf{e}_i), i = 0, \ldots, N$. It follows from the proof of Lemma 8.2.22, that each e_i is an effective class. Assume e_N is the class of a (-1)-curve E. Let $\pi_N : S \to S_{N-1}$ be the blowing down of E. Then $e_0, e_1, \ldots, e_{N-1}$ are equal to the preimages of the divisor classes $e'_0, e'_1, \ldots, e'_{N-1}$ on S_{N-1} which define a marking of S_{N-1}. By induction, there exists an element $w \in W(S_{N-1})^n$ such that $w(e'_0), w(e'_1), \ldots, w(e'_{N-1})$ defines a geometric marking. Since $\pi_N(e_N)$ does not lie on any (-2)-curve (otherwise S is not a weak del Pezzo surface), we see that for any (-2)-curve R on S_{N-1}, $\pi_N^*(R)$ is a (-2)-curve on S. Thus, under the canonical isomorphism $\mathrm{Pic}(S) \cong \pi_N^*(\mathrm{Pic}(S_{N-1})) \perp \mathbb{Z}e_N$, we can identify $W(S_{N-1})^n$ with a subgroup of $W(S)^n$. Applying w to $(e_0, \ldots, e_{N-1})$ we get a geometric marking of S.

If e_N is not a (-1)-curve, then we apply an element $w \in W(S)^n$ such that $w(e_N) \in C^n$. By Lemma 8.2.22, $w(e_N)$ is a (-1)-curve. Now we have a basis $w(e_0), \ldots, w(e_N)$ satisfying the previous assumption. □

Corollary 8.2.36 *There is a bijection from the set of geometric markings on S and the set of left cosets $W(S)/W(S)^n$.*

Proof The group $W(S)$ acts simply transitively on the set of markings. By Theorem 8.2.35, each orbit of $W(S)^n$ contains a unique geometric marking. □

Corollary 8.2.37 *The group* $\mathrm{Cris}(S)$ *acts on the set of geometric markings of S.*

Proof Let $(e_0, \ldots, e_N)$ defines a geometric marking, and $\sigma \in \mathrm{Cris}(S)$. Then there exists $w \in W(S)^n$ such that $\omega(\sigma(e_0)), \ldots, \omega(\sigma(e_N))$ defines a geometric marking. By Proposition 8.2.32, $\sigma(e_N)$ is the divisor class of a (-1)-curve E, hence it belongs to C^n. By Lemma 8.2.31, we get $w(\sigma(e_N)) = \sigma(e_1)$. This shows that $w \in W^n(\bar{S})$, where $S \to \bar{S}$ is the blow-down $\sigma(E)$. Continuing in this way, we see that $w \in W(\mathbb{P}^2)^n = \{1\}$. Thus $w = 1$ and we obtain that σ sends a geometric marking to a geometric marking. □

Let $\varphi : I^{1,N} \to \mathrm{Pic}(S)$ and $\varphi' : I^{1,N} \to \mathrm{Pic}(S)$ be two geometric markings corresponding to two blowing-down structures $\pi = \pi_1 \circ \ldots \circ \pi_N$ and $\pi' = \pi'_1 \circ \ldots \circ \pi'_N$. Then $T = \pi' \circ \pi^{-1}$ is a Cremona transformation of $\mathbb{P}^2$ and $w = \varphi \circ \varphi'^{-1} \in W(\mathbf{E}_N)$ is its characteristic matrix. Conversely, if T is a Cremona transformation with fundamental points $x_1, \ldots, x_N$ such that their blow-up is a weak del Pezzo surface S, a characteristic matrix of T defines a pair of geometric markings φ, φ' of S and an element $w \in W(\mathbf{E}_N)$ such that

$$\varphi = \varphi' \circ w.$$

Example 8.2.38 Let S be a nonsingular del Pezzo surface of degree 3 and let $\pi : S \to \mathbb{P}^2$ be the blow-up of six points. Let $e_0, e_1, \ldots, e_6$ be the geometric marking and $\alpha = 2e_0 - e_1 - \cdots - e_6$. The reflection $w = s_\alpha$ transforms the geometric marking $e_0, e_1, \ldots, e_6$ to the geometric marking $e'_0, e'_1, \ldots, e'_6$, where $e'_0 = 5e_0 - 2(e_1 + \cdots + e_6), e'_i = 2e_0 - (e_1 + \cdots + e_6) + e_i, i = 1, \ldots, 6$. The corresponding Cremona transformation is the symmetric involutorial transformation of degree 5 with characteristic matrix equal to

$$\begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}. \tag{8.12}$$

Let S be a weak del Pezzo surface of degree d and $\mathrm{Aut}(S)$ be its group of biregular automorphisms. By functoriality $\mathrm{Aut}(S)$ acts on $\mathrm{Pic}(S)$ leaving the canonical class K_S invariant. Thus $\mathrm{Aut}(S)$ acts on the lattice $K_X^\perp = (\mathbb{Z}K_S)^\perp$ preserving the intersection form. Let

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(K_X^\perp), \quad \sigma \mapsto \sigma^*,$$

be the corresponding homomorphism.

Proposition 8.2.39 *The image of ρ is contained in the group* $\mathrm{Cris}(S)$. *If S is a nonsingular del Pezzo surface, the kernel of ρ is trivial if $d \leq 5$. If $d \geq 6$, then the kernel is a linear algebraic group of dimension* $2d - 10$.

Proof Clearly, any automorphism induces a Cremona isometry of $\mathrm{Pic}(S)$. We know that it is contained in the Weyl group. An element in the kernel does not change any geometric basis of $\mathrm{Pic}(S)$. Thus it descends to an automorphism of $\mathbb{P}^2$ which fixes an ordered set of $k = 9 - d$ points in general linear position. If $k \geq 4$ it must be the identity transformation. Assume $k \leq 3$. The assertion is obvious when $k = 0$.

If $k = 1$, the surface S is the blow-up of one point. Each automorphism leaves the unique exceptional curve invariant and acts trivially on the Picard group. The group $\mathrm{Aut}(S)$ is the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ fixing a point. It is a connected linear algebraic group of dimension 6 isomorphic to the semi-direct product $\mathbb{C}^2 \rtimes \mathrm{GL}(2)$.

If $k = 2$, the surface S is the blow-up of two distinct points p_1, p_2. Each automorphism leaves the proper inverse transform of the line $\overline{p_1 p_2}$ invariant. It either leaves the exceptional curves E_1 and E_2 invariant, or switches them.

The kernel of the Weyl representation consists of elements that do not switch E_1 and E_2. It is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ which fixes two points in $\mathbb{P}^2$ and is isomorphic to the group G of invertible matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Its dimension is equal to 4. The image of the Weyl representation is a group of order 2. So $\mathrm{Aut}(S) = G \rtimes C_2$.

If $k = 3$, the surface S is the blow-up of three, non-collinear points. The kernel of the Weyl representation is isomorphic to the group of invertible diagonal 3×3 matrices modulo scalar matrices. It is isomorphic to the 2-dimension torus $(\mathbb{C}^*)^2$. □

Corollary 8.2.40 *Let S be a nonsingular del Pezzo surface of degree $d \leq 5$, then* $\mathrm{Aut}(S)$ *is isomorphic to a subgroup of the Weyl group $W(\mathbf{E}_{9-d})$.*

We will see later examples of automorphisms of weak del Pezzo surfaces of degree 1 or 2 which act trivially on $\mathrm{Pic}(S)$.

8.3 Anticanonical models

8.3.1 Anticanonical linear systems

In this Section we will show that any weak del Pezzo surface of degree $d \geq 3$ is isomorphic to a minimal resolution of a del Pezzo surface of degree d in $\mathbb{P}^d$. In particular, any nonsingular del Pezzo surface of degree $d \geq 3$ is isomorphic to a nonsingular surface of degree d in $\mathbb{P}^d$.

Lemma 8.3.1 *Let S be a weak del Pezzo surface with $K_S^2 = d$. Then*

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = 1 + \tfrac{1}{2}r(r+1)d.$$

Proof By Ramanujam's Vanishing Theorem, which we already used, for any $r \geq 0$ and $i > 0$,

$$H^i(S, \mathcal{O}_S(-rK_S)) = H^i(S, \mathcal{O}_S(K_S + (-r-1)K_S)) = 0. \tag{8.13}$$

The Riemann-Roch Theorem gives

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = \tfrac{1}{2}(-rK_S - K_S) \cdot (-rK_S) + 1 = 1 + \tfrac{1}{2}r(r+1)d.$$

□

Theorem 8.3.2 *Let S be a weak del Pezzo surface of degree d and $\mathcal{R}$ be the union of (-2)-curves on S. Then we have the following.*

(i) $|-K_S|$ *has no fixed part.*
(ii) If $d > 1$, *then* $|-K_S|$ *has no base points.*
(iii) If $d > 2$, $|-K_S|$ *defines a regular map* ϕ *to* $\mathbb{P}^d$ *which is an isomorphism outside* $\mathcal{R}$. *The image surface* $\bar{S}$ *is a del Pezzo surface of degree* d *in* $\mathbb{P}^d$. *The image of each connected component of* $\mathcal{R}$ *is a RDP of* $\phi(S)$.
(iv) If $d = 2$, $|-K_S|$ *defines a regular map* $\phi : S \to \mathbb{P}^2$. *It factors as a birational morphism* $f : S \to \bar{S}$ *onto a normal surface and a finite map* $\pi : \bar{S} \to \mathbb{P}^2$ *of degree 2 branched along a curve* B *of degree 4. The image of each connected component of* $\mathcal{N}$ *is a RDP of* $\bar{S}$. *The curve* B *is either nonsingular or has only simple singularities.*
(v) If $d = 1$, $|-2K_S|$ *defines a regular map* $\phi : S \to \mathbb{P}^3$. *It factors as a birational morphism* $f : S \to \bar{S}$ *onto a normal surface and a finite map* $\pi : \bar{S} \to Q \subset \mathbb{P}^3$ *of degree 2, where* Q *is a quadric cone. The morphism* π *is branched along a curve* B *of degree 6 cut out on* Q *by a cubic surface. The image of each connected component of* $\mathcal{N}$ *under* f *is a RDP of* $\bar{S}$. *The curve* B *either nonsingular or has only simple singularities.*

Proof The assertions are easily verified if $S = \mathbf{F}_0$ or $\mathbf{F}_2$. So we assume that S is obtained from $\mathbb{P}^2$ by blowing up $k = 9 - d$ points t_i.

(i) Assume there is a fixed part F of $|-K_S|$. Write $|-K_S| = F + |M|$, where $|M|$ is the mobile part. If $F^2 > 0$, by Riemann-Roch,

$$\dim |F| \geq \tfrac{1}{2}(F^2 - F \cdot K_S) \geq \tfrac{1}{2}(F^2) > 0,$$

and hence F moves. Thus $F^2 \leq 0$. If $F^2 = 0$, we must also have $F \cdot K_S = 0$. Thus $F = \sum n_i R_i$, where R_i are (-2)-curves. Hence $[f] \in (\mathbb{Z}K_S)^\perp$ and hence $F^2 \leq -2$ (the intersection form on $(\mathbb{Z}K_S)^\perp$ is negative definite and even). Thus $F^2 \leq -2$. Now

$$M^2 = (-K_S - F)^2 = K_S^2 + 2K_S \cdot F + F^2 \leq K_S^2 + F^2 \leq d - 2,$$
$$-K_S \cdot M = K_S^2 + K_S \cdot F \leq d.$$

Suppose $|M|$ is irreducible. Since $\dim |M| = \dim |-K_S| = d$, the linear system $|M|$ defines a rational map to $\mathbb{P}^d$ whose image is a nondegenerate irreducible surface of degree $\leq d - 3$ (strictly less if $|M|$ has base points). This contradicts Theorem 8.1.1.

Now assume that $|M|$ is reducible, i.e. defines a rational map to a nondegenerate curve $W \subset \mathbb{P}^d$ of some degree t. By Theorem 8.1.1, we have $t \geq d$. Since S is rational, W is a rational curve, and then the preimage of a general

hyperplane section is equal to the disjoint sum of t linearly equivalent curves. Thus $M \sim tM_1$ and

$$d \geq -K_S \cdot M = -tK_S \cdot M_1 \geq d(-K_S \cdot M_1).$$

Since $-K_S \cdot M = 0$ implies $M^2 < 0$ and a curve with negative self-intersection does not move, this gives $-K_S \cdot M_1 = 1, d = t$. But then $M^2 = d^2 M_1^2 \leq d-2$ gives a contradiction.

(ii) Assume $d > 1$. We have proved that $|-K_S|$ is irreducible. A general member of $|-K_S|$ is an irreducible curve C with $\omega_C = \mathcal{O}_C(C + K_S) = \mathcal{O}_C$. If C is smooth, then it is an elliptic curve and the linear system $|\mathcal{O}_C(C)|$ is of degree $d > 1$ and has no base points. The same is true for a singular irreducible curve of arithmetic genus 1. This is proved in the same way as in the case of a smooth curve. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Applying the exact sequence of cohomology, we see that the restriction of the linear system $|C| = |-K_S|$ to C is surjective. Thus we have an exact sequence of groups

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(S, \mathcal{O}_C(C)) \to 0.$$

Since $|\mathcal{O}_C(C)|$ has no base points, we have a surjection

$$H^0(S, \mathcal{O}_C(C)) \otimes \mathcal{O}_C \to \mathcal{O}_C(C).$$

This easily implies that the homomorphism

$$H^0(S, \mathcal{O}_S(C)) \otimes \mathcal{O}_C \to \mathcal{O}_S(C)$$

is surjective. Hence $|C| = |-K_S|$ has no base points.

(iii) Assume $d > 2$. Let $x, y \in S$ be two points outside $\mathcal{R}$. Let $f : S' \to S$ be the blowing up of x and y with exceptional curves E_x and E_y. By Proposition 8.1.23, blowing them up, we obtain a weak del Pezzo surface S' of degree $d-2$. We know that the linear system $|-K_{S'}|$ has no fixed components. Thus

$$\dim |-K_S - x - y| = \dim |-K_{S'} - E_x - E_y| \geq 1.$$

This shows that $|-K_S|$ separates points. Also, the same is true if $y \succ_1 x$ and x does not belong to any (-1)-curve E on S or $x \in E$ and y does not correspond to the tangent direction defined by E. Since $-K_S \cdot E = 1$ and $x \in E$, the latter case does not happen.

Since $\phi : S \dashrightarrow \bar{S}$ is a birational map given by a complete linear system $|-K_S|$, its image is a nondegenerate surface of degree $d = (-K_S)^2$. Since

$-K_S \cdot R = 0$ for any (-2)-curve, we see that ϕ blows down R to a point p. If $d = 3$, then $\bar{S}$ is a cubic surface with isolated singularities (the images of connected components of $\mathcal{N}$). It is well known that a hypersurface with no singularities in codimension 1 is a normal variety. Thus $\bar{S}$ is a normal surface. If $d = 4$, then S is obtained by a blow-up one point on a weak del Pezzo surface S' of degree 3. This point does not lie on a (-2)-curve. Thus, $\bar{S}'$ is obtained from $\bar{S}$ by a linear projection from a nonsingular point. We have explained already in Proposition 8.1.8 that this implies that $\bar{S}$ is a normal surface.

The fact that singular points of $\bar{S}$ are RDP is proven in the same way as we have proved assertion (iii) of Theorem 8.2.27.

(iv) Assume $d = 2$. By (ii), the linear system $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. Since $K_S^2 = 2$, the map is of degree 2. Using Stein's factorization [282], Chapter 3, Corollary 11.5, it factors through a birational morphism onto a normal surface $f : S \to \bar{S}$ and a finite degree 2 map $\pi : \bar{S} \to \mathbb{P}^2$. Also we know that $f_*\mathcal{O}_S = \mathcal{O}_{\bar{S}}$. A standard Hurwitz's formula gives

$$\omega_{\bar{S}} \cong \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}), \tag{8.14}$$

where $s \in H^0(\mathbb{P}^2, \mathcal{L}^{\otimes 2})$ vanishes along the branch curve W of π. We have

$$\mathcal{O}_S(K_S) = \omega_S = (\pi \circ f)^*\mathcal{O}_{\mathbb{P}^2}(-1) = f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

It follows from the proof of Theorem 8.2.27 (iii) that singular points of $\bar{S}$ are RDP. Thus $f^*\omega_{\bar{S}} = \omega_S$, and hence

$$f^*\omega_{\bar{S}} \cong f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

Applying f_* and using the projection formula and the fact that $f_*\mathcal{O}_X = \mathcal{O}_Y$, we get $\omega_{\bar{S}} \cong \pi^*\mathcal{O}_{\mathbb{P}^2}(-1)$. It follows from (8.14) that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $\deg W = 4$.

Proof of (v). Let $\pi : S \to \mathbb{P}^2$ be the blow-up of 8 points $x_1, \dots, x_8$. Then $|-K_S|$ is the proper inverse transform of the pencil $|3h - x_1 - \cdots - x_8|$ of plane cubics passing through the points $x_1, \dots, x_8$. Let x_9 be the ninth intersection point of two cubics generating the pencil. The point $x_9' = \pi^{-1}(x_9)$ is the base point of $|-K_S|$. By Bertini's Theorem, all fibres except finitely many, are nonsingular curves (the assumption that the characteristic is zero is important here). Let F be a nonsingular member from $|-K_S|$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-K_S) \to \mathcal{O}_S(-2K_S) \to \mathcal{O}_F(-2K_S) \to 0. \tag{8.15}$$

The linear system $|\mathcal{O}_F(-2K_S)|$ on F is of degree 2. It has no base points. We know from (8.13) that $H^1(S, \mathcal{O}_S(-K_S)) = 0$. Thus the restriction map

$$H^0(S, \mathcal{O}_S(-2K_S)) \to H^0(F, \mathcal{O}_F(-2K_S))$$

is surjective. By the same argument as we used in the proof of (ii), we obtain that $|-2K_S|$ has no base points. By Lemma 8.3.1, $\dim|-2K_S| = 3$. Let $\phi : S \to \mathbb{P}^3$ be a regular map defined by $|-2K_S|$. Its restriction to any nonsingular member F of $|-K_S|$ is given by the linear system of degree 2 and hence is of degree 2. Therefore, the map f is of degree $t > 1$. The image of ϕ is a surface of some degree k. Since $(-2K_S)^2 = 4 = kt$, we conclude that $k = t = 2$. Thus the image of ϕ is a quadric surface Q in $\mathbb{P}^3$ and the images of members F of $|-K_S|$ are lines l_F on Q. I claim that Q is a quadric cone. Indeed, all lines l_F intersect at the point $\phi(t_9')$. This is possible only if Q is a cone.

Let $S \xrightarrow{\pi} S' \xrightarrow{\phi'} Q$ be the Stein factorization. Note that a (-2)-curve R does not pass through the base point x_9' of $|-K_S|$ (because $-K_S \cdot R = 0$). Thus $\pi(x_9')$ is a nonsingular point q' of S'. Its image in Q is the vertex q of Q. Since ϕ' is a finite map, the local ring $\mathcal{O}_{S',q'}$ is a finite algebra over $\mathcal{O}_{Q,q}$ of degree 2. After completion, we may assume that $\mathcal{O}_{S',q'} \cong \mathbb{C}[[u,v]]$. If $u \in \mathcal{O}_{Q,q}$, then v satisfies a monic equation $v^2 + av + b$ with coefficients in $\mathcal{O}_{Q,q}$, where after changing v to $v + \frac{1}{2}a$ we may assume that $a = 0$. Then $\mathcal{O}_{Q,q}$ is equal to the ring of invariants in $\mathbb{C}[[u,v]]$ under the automorphism $u \mapsto u, v \mapsto -v$ which as easy to see isomorphic to $\mathbb{C}[[u,v^2]]$. However, we know that q is a singular point so the ring $\mathcal{O}_{Q,q}$ is not regular. Thus we may assume that $u^2 = a, v^2 = b$ and then $\mathcal{O}_{Q,q}$ is the ring of invariants for the action $(u,v) \mapsto (-u,-v)$. This action is free outside the maximal ideal (u,v). This shows that the finite map ϕ' is unramified in a neighborhood of q' with q' deleted. In particular, the branch curve Q of ϕ' does not pass through q. We leave it to the reader to repeat the argument from the proof of (iv) to show that the branch curve W of ϕ belongs to the linear system $|\mathcal{O}_Q(3)|$. □

Let X be a weal del Pezzo surface of degree $d \leq 3$. The image of a (-1)-curve on X under the antticanonical map is a line on the anti canonical model S of X in $\mathbb{P}^d$. Conversely, any line ℓ on a del Pezzo surface S of degree d in $\mathbb{P}^d$ is the image of a (-1)-curve E on its minimal resolution X. It passes through a singular point if and only if E intersects a component of a Dynkin curve blown down to this singular point. By Proposition 8.2.34, the set of lines on S is in a bijective correspondence to the set of orbits of exceptional vectors in the lattice $K_X^{\perp} \cong \mathbf{E}_{9-d}$ with respect to the Weyl group of the root sublattice of generated by (-2)-curves. This justifies calling a (-1)-curve on a weak del Pezzo surface a *line*.

8.3.2 Anticanonical model

Let X be a normal projective algebraic variety and let D be a Cartier divisor on X. It defines the graded algebra

$$R(X,D) = \bigoplus_{r=0}^{\infty} H^0(S, \mathcal{O}_S(rD)),$$

which depends only (up to isomorphism) on the divisor class of D in $\mathrm{Pic}(X)$. Assume $R(X,D)$ is finitely generated, then $X_D = \mathrm{Proj}\ R(X,D)$ is a projective variety. If $s_0, \dots, s_n$ are homogeneous generators of $R(X,D)$ of degrees $q_0, \dots, q_n$ there is a canonical closed embedding into the weighted projective space

$$X_D \hookrightarrow \mathbb{P}(q_0, \dots, q_n).$$

Also, the evaluation homomorphism of sheaves of graded algebras

$$R(X,D) \otimes \mathcal{O}_X \to S(\mathcal{L}) = \bigoplus_{r=0}^{\infty} \mathcal{O}_S(rD)$$

defines a morphism

$$\varphi_{\mathrm{can}} : X = \mathrm{Proj}(S(\mathcal{L})) \to X_D.$$

For every $r > 0$, the inclusion of subalgebras

$$S(H^0(X, \mathcal{O}_X(rD))) \to R(X,D)$$

defines a rational map

$$\tau_r : X_D \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rD))).$$

The rational map $\phi_{|rD|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rD)))$ is given by the complete linear system $|rD|$ factors through φ

$$\phi_{|rd|} : X \overset{\varphi}{\dashrightarrow} X_D \overset{\tau_r}{\dashrightarrow} \mathbb{P}(H^0(X, \mathcal{O}_X(rD))).$$

A proof of the following Proposition can be found in [157], 7.1.

Proposition 8.3.3 *Suppose $|rD|$ has no base points for some $r > 0$ and $D^{\dim X} > 0$. Then*

(i) $R(X,D)$ is a finitely generated algebra;
(ii) X_D is a normal variety;
(iii) $\dim X_D = \max_{r>0} \dim \phi_{|rD|}(X)$;
(iv) if $\dim X_D = \dim X$, then φ is a birational morphism.

We apply this to the case when $X = S$ is a weak del Pezzo surface and $D = -K_S$. Applying the previous Proposition, we easily obtain that

$$X_{-K_S} \cong \bar{S},$$

where we use the notation of Theorem 8.3.2. The variety $\bar{S}$ is called the *anticanonical model* of S. If S is of degree $d > 2$, the map $\tau_1 : \bar{S} \to \mathbb{P}^d$ is a closed embedding, hence $R(S, -K_S)$ is generated by $d + 1$ elements of order 1. If $d = 2$, the map τ_1 is the double cover of $\mathbb{P}^2$. This shows that $R(S, -K_S)$ is generated by three elements s_0, s_1, s_2 of degree 1 and one element s_3 of degree 2 with a relation $s_3^2 + f_4(s_0, s_1, s_2) = 0$ for some homogeneous polynomial f_4 of degree 2. This shows that $\bar{S}$ is isomorphic to a hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 2)$ given by an equation

$$t_3^2 + f_4(t_0, t_1, t_2) = 0. \tag{8.16}$$

In the case $d = 1$, by Lemma 8.3.1 we obtain that

$$\dim R(S, -K_S)_1 = 2,\ \dim R(S, -K_S)_2 = 4,\ \dim R(S, -K_S)_3 = 7.$$

Let s_0, s_1 be generators of degree 1, let s_2 be an element of degree 2 that is not in $S^2(R(S, -K_S)_1)$ and let s_3 be an element of degree 3 that is not in the subspace generated by $s_0^3, s_0 s_1^2, s_0^2 s_1, s_1^3, s_2 s_0, s_2 s_1$. The subring $R(S, -K_S)'$ generated by s_0, s_1, s_2, s_3 is isomorphic to $\mathbb{C}[t_0, t_1, t_2, t_3]/(F(t_0, t_1, t_2, t_3))$, where

$$F = t_3^2 + t_2^3 + f_4(t_0, t_1)t_2 + f_6(t_0, t_1),$$

and $f_4(t_0, t_1)$ and $f_6(t_0, t_1)$ are binary forms of degrees 4 and 6. The projection $[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, t_2]$ is a double cover of the quadratic cone $Q \subset \mathbb{P}^3$ which is isomorphic to the weighted projective plane $\mathbb{P}(1, 1, 2)$. Using Theorem 8.3.2, one can show that the rational map $\bar{S} \dashrightarrow \mathrm{Proj}\ R(S, -K_S)'$ is an isomorphism. This shows that the anticanonical model $\bar{S}$ of a weak del Pezzo surface of degree 1 is isomorphic to a hypersurface $V(F)$ of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

Recall that a nondegenerate subvariety X of a projective space $\mathbb{P}^n$ is called *projectively normal* if X is normal and the natural restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_X(m))$$

is surjective for all $m \geq 0$. This can be restated in terms of vanishing of cohomology

$$H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0,\ m > 0\ (\text{resp. } m = 1),$$

where $\mathcal{I}_X$ is the ideal sheaf of X. If X is a normal surface, this is equivalent to

that the ideal sheaf $\mathcal{I}_X$ is an aCM sheaf. As we saw earlier, in our discussion of aCM sheaves, this is also equivalent to the ring $\oplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m))$ being a Cohen-Macaulay ring, and in dimension 2, this the same as a normal ring.

Theorem 8.3.4 *Let S be a weak del Pezzo surface, then the anticanonical ring $R(S, -K_S)$ is a normal domain. In particular, a del Pezzo surface of degree d in $\mathbb{P}^d$ is projectively normal.*

Proof For $d \leq 2$, this follows from the explicit description of the ring. It is quotient of a ring of polynomials by a principal ideal, and it has singularities in codimension ≥ 2. By Serre's criterion, it is a normal domain (see [208], 11.2). For $d \geq 3$, we have to show that a del Pezzo surface of degree d in $\mathbb{P}^d$ is projectively normal.

Let H be a general hyperplane. Tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_H(m) \to 0$$

with $\mathcal{I}_X$ we get an exact sequence

$$0 \to \mathcal{I}_X(m-1) \to \mathcal{I}_X(m) \to \mathcal{I}_{H\cap X}(m) \to 0. \tag{8.17}$$

We know that a general hyperplane section $C = S \cap H$ is an elliptic curve of degree d in H which is a projectively normal curve in H. Thus

$$H^1(C, \mathcal{I}_C(m)) = 0,\ m > 0.$$

We know that S is linearly normal surface in $\mathbb{P}^d$. This implies that

$$H^1(\mathbb{P}^d, \mathcal{I}_X(1)) = 0.$$

The exact sequence gives that $H^1(\mathbb{P}^d, \mathcal{I}_X(2)) = 0$. Continuing in this way, we get that $H^1(\mathbb{P}^d, \mathcal{I}_X(m)) = 0, m > 0$. □

8.4 Del Pezzo surfaces of degree ≥ 6

8.4.1 Del Pezzo surfaces of degree $7, 8, 9$

A weak del Pezzo surface of degree 9 is isomorphic to $\mathbb{P}^2$. Its anticanonical model is a Veronese surface V_3^2. It does not contain lines.

A weak del Pezzo surface is isomorphic to either $\mathbf{F}_0$, or $\mathbf{F}_1$, or $\mathbf{F}_2$. In the first two cases it is a del Pezzo surface isomorphic to its anticanonical model in $\mathbb{P}^8$. If $S \cong \mathbf{F}_0$, the anticanonical model is a Veronese-Segre surface embedded by the complete linear system of divisors of type $(2, 2)$. It does not contain lines. If $S \cong \mathbf{F}_1$, the anticanonical model is isomorphic to the projection of

the Veronese surface V_2^2 from one point on the surface. It contains one line. If $S \cong \mathbf{F}_2$, the anticanonical model is isomorphic to the quadratic cone Q embedded in $\mathbb{P}^8$ by the complete linear system $|\mathcal{O}_Q(2)|$. It does not contain lines.

A weak del Pezzo surface of degree 7 is isomorphic to the blow-up of two points x_1, x_2 in $\mathbb{P}^2$. If the points are proper, the anticanonical model of S is a nonsingular surface which contains three lines representing the divisor classes $e_1, e_2, e_0 - e_1 - e_2$. If only one point is proper, then it has one singular point of type A_1 and contains two intersecting lines representing the classes e_1 and $e_0 - e_1 - e_2$. In both cases the surface is isomorphic to a projection of the Veronese surface V_3^2 from a secant line of the surface. In the second case, the secant line is tangent to the Veronese surface.

The automorphism groups of a nonsingular del Pezzo surfaces of degree ≥ 7 were described in Subsection 8.2.8.

8.4.2 Del Pezzo surfaces of degree 6

A weak del Pezzo surface S of degree 6 is isomorphic to the blow-up of a bubble cycle $\eta = x_1 + x_2 + x_3$. Up to a change of an admissible order, we have the following possibilities:

(i,i') x_1, x_2, x_3 are three proper non-collinear (collinear) points;
(ii, ii') $x_2 \succ x_1, x_3$ are non-collinear (collinear) points;
(iii, iii') $x_3 \succ x_2 \succ x_1$ are non-collinear (collinear) points.

In cases (i), (ii) and (iii) the net of conics $|\mathcal{O}_{\mathbb{P}^2}(2) - \eta|$ is homaloidal and the surface S is isomorphic to a minimal resolution of the graph of the Cremona transformation T defined by this net. Since a quadratic Cremona transformation is a special case of a bilinear Cremona transformation, its graph is a complete intersection of two hypersurfaces of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Under the Segre map, the graph embeds in $\mathbb{P}^6$ and the composition of the maps

$$\Phi : S \to \Gamma_T \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \overset{s}{\hookrightarrow} \mathbb{P}^6,$$

is the map given by the anticanonical linear system. Its image is a del Pezzo surface of degree 6 embedded in $\mathbb{P}^6$. It is a nonsingular surface in case (i) and it has one singular point of type A_1 in case (ii) and type A_2 in case (iii). The two maps $S \to \mathbb{P}^2$ are defined by the linear systems $|e_0|$ and $|2e_0 - e_1 - e_2 - e_3|$.

The set of (-1)-curves and (-2)-curves on a weak del Pezzo surface of types (i) (resp. (ii), resp. (iii)) is pictured in Figure 7.1 (resp. Figure 7.2, resp. Figure 7.3).

In the cases where the points x_1, x_2, x_3 are collinear, S has only one map to

$\mathbb{P}^2$ defined by the linear system $|e_0|$ and is not related to Cremona transformations.

Surfaces of types (i), (ii), (ii'), (iii') are examples of *toric surfaces*. They contain an algebraic torus as its open Zariski set U, and the action of U on itself by translations extends to a biregular action of U on S. The complement of U is the union of orbits of dimension 0 and 1. It supports an anticanonical divisor. For example, in case (i), the complement of U is the union of six lines on the surface.

The anticanonical model of a weak toric del Pezzo surface is a toric del Pezzo surface of degree 6 in $\mathbb{P}^6$. It is nonsingular only in case (i).

The types of singular points and the number of lines on a del Pezzo surface of degree 6 is given in Table 8.4.

Bubble cycle	(i)	(ii)	(iii)	(i')	(ii')	(iii')
Singular points	$\emptyset$	A_1	A_2	A_1	$2A_1$	$A_1 + A_2$
Lines	6	4	2	3	2	1

Table 8.4 *Lines and singular points on a del Pezzo surface of degree 6*

The secant variety of a nonsingular del Pezzo surface of degree 6 in $\mathbb{P}^6$ is of expected dimension 5. In fact, projecting from a general point, we obtain a nonsingular surface of degree 6 in $\mathbb{P}^5$. It follows from Zak's classification of Severi varieties that a surface in $\mathbb{P}^5$ with secant variety of dimension 4 is a Veronese surface. More precisely, we have the following description of the secant variety.

Theorem 8.4.1 *Let S be a nonsingular del Pezzo surface of degree 6 in $\mathbb{P}^6$. Then S is projectively equivalent to the subvariety given by equations expressing the rank condition*

$$\operatorname{rank} \begin{pmatrix} t_0 & t_1 & t_2 \\ t_3 & t_0 & t_4 \\ t_5 & t_6 & t_0 \end{pmatrix} \leq 1.$$

The secant variety $\operatorname{Sec}(X)$ is the cubic hypersurface defined by the determinant of this matrix.

Proof We know that S is isomorphic to the intersection of the Segre variety $\mathsf{S}_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ by a linear subspace L of codimension 2. If we identify $\mathbb{P}^8$ with the projectivization of the space of 3×3-matrices, then the Segre variety $\mathsf{S}_{2,2}$ is the locus of matrices of rank 1, hence given, even scheme-theoretically, by the 2×2-matrices. A secant of S is contained in L and is a secant of $\mathsf{S}_{2,2}$. It represents a matrix equal to the sum of matrices of rank 1.

Hence each secant is contained in the determinantal cubic hypersurface. Thus the secant variety of S is the intersection of the cubic by the linear subspace L, so it is a cubic hypersurface in $\mathbb{P}^6$.

Explicitly, we find the linear space L as follows. The map $S \to \mathbb{P}^2 \times \mathbb{P}^2$ is given by the map (π_1, π_2), where $\pi_i : S \to \mathbb{P}^2$ are given by the linear systems $|e_0|$ and $|2e_0 - e_1 - e_2 - e_3|$. Choose a basis z_0, z_1, z_2 in $|e_0|$ and a basis z_1z_2, z_0z_1, z_0z_1 in $|2e_0 - e_1 - e_2 - e_3|$ corresponding to the standard quadratic transformation T_{st}. Then the graph of T_{st} is equal to the intersection of $\mathsf{S}_{2,2} \subset |\text{Mat}_{3,3}|$ with equal diagonal entries $a_{11} = a_{22} = a_{33}$ corresponding to the relations $z_0(z_1z_2) = z_1(z_0z_2) = z_2(z_0z_1)$. This gives the equations from the assertion of the Theorem. □

Let us describe the group of automorphisms of a nonsingular del Pezzo surface of degree 6. The surface is obtained by blowing up three non-collinear points x_1, x_2, x_3. We may assume that their coordinates are $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. We know from Section 8.2.8 that the kernel of the representation $\rho : \text{Aut}(S) \to \text{O}(\text{Pic}(S))$ is a 2-dimensional torus. The root system is of type $A_2 + A_1$, so the Weyl group is isomorphic to $2 \times \mathfrak{S}_3 \cong D_{12}$, where D_{12} is the dihedral group of order 12. Let us show that the image of the Weyl representation is the whole Weyl group.

We choose the standard generators s_1, s_2, s_3 of $W(S) \cong W(\mathbf{E}_3)$ defined by the reflections with respect to the roots $e_0 - e_1 - e_2, e_1 - e_2, e_2 - e_3$. The reflection s_1 acts as the standard quadratic transformation T_{st}, which is lifted to an automorphism of S. It acts on the hexagon of lines on S by switching the opposite sides. The reflection s_2 (resp. s_3) acts as a projective transformations which permutes the points x_1, x_2 and fixes x_3 (resp. permutes x_2 and x_3 and fixes x_1). The subgroup $\langle s_2, s_3\rangle \cong D_6 \cong \mathfrak{S}_3$ acts on the hexagon of lines by natural embedding $D_6 \hookrightarrow \text{O}(2)$.

We leave it to the reader to verify the following.

Theorem 8.4.2 *Let S be a del Pezzo surface of degree 6. Then*

$$\text{Aut}(S) \cong (\mathbb{C}^*)^2 \rtimes \mathfrak{S}_3 \times \mathfrak{S}_2.$$

If we represent the torus as the quotient group of $(\mathbb{C}^*)^3$ by the diagonal subgroup $\Delta \cong \mathbb{C}^*$, then the subgroup $\mathfrak{S}_3$ acts by permutations of factors, and the cyclic subgroup $\mathfrak{S}_2$ acts by the inversion automorphism $z \mapsto z^{-1}$.

Finally, we mention that the Gosset polytope $\Sigma_3 = -1_{21}$ corresponding to a nonsingular del Pezzo surface of degree 6 is an octahedron. This agrees with the isomorphism $W(\mathbf{E}_3) \cong D_{12}$. The surface has two blowing-down morphisms $S \to \mathbb{P}^2$ corresponding to two α-facets and three conic bundle

structures corresponding to the pencils of lines through three points on the plane.

8.5 Del Pezzo surfaces of degree 5

8.5.1 Lines and singularities

A weak del Pezzo surface S of degree 5 is isomorphic to the blow-up of a bubble cycle $\eta = x_1 + x_2 + x_3 + x_4$. The only assumption on the cycle is that $|h - \eta| = \emptyset$. Let e_0, e_1, e_2, e_3, e_4 be a geometric basis defined by an admissible order of η. There are the following four possibilities:

(i) x_1, x_2, x_3, x_4 are proper points;
(ii) $x_2 \succ x_1, x_3, x_4$;
(iii) $x_3 \succ x_2 \succ x_1, x_4$;
(iv) $x_2 \succ x_1, x_4 \succ x_3$;
(v) $x_4 \succ x_3 \succ x_2 \succ x_1$.

There are the following root sublattices in a root lattice of type A_4:

$$A_1, A_1 + A_1, A_2, A_1 + A_2, A_3, A_4.$$

In case (i), S is a del Pezzo surface or has one Dynkin curve of type A_1 if three points are collinear.

In case (ii), we have three possibilities for Dynkin curves: A_1 if no three points are collinear, $A_1 + A_1$ if x_1, x_2, x_3 are collinear, A_2 if x_1, x_3, x_4 are collinear.

In case (iii), we have three possibilities: A_2 if no three points are collinear, A_3 if x_1, x_2, x_3 are collinear, $A_1 + A_2$ if x_1, x_2, x_4 are collinear.

In case (iv), we have two possibilities: $A_1 + A_1$ if no three points are collinear, $A_2 + A_1$ if x_2, x_3, x_4 or x_1, x_2, x_3 are collinear,

In case (v), we have two possibilities: A_3 if x_1, x_2, x_3 are not collinear, A_4 otherwise.

It can be checked that the cases with the same root bases are obtained from each other by Cremona isometries. So, they lead to isomorphic surfaces.

Table 8.5 gives the possibilities of lines and singular points on the anticanonical model of a del Pezzo surface of degree 5 in $\mathbb{P}^5$.

From now on we will study nonsingular del Pezzo surfaces of degree 5. Since any set of four points in general position is projectively equivalent to the set of reference points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 1]$, we obtain that all nonsingular del Pezzo surfaces of degree 5 are isomorphic. A nonsingular

Singular points	$\emptyset$	A_1	$2A_1$	A_2	A_1+A_2	A_3	A_4
Lines	10	7	5	4	3	2	1

Table 8.5 *Lines and singular points on a del Pezzo surface of degree 5*

del Pezzo surface of degree 5 has 10 lines. The union of them is a divisor in $|-2K_S|$. The incidence graph of the set of 10 lines is the famous *Petersen graph*.

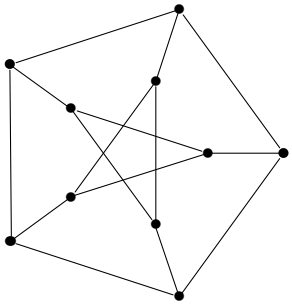


Figure 8.4 Petersen graph

The Gosset polytope $\Sigma_4 = 0_{21}$ has five facets of type α corresponding to contractions of five disjoint lines on S and five pencils of conics corresponding to the pencils of lines through a point in the plane and the pencil of conics through the four points.

8.5.2 Equations

In this Subsection we use some elementary properties of Grassmann varieties $G_k(\mathbb{P}^n) = G(k+1, n+1)$ of k-dimensional subspaces in $\mathbb{P}^n$ (equivalently, $(k+1)$-dimensional linear subspaces of $\mathbb{C}^{n+1}$). We refer to Chapter 10 for the proof of all properties we will use.

Proposition 8.5.1 *Let S be a nonsingular del Pezzo surface of degree 5 in $\mathbb{P}^5$. Then S is isomorphic to a linear section of the Grassmann variety $G_1(\mathbb{P}^4)$ of lines in $\mathbb{P}^4$.*

Proof It is known that the degree of $G = G_1(\mathbb{P}^4)$ in the Plücker embedding is equal to 5 and $\dim G = 6$. It is also known is that the canonical sheaf is equal to $\mathcal{O}_G(-5)$. By the adjunction formula, the intersection of G with a general linear subspace of codimension 4 is a nonsingular surface X with

$\omega_X \cong \mathcal{O}_X(-1)$. This must be a del Pezzo surface of degree 5. Since all del Pezzo surfaces of degree 5 are isomorphic, the assertion follows. □

Corollary 8.5.2 *Let S be a nonsingular del Pezzo surface of degree 5 in $\mathbb{P}^5$. Then its homogeneous ideal is generated by five linearly independent quadrics.*

Proof Since S is projectively normal, applying Lemma 8.3.1, we obtain that the linear system of quadrics containing S has dimension equal to 4. It is known that the homogeneous ideal of the Grassmannian $G(2,5)$ is generated by five quadrics. In fact, the Grassmannian is defined by five pfaffians of principal 4×4 minors of a general skew-symmetric 5×5-matrix. So, restricting this linear system to the linear section of the Grassmannian, we obtain that the quadrics containing S define S scheme-theoretically. □

Let $\mathbb{P}^4 = |E|$ for some linear space E of dimension 5. For any line $\ell = |U|$ in $|E|$, the dual subspace $U^\perp$ in $E^\vee$ defines a plane $|U^\perp|$ in $\mathbb{P}(E)$. This gives a natural isomorphism between the Grassmannians $G_1(|E|)$ and $G_2(\mathbb{P}(E))$. Dually, we get an isomorphism $G_2(|E|) \cong G_1(\mathbb{P}(E))$.

Fix an isomorphism $\bigwedge^5 E \cong \mathbb{C}$, and consider the natural pairing

$$\bigwedge^2 E \times \bigwedge^3 E \to \bigwedge^5 E \cong \mathbb{C}$$

defined by the wedge product. It allows one to identify $(\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$ with $\bigwedge^3 E$. The corresponding identification of the projective spaces does not depend on the choice of an isomorphism $\bigwedge^5 E \cong \mathbb{C}$. A point $U \in G(2,E)$ is orthogonal to a point $V \in G(3,E)$ if and only if $|U| \cap |V| \neq \emptyset$. We know that a quintic del Pezzo surface S is contained in the base locus of a web W of hyperplanes in $|\bigwedge^2 E|$. The web of hyperplanes, considered as a 3-dimensional subspace of $|\bigwedge^2 E^\vee| \cong |\bigwedge^3 E|$ intersects $G_3(|E|)$ at 5 points $\Lambda_1, \dots, \Lambda_5$. Thus any point in S intersects $\Lambda_1, \dots, \Lambda_5$.

Conversely, let $\Lambda_1, \dots, \Lambda_5$ be five planes in $|E|$ such that, considered as points in the space $|\bigwedge^3 E|$, they span a general 3-dimensional subspace W. Then $W^\vee \cap G_2(|E|)$ is a general 5-dimensional subspace in $|\bigwedge^2 E|$ which cuts $G_2(|E|)$ along a quintic del Pezzo surface.

Let us record this.

Proposition 8.5.3 *A nonsingular del Pezzo quintic is isomorphic to the variety of lines in $\mathbb{P}^4$ that intersect five planes in $\mathbb{P}^4$ that span a general 3-dimensional subspace in the Plücker space $\mathbb{P}^9$. Via duality, it is also isomorphic to the variety of planes in $\mathbb{P}^4$ that intersect five lines in $\mathbb{P}^4$ that span a general 3-dimensional subspace of the Plücker space.*

8.5.3 OADP varieties

Let S be a del Pezzo surface of degree 5 in $\mathbb{P}^5$. The linear system of cubics in $\mathbb{P}^5$ containing S has dimension 24. Let us see that any nonsingular cubic fourfold containing X is rational (the rationality of a general cubic fourfold is unknown at the moment).

Lemma 8.5.4 *Let S be a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$. For any point x outside S there exists a unique secant line of S containing x.*

Proof It is known that $\mathrm{Sec}(X) = \mathbb{P}^5$ since any nondegenerate nonsingular surface in $\mathbb{P}^5$ with secant variety of dimension 4 is a Veronese surface. Let $a, b \in S$ such that $x \in \ell = \overline{ab}$. Consider the projection $p_\ell : X \dashrightarrow \mathbb{P}^3$ from the line ℓ. Its image is a cubic surface S_3 isomorphic to the anticanonical model of the blow-up of S at a, b. If $a = b$, the line ℓ is tangent to S, and one of the points is infinitely near the other. In this case the cubic surface is singular. The map $p_\ell : S \setminus \ell$ is an isomorphism outside a, b. Suppose x belongs to another secant $\ell' = \overline{cd}$. Then the projection of the plane $\langle \ell, \ell' \rangle$ spanned by ℓ and ℓ' is a point on the cubic surface whose preimage contains c, d. This shows that p_ℓ is not an isomorphism outside $\ell \cap S$. This contradiction proves the assertion. □

Theorem 8.5.5 *Let F be an irreducible cubic fourfold containing a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$. Then F is a rational variety.*

Proof Consider the linear system $|\mathcal{I}_S(2)|$ of quadrics containing S. It defines a regular map $Y \to \mathbb{P}^4$, where Y is the blow-up of S. Its fibres are proper transforms of secants of X. This shows that the subvariety of $G_1(\mathbb{P}^5)$ parameterizing secants of S is isomorphic to $\mathbb{P}^4$. Let take a general point z in F. By the previous Lemma, there exists a unique secant of X passing through z. By Bezout's Theorem, no other point outside S lies on this secant. This gives a rational injective map $F \dashrightarrow \mathbb{P}^4$ defined outside S. Since a general secant intersects F at three points, with two of them on S, we see that the map is birational. □

Remark 8.5.6 According to a result of A. Beauville [37], Proposition 8.2, any smooth cubic fourfold containing S is a pfaffian cubic hypersurface, i.e. is given by the determinant of a skew-symmetric matrix with linear forms as its entries. Conversely, any pfaffian cubic fourfold contains a nondegenerate surface of degree 5, i.e. an anticanonical weak del Pezzo surface or a scroll.

Remark 8.5.7 A closed subvariety X of $\mathbb{P}^n$ is called a *subvariety with one apparent double point* (*OADP subvariety*, for short) if a general point in $\mathbb{P}^n$

lies on a unique secant line of X. Thus we see that a nonsingular del Pezzo surface of degree 5 is an OADP variety of dimension 2.

An OADP subvariety X of $\mathbb{P}^n$ defines a Cremona involution of $\mathbb{P}^n$ in a way similar to the definition of a de Jonquières involution. For a general point $x \in \mathbb{P}^n$ we find a unique secant line of X intersecting X at two points (a, b), and then define the unique $T(x)$ such that the pair $\{x, T(x)\}$ is harmonically conjugate to $\{a, b\}$.

An infinite series of examples of OADP subvarietes was given by D. Babbage [19] and W. Edge [205]. They are now called the *Edge varieties*. The Edge varieties are of two kinds. The first kind is a general divisor $E_{n,2n+1}$ of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$ embedded by Segre in $\mathbb{P}^{2n+1}$. Its degree is equal to $2n + 1$. For example, when $n = 1$, we obtain a twisted cubic in $\mathbb{P}^3$. If $n = 2$, we obtain a del Pezzo surface in $\mathbb{P}^5$. The second type is a general divisor of bidegree $(0, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$. For example, when $n = 1$, we get the union of two skew lines. When $n = 2$, we get a quartic ruled surface $S_{2,5}$ in $\mathbb{P}^5$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by the linear system of divisors of bidegree $(1, 2)$. A smooth OADP surface in $\mathbb{P}^5$ is either an Edge variety of dimension 2, or a scroll $S_{1,5}$ of degree 4 [483].

We refer to [101] for more information about OADP subvarieties.

8.5.4 Automorphism group

Let us study automorphisms of a nonsingular del Pezzo surface of degree 5. Recall that the Weyl group $W(\mathbf{E}_4)$ is isomorphic to the Weyl group $W(A_4) \cong \mathfrak{S}_5$. By Proposition 8.2.39, we have a natural injective homomorphism

$$\rho : \mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Theorem 8.5.8 *Let S be a nonsingular del Pezzo surface of degree 5. Then*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Proof We may assume that S is isomorphic to the blow-up of the reference points $x_1 = [1, 0, 0], x_2 = [0, 1, 0], x_3 = [0, 0, 1]$ and $x_4 = [1, 1, 1]$. The group $\mathfrak{S}_5$ is generated by its subgroup isomorphic to $\mathfrak{S}_4$ and an element of order 5. The subgroup $\mathfrak{S}_4$ is realized by projective transformations permuting the points $x_1, \dots, x_4$. The action is realized by the standard representation of $\mathfrak{S}_4$ in the hyperplane $z_1 + \cdots + z_4 = 0$ of $\mathbb{C}^4$ identified with $\mathbb{C}^3$ by the projection to the first three coordinates. An element of order 5 is realized by a quadratic transformation with fundamental points x_1, x_2, x_3 defined by the formula

$$T : [t_0, t_1, t_2] \mapsto [t_0(t_2 - t_1), t_2(t_0 - t_1), t_0 t_2]. \tag{8.18}$$

It maps the line $V(t_0)$ to the point x_2, the line $V(t_1)$ to the point x_4, the line $V(x_2)$ to the point x_1, the point x_4 to the point x_3. □

Note that the group of automorphisms acts on the Petersen graph of 10 lines and defines an isomorphism with the group of symmetries of the graph.

Let S be a del Pezzo surface of degree 5. The group $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ acts linearly on the space $V = H^0(S, \mathcal{O}_S(-K_S)) \cong \mathbb{C}^6$. Let us compute the character of this representation. Choose the following basis in the space V:

$$(t_0^2t_1 - t_0t_1t_2, t_0^2t_2 - t_0t_1t_2, t_1^2t_0 - t_0t_1t_2, t_1^2t_2 - t_0t_1t_2, t_2^2t_0 - t_0t_1t_2, t_2^2t_1 - t_0t_1t_2). \tag{8.19}$$

Let $s_1 = (12), s_2 = (23), s_3 = (34), s_4 = (45)$ be the generators of $\mathfrak{S}_5$. It follows from the proof of Theorem 8.5.8 that s_1, s_2, s_3 generate the subgroup of $\mathrm{Aut}(S)$ which is realized by projective transformations permuting the points x_1, x_2, x_3, x_4, represented by the matrices

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The last generator s_4 is realized by the standard quadratic transformation T_{st}. Choose the following representatives of the conjugacy classes in $\mathfrak{S}_5$ different from the conjugacy class of the identity element id:

$$g_1 = (12), \ g_2 = (123) = s_2s_1, g_3 = (1234) = s_3s_2s_1,$$

$$g_4 = (12345) = s_4s_3s_2s_1, g_5 = (12)(34) = s_1s_3, g_6 = (123)(45) = s_3s_2s_1s_4.$$

The subgroup generated by s_1, s_2 acts by permuting the coordinates t_0, t_1, t_2. The generator s_3 acts as the projective transformation

$$(y_1, \dots, y_6) \mapsto (y_1 - y_2 + y_5, -y_2 + y_5 - y_6, y_3 - y_4 + y_6, -y_4 - y_5 + y_6, -y_6, -y_5),$$

where $(y_1, \dots, y_6)$ is the basis from (8.19). Finally, s_4 acts by

$$(y_1, y_2, y_3, y_4, y_5, y_6) \mapsto (y_6, y_4, y_5, y_2, y_3, y_1).$$

Simple computation gives the character vector of the representation

$$\chi = (\chi(1), \chi(g_1), \chi(g_2), \chi(g_3), \chi(g_4), \chi(g_5), \chi(g_6)) = (6, 0, 0, 0, 1, -2, 0).$$

Using the character table of $\mathfrak{S}_5$, we find that χ is the character of an irreducible 6-dimensional irreducible representation isomorphic to $V = \bigwedge^2 R_{\mathrm{st}}$, where R_{st} is the standard 4-dimensional irreducible linear representation of $\mathfrak{S}_5$ with

character vector $(4, 2, 1, 0, -1, 0, -1)$ (see [232], p. 28). The linear system $|-K_S|$ embeds S in $\mathbb{P}(V)$. Since V is isomorphic to its dual representation, we can identify $\mathbb{P}(V)$ with $|V|$.

We will see later in Chapter 10 that $G_1(\mathbb{P}^4)$, embedded in $\mathbb{P}^9$, is defined by five pfaffians of principal minors of a skew-symmetric 5×5-matrix (p_{ij}), where $p_{ji} = -p_{ij}, i < j$, are the Plücker coordinates. The group $\mathfrak{S}_5$ acts on $\mathbb{P}^9$ via its natural representation on $\bigwedge^2 W$, where W is an irreducible representation of $\mathfrak{S}_5$ with character $(5, 1, -1, -1, 0, 1, 1)$. The representation $\bigwedge^2 W$ decomposes into irreducible representation $V \oplus R'_{\text{st}}$, where R'_{st} is the standard 4-dimensional representation of $\mathfrak{S}_5$ tensored with the sign representation U'.

Now let us consider the linear representation of $\mathfrak{S}_5$ on the symmetric square $S^2(V^v ee)$. Using the formula

$$\chi_{S^2(V)}(g) = \tfrac{1}{2}(\chi(g)^2 + \chi(g^2)),$$

we get

$$\chi_{S^2(V)} = (21, 3, 0, -1, 1, 5, 0).$$

Taking the inner product with the character of the trivial representation, we get 1. This shows that the subspace of invariant vectors $U = S^2(V)^{\mathfrak{S}_5}$ is one-dimensional. Similarly, we find that $\dim S^2(V)$ contains one copy of the 1-dimensional sign representation U' of $\mathfrak{S}_5$. The equation of the union of ten lines, considered as an element of $S^2(V)$, is represented by the equation of the union of six lines $\overline{x_i x_j}$, where $x_1, \dots, x_4$ are the reference points. It is

$$F = t_0 t_1 t_2 (t_0 - t_1)(t_0 - t_2)(t_1 - t_2) = 0.$$

It is easy to check that F transforms under $\mathfrak{S}_5$ as the sign representation. It is less trivial, but straightforward, to find a generator of the vector space $S^2(V)^{\mathfrak{S}_5}$. It is equal to

$$G = 2\sum t_i^4 t_j^2 - 2\sum t_i^4 t_j t_k - 2\sum t_i^3 t_j^3 - \sum t_i^3 t_j^2 t_k + 6 t_0^2 t_1^2 t_2^2. \quad (8.20)$$

Its singular points are the reference points $x_1, \dots, x_4$. In another coordinate system, the equation looks even better:

$$t_0^6 + t_1^6 + t_2^6 + (t_0^2 + t_1^2 + t_2^2)(t_0^4 + t_1^4 + t_2^4) - 12 t_0^2 t_1^2 t_2^2 = 0.$$

(see [203]). The singular points here are the points $[1, -1, -1], [-1, 1, -1], [-1, -1, 1], [1, 1, 1]$. The equation $G = 0$ reveals obvious symmetry with respect to the group generated by the permutation of the coordinates corresponding to the generators s_1 and s_2. It is also obviously invariant with respect to the standard quadratic transformation T_{st} which we can write in the form $[t_0, t_1, t_2] \mapsto$

$[1/t_0, 1/t_1, 1/t_2]$. Less obvious is the invariance with respect to the generator s_3.

The $\mathfrak{S}_5$-invariant plane sextic $W_6 = V(G)$ is called the *Wiman sextic*. Its proper transform on S is a smooth curve of genus 6 in $|-2K_S|$. All curves in the pencil of sextics spanned by $V(\lambda F + \mu G)$ (the *Wiman pencil*) are $\mathfrak{A}_5$-invariant. It contains two $\mathfrak{S}_5$-invariant members $V(F)$ and $V(G)$.

Remark 8.5.9 It is known that a del Pezzo surface of degree 5 is isomorphic to the GIT-quotient P_1^5 of the space $(\mathbb{P}^1)^5$ by the group $\mathrm{SL}(2)$ (see [176]). The group $\mathfrak{S}_5$ is realized naturally by the permutation of factors. The isomorphism is defined by assigning to any point x on the surface the five ordered points $(x_1, \ldots, x_4, x_5 = x)$, where $p_1, \ldots, p_4$ are the tangent directions of the conic in the plane passing through the points x_1, x_2, x_3, x_4, x. The isomorphism from P_1^5 onto a quintic surface in $\mathbb{P}^5$ is given by the linear system of bracket-functions $(ab)(cd)(ef)(hk)(lm)$, where $a, b, c, d, e, f, h, k, l, m$ belong to the set $\{1, 2, 3, 4, 5\}$ and each number from this set appears exactly 2 times.

Remark 8.5.10 Let S be a weak del Pezzo surface and D be a smooth divisor in $|-2K_S|$. The double cover X of S branched over D is a *K3 surface*. If we take S to be a nonsingular del Pezzo surface of degree 5 and D to be the proper transform of the Wiman sextic, we obtain a K3 surface with automorphism group containing the group $\mathfrak{S}_5 \times 2$. The cyclic factor here acts on the cover as the deck transformation. Consider the subgroup of $\mathfrak{S}_5 \times 2$ isomorphic to $\mathfrak{S}_5$ that consists of elements $(\sigma, \epsilon(\sigma))$, where $\epsilon : \mathfrak{S}_5 \to \{\pm 1\}$ is the sign representation. This subgroup acts on X symplectically, i.e. leaves a nonzero holomorphic 2-form on X invariant. The list of maximal groups of automorphisms of K3 surfaces which act symplectically was given by S. Mukai [398]. We find the group $\mathfrak{S}_5$ in this list (although the example in the paper is different).

Remark 8.5.11 If we choose one of the nonsingular quadrics containing a nonsingular del Pezzo quintic surface S to represent the Grassmannnian $G_1(\mathbb{P}^3)$, then S can be viewed as a congruence of lines in $\mathbb{P}^3$ of order 2 and class 3. It is equal to one of the irreducible components of the variety of bitangent lines of a quartic surface X with 15 nodes and 10 *tropes* (planes which touch the quartic along a conic). Each ray of the congruence contains two tangency points with X. This defines a double cover of S ramified along a curve Γ cut out by a quadric. This curve is touching all ten lines. Their pre-images split into the set of 20 lines on X. The image of Γ in the plane is a curve of degree 6 with four cusps. For each line joining two cusps, the two residual points coincide.

8.6 Quartic del Pezzo surfaces

8.6.1 Equations

Here we study in more details del Pezzo surfaces of degree 4. Their minimal resolutions of singularities are obtained by blowing up five points in $\mathbb{P}^2$ and hence vary in a 2-dimensional family.

Lemma 8.6.1 *Let X be the complete intersection of two quadrics in $\mathbb{P}^n$. Then X is nonsingular if and only if it is isomorphic to the variety*

$$\sum_{i=0}^{n} t_i = \sum_{i=0}^{n} a_i t_i^2 = 0$$

where the coefficients $a_0, \dots, a_n$ are all distinct.

Proof The pencil of quadrics has the discriminant hypersurface Δ defined by a binary form of degree $n+1$. If all quadrics are singular, then, by Bertini's Theorem they share a singular point. This implies that X is a cone, and hence singular. Conversely, if X is a cone, then all quadrics in the pencil are singular. Suppose Δ consists of less than $n+1$ points. The description of the tangent space of the discriminant hypersurface of a linear system of quadrics (see Example 1.2.3) shows that a multiple point corresponds to either a quadric of corank ≥ 2 or to a quadric of corank 1 such that all quadrics in the pencil contain its singular point. In both cases, X contains a singular point of one of the quadrics in the pencil causing X to be singular. Conversely, if X has a singular point, all quadrics in the pencil are tangent at this point. One of them must be singular at this point causing Δ to have a multiple point.

So, we see that X is nonsingular if and only if the pencil contains exactly $n+1$ quadrics of corank 1. It is a standard fact from linear algebra that in this case the quadrics can be simultaneously diagonalized (see, for example, [233] or [302], vol. 2, Chapter XIII). Thus we see that, after a linear change of coordinates, X can be given by equations from the assertion of the Lemma. If two coefficients a_i are equal, then the pencil contains a quadrics of corank ≥ 2, and hence Δ has a multiple point. □

Theorem 8.6.2 *Let S be a del Pezzo surface S of degree 4. Then S is a complete intersection of two quadrics in $\mathbb{P}^4$. Moreover, if S is nonsingular, then the equations of the quadrics can be reduced, after a linear change of variables, to the diagonal forms:*

$$\sum_{i=0}^{4} t_i^2 = \sum_{i=0}^{4} a_i t_i^2 = 0,$$

where $a_i \neq a_j$ for $i \neq j$.

Proof By Theorem 8.3.4, S is projectively normal in $\mathbb{P}^4$. This gives the exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}) \to H^0(S, \mathcal{O}_S(2)) \to 0.$$

By Lemma 8.3.1,

$$\dim H^0(S, \mathcal{O}_S(2)) = \dim H^0(S, \mathcal{O}_S(-2K_S)) = 13.$$

This implies that S is the base locus of a pencil of quadrics. Now the assertion follows from the previous Lemma. □

Following the classical terminology, a del Pezzo surface of degree 4 in $\mathbb{P}^4$ is called a *Segre quartic surface*.

One can say more about equations of singular del Pezzo quartics. Let $\mathcal{Q}$ be a pencil of quadrics in $\mathbb{P}^n$. We view it as a line in the space of symmetric matrices of size $n+1$ spanned by two matrices A, B. Assume that $\mathcal{Q}$ contains a nonsingular quadric, so that we can choose B to be a nonsingular matrix. Consider the λ-matrix $A + \lambda B$ and compute its elementary divisors. Let $\det(A + \lambda B) = 0$ have r distinct roots $\alpha_1, \ldots, \alpha_r$. For every root α_i we have elementary divisors of the matrix $A + \lambda B$

$$(\lambda - \alpha_i)^{e_i^{(1)}}, \ldots, (\lambda - \alpha_i)^{e_i^{(s_i)}}, \quad e_i^{(1)} \leq \ldots \leq e_i^{(s_i)}.$$

The *Segre symbol* of the pencil $\mathcal{Q}$ is the collection

$$[(e_1^{(1)} \ldots e_1^{(s_1)})(e_2^{(1)} \ldots e_2^{(s_2)}) \ldots (e_r^{(1)} \ldots e_r^{(s_r)})].$$

It is a standard result in linear algebra (see, the references in the proof of Lemma 8.6.1) that one can simultaneously reduce the pair of matrices (A, B) to the form (A', B') (i.e. there exists an invertible matrix C such that $CAC^t = A', CBC^t = B'$) such that the corresponding quadratic forms Q_1', Q_2' have the following form

$$Q_1' = \sum_{i=1}^{r} \sum_{j=1}^{s_i} p(\alpha_i, e_i^{(j)}), \quad Q_2' = \sum_{i=1}^{r} \sum_{j=1}^{s_i} q(e_i^{(j)}), \tag{8.21}$$

where

$$p(\alpha, e) = \alpha \sum_{i=1}^{e} t_i t_{e+1-i} + \sum_{i=1}^{e-1} t_{i+1} t_{e+1-i},$$

$$q(e) = \sum_{i=1}^{e} t_i t_{e+1-i}.$$

It is understood here that each $p(\alpha, e)$ and $q(e)$ are written in disjoint sets of variables. This implies the following.

Theorem 8.6.3 *Let X and X' be two complete intersections of quadrics and $\mathcal{P}, \mathcal{P}'$ be the corresponding pencils of quadrics. Assume that $\mathcal{P}$ and $\mathcal{P}'$ contains a nonsingular quadric. Let H and H' be the set of singular quadrics in $\mathcal{P}$ and $\mathcal{P}'$ considered as sets marked with the corresponding part of the Segre symbol. Then X is projectively equivalent to X' if and only if the Segre symbols of $\mathcal{P}$ and $\mathcal{P}'$ coincide and there exists a projective isomorphism $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi(H) = H'$ and the marking is preserved.*

Applying this to our case $n = 4$, we obtain the following possible Segre symbols:

$r = 5$: $[11111]$;
$r = 4$: $[(11)111]$, $[2111]$;
$r = 3$: $[(11)(11)1]$, $[(11)21]$, $[311]$, $[221]$, $[(12)11]$;
$r = 2$: $[14]$, $[(31)1]$, $[3(11)]$, $[32]$, $[(12)2]$, $[(12)(11)]$;
$r = 1$: $[5]$, $[(14)]$.

Here r is the number of singular quadrics in the pencil. Note that the case $[(1,1,1,1,1)]$ leads to linearly dependent matrices A, B, so it is excluded for our purpose. Also in cases $[(111)11], [(1111)1], [(112)1], [(22)1]$, there is a reducible quadric in the pencil, so the base locus is a reducible. Finally, the cases $[(23)], [(113)]$, $[(122)]$, and $[(1112)]$ correspond to cones over a quartic elliptic curve.

8.6.2 Cyclid quartics

Let S be a nonsingular del Pezzo quartic surface in $\mathbb{P}^4$. Let us project S to $\mathbb{P}^3$. First assume that the center of the projection p lies on S. Then the image of the projection is a cubic surface S_3 in $\mathbb{P}^3$. It is nonsingular if the center of the projection does not belong to a line on S, has one node if it lies on one line, and has an A_2 singularity if it lies on two lines. Note that no three lines on S are coplanar because otherwise the pencil of hyperplanes through this line cuts out, residually, a pencil of lines on S. So, no point lies on three lines.

Now let us assume that the center of the projection p does not lie on S. Let Q_p be the unique quadric from the pencil which contains p.

Theorem 8.6.4 *Assume that the quadric Q_p is nonsingular. Then the projection X of S from p is a quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic. Any irreducible quartic surface in $\mathbb{P}^3$ which is singular along*

a nonsingular conic arises in this way from a Segre quartic surface S in $\mathbb{P}^4$. The surface S is nonsingular if and only if X is nonsingular outside the conic.

Proof First of all let us see that X is indeed a quartic surface. If not, the projection is a finite map of degree 2 onto a quadric. In this case, the preimage of the quadric in $\mathbb{P}^4$ is a quadratic cone containing S with the vertex at the center of the projection. This is excluded by the assumption.

Let H be the tangent hyperplane of Q_p at p and $C = H \cap S$. The intersection $H \cap Q_p$ is an irreducible quadric in H with singular point at p. The curve C lies on this quadric and is cut out by a quadric $Q' \cap H$ for some quadric $Q' \neq Q$ from the pencil. Thus the projection from p defines a degree 2 map from C to a nonsingular conic K equal to the projection of the cone $H \cap Q_p$. It spans the plane in $\mathbb{P}^3$ equal to the projection of the hyperplane H. Since the projection defines a birational isomorphism from S to X that is not an isomorphism over the conic K, we see that X is singular along K. It is also nonsingular outside K (since we assume that S is nonsingular).

Conversely, let K be a nonsingular conic in $\mathbb{P}^3$. As we saw in Subsection 7.2.1, the linear system $|\mathcal{I}_K(2)|$ of quadrics through K maps $\mathbb{P}^3$ onto a quadric Q_1 in $\mathbb{P}^4$. The preimage of a quadric $Q_2 \neq Q_1$ under this rational map is a quartic surface X containing K as a double curve. The intersection $S = Q_1 \cap Q_2$ is a Segre quartic surface. The image of the plane Π containing K is a point p on Q_1. The inverse map $S \dashrightarrow X$ is the projection from p. Since the rational map $\mathbb{P}^3 \dashrightarrow Q_1$ is an isomorphism outside Π, the quartic X is nonsingular outside K if and only if S is nonsingular.

□

In classical literature a quartic surface in $\mathbb{P}^3$ singular along a conic is called a *cyclide quartic surface*.

Remark 8.6.5 If we choose the equation of the conic K in the form $V(t_0^2 + t_1^2 + t_2^2)$, then formula (7.17) shows that the equation of the quartic can be written in the form $V(\sum a_{ij} z_i z_j)$, where $(z_0, z_1, z_2, z_3) = (t_0^2, t_0 t_1, t_0 t_2, t_0 t_3)$. Since the quartic is irreducible, we may assume that $a_{00} \neq 0$, hence the equation of a cyclide surface can be reduced to the form

$$(t_1^2 + t_2^2 + t_3^2)^2 + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0. \tag{8.22}$$

Note that this can generalized to any dimension. We obtain a quartic hypersurface

$$(\sum_{i=1}^{n} t_i^2)^2 + t_0^2 g_2(t_0, \dots, t_n) = 0$$

singular along the quadric $V(t_0) \cap V(\sum_{i=1}^{n} t_i^2)$. In dimension 1, we obtain a

quartic curve with two double points (a *cyclide curve*). Let ℓ be the line through the nodes. We may assume that its equation is $t_0 = 0$ and the coordinates of the points are $[0, 1, i], [0, 1, -i]$. By definition, a *complex n-sphere* in $\mathbb{P}^{n+1}$ is a quadric containing a fixed nonsingular quadric Q_0 in a fixed hyperplane in $\mathbb{P}^{n+1}$. We already discussed complex circles in Chapter 2. Thus we see that complex spheres are preimages of quadrics in $\mathbb{P}^{n+2}$ under a map given by the linear system of quadrics in $\mathbb{P}^{n+1}$ through Q_0. The equation of a complex n-ball in $\mathbb{P}^{n+1}$ becomes a linear equation in $\mathbb{P}^{n+2}$. Over reals, we obtain that the geometry of real spheres is reduced to the geometry of hyperplane sections of a fixed quadric in a higher-dimensional space (see [335]).

Next we consider the projection of a nonsingular Segre surface from a nonsingular point p on a singular quadric Q from the pencil containing S. The tangent hyperplane H of Q at p intersects Q along the union of two planes. Thus H intersects S along the union of two conics intersecting at two points. This is a degeneration of the previous case. The projection is a *degenerate cyclide surface*. It is isomorphic to the preimage of a quadric in $\mathbb{P}^4$ under a map given by the linear system of quadrics in $\mathbb{P}^3$ containing the union of two coplanar lines (a degeneration of the conic K from above). Its equation can be reduced to the form

$$t_1^2 t_2^2 + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0.$$

Finally, let us assume that the center of the projection is the singular point p of a cone Q from the pencil. In this case the projection defines a degree 2 map $S \to \bar{Q}$, where $\bar{Q}$ is a nonsingular quadric in $\mathbb{P}^3$, the projection of Q. The branch locus of this map is a nonsingular quartic elliptic curve of bidegree $(2, 2)$. If we choose the diagonal equations of S as in Theorem 8.6.2, and take the point $p = [1, 0, 0, 0, 0]$, then Q is given by the equation

$$(a_2 - a_1)t_1^2 + (a_3 - a_1)t_2^2 + (a_3 - a_1)t_3^2 + (a_4 - a_1)t_4^2 = 0.$$

It is projected to the quadric with the same equations in coordinates $[t_1, \dots, t_4]$ in $\mathbb{P}^3$. The branch curve is cut out by the quadric with the equation

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0.$$

A more general cyclid quartic surfaces are obtained by projection from singular quartic surfaces in $\mathbb{P}^3$. They have been all classified by C. Segre [517].

8.6.3 Lines and singularities

Let S be a quartic del Pezzo surface and X be its minimal resolution of singularities. The surface X is obtained by blowing up a bubble cycle $\eta = x_1 + \cdots + x_5$ of points in almost general position. Applying the procedure of Borel-De Sibenthal-Dynkin, we obtain the following list of types of root bases in the lattice $K_X^{\perp} \cong \mathbf{E}_5$:

$$D_5, A_3{+}2A_1, D_4, A_4, 4A_1, A_2{+}2A_1, A_3{+}A_1, A_3, 3A_1, A_2{+}A_1, A_2, 2A_1, A_1.$$

All of these types can be realized as the types of root bases defined by Dynkin curves.

$D_5 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1,$ and x_1, x_2, x_3 are collinear;

$A_3 + 2A_1 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4,\ x_1, x_4, x_5$ and x_1, x_2, x_3 are collinear;

$D_4 : x_4 \succ x_3 \succ x_2 \succ x_1,$ and x_1, x_2, x_5 are collinear;

$A_4 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1$;

$4A_1 : x_2 \succ x_1, x_4 \succ x_3,\ x_1, x_2, x_5$ and x_3, x_4, x_5 are collinear;

$2A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4$ and x_1, x_2, x_3 are collinear;

$A_1 + A_3 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4$, and x_1, x_4, x_5 are collinear;

$A_3 : x_4 \succ x_3 \succ x_2 \succ x_1$, or $x_3 \succ x_2 \succ x_1$ and x_1, x_2, x_4 are collinear;

$A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4$;

$3A_1 : x_2 \succ x_1,\ x_4, \succ x_3$, and x_1, x_3, x_5 are collinear;

$A_2 : x_3 \succ x_2 \succ x_1$;

$2A_1 : x_2 \succ x_1, x_4 \succ x_3$, or x_1, x_2, x_3 and x_1, x_3, x_4 are collinear;

$A_1 : x_1, x_2, x_3$ are collinear.

This can be also stated in terms of equations indicated in the next table. The number of lines is also easy to find by looking at the blow-up model. We have the following table (see [565]).

$\emptyset$	A_1	$2A_1$	$2A_1$	A_2	$3A_1$
[11111]	[2111]	[221]	[(11)111]	[311]	[(11)21]
16	12	9	8	8	6
$A_1 + A_2$	A_3	A_3	$A_1 + A_3$	$A_2 + 2A_1$	$4A_1$
[32]	[41]	[(21)11]	[(21)2]	[3(11)]	[(11)(11)1]
6	5	4	3	4	4
A_4	D_4	$2A_1 + A_3$	D_5		
[5]	[(31)1]	[(21)(11)]	[(41)]		
3	2	2	1		

Table 8.6 *Lines and singularities on a weak del Pezzo surface of degree 4*

Example 8.6.6 The quartic surfaces with singular points of type $4A_1$ or $2A_1+A_3$ have a remarkable property that they admit a double cover ramified only at the singular points. We refer to [133] for more details about these quartic surfaces. The projections of these surfaces to $\mathbb{P}^3$ are cubic symmetroid surfaces which will be discussed in the next Chapter. The cover is the quadric surface $\mathbf{F}_0$ in the first case and the quadric cone Q in the second case (see Figure 8.5.1).

The Gosset polytope $\Sigma_5 = 1_{21}$ has 16 facets of type α and ten facets of type β. They correspond to contractions of 5 disjoint lines and pencils of conics arising from the pencils of lines through one of the five points in the plane and pencils of conics through four of the five points.

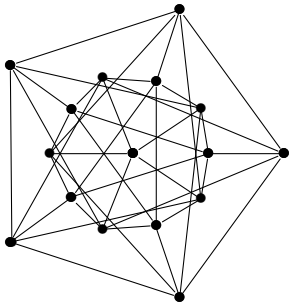


Figure 8.5 Lines on a del Pezzo quartic

8.6.4 Automorphisms

Let S be a nonsingular del Pezzo surface. We know that the natural homomorphism

$$\mathrm{Aut}(S) \to W(S) \cong W(D_5)$$

is injective.

Proposition 8.6.7

$$W(D_5) \cong 2^4 \rtimes \mathfrak{S}_5,$$

where 2^k *denotes the elementary abelian group* $(\mathbb{Z}/2\mathbb{Z})^k$.

Proof This is a well-known fact from the theory of reflection groups. However, we give a geometric proof exhibiting the action of $W(D_5)$ on $\mathrm{Pic}(S)$. Fix a geometric basis $e_0, \dots, e_5$ corresponding to an isomorphism S and the

blow-up of five points $x_1, \ldots, x_5$ in general position. Consider five pairs of pencils of conics defined by the linear systems

$$L_i = |e_0 - e_i|, \quad L_i' = |2e_0 - \sum_{i=1}^{5} e_i|, \quad i = 1, \ldots, 5.$$

Let $\alpha_1, \ldots, \alpha_5$ be the canonical root basis defined by the geometric basis and $r_i = r_{\alpha_i}$ be the corresponding reflections. Then $r_2, \ldots, r_5$ generate $\mathfrak{S}_5$ and act by permuting L_i's and L_i'. Consider the product $r_1 \circ r_5$. It is immediately checked that it switches L_4 with L_4' and L_5 with L_5' leaving L_i, L_i' invariant for $i = 1, 2, 3$. Similarly, a conjugate of $r_1 \circ r_5$ in $W(D_5)$ does the same for some other pair of the indices. The subgroup generated by the conjugates is isomorphic to 2^4. Its elements switch the L_i with L_i' in an even number of pairs of pencils. This defines a surjective homomorphism $W(D_5) \to \mathfrak{S}_5$ with a kernel containing 2^4. Comparing the orders of the groups we see that the kernel is 2^4 and we have an isomorphism of groups asserted in the Proposition. □

We know that the pencil of quadrics containing S has exactly five singular members Q_i of corank 1. Each quadric Q_i is a cone over a nonsingular quadric in $\mathbb{P}^3$. It contains two rulings of planes containing the vertex of Q_i. Since $S = Q_i \cap Q$ for some nonsingular quadric Q, we see that S contains two pencils of conics $|C_i|$ and $|C_i'|$ such that $C_i \cap C_i' = 2$. In the blow-up model of S these are the pencils of conics $|L_i|$ and $L_i'|$ which we used in the proof of the previous Proposition. The group $W(S)$ acts on pairs of pencils of conics, and on the set of five singular quadrics Q_i. The subgroup 2^4 acts trivially on the set of singular quadrics.

Theorem 8.6.8 *Let S be a nonsingular del Pezzo surface of degree 4. The image of the homomorphism*

$$\mathrm{Aut}(S) \to 2^4 \rtimes \mathfrak{S}_5$$

contains the normal subgroup 2^4. The quotient group is isomorphic to either a cyclic group of order 1, 2 or 4, or to the dihedral group D_6 or D_{10}.

Proof Consider the map

$$|L_i| \times |L_i'| \to |-K_S|, \quad (D, D') \mapsto D + D'.$$

Its image generates a 3-dimensional linear system contained in $|-K_S|$. This linear system defines the projection map $\psi_i : S \to \mathbb{P}^3$. Since $D_i \cdot D_i' = 2$ for $D_i \in L_i, D_i' \in L_i'$, the degree of the map is equal to 2. So the image of ψ is a quadric in $\mathbb{P}^3$. This shows that the center of the projection is the vertex of one of the five singular quadric cones in the pencil of quadrics containing

S. The deck transformation $g_i, i = 1, \ldots, 5$, of the cover is an automorphism and these five automorphisms generate a subgroup H of $\mathrm{Aut}(S)$ isomorphic to 2^4. One can come to the same conclusion by looking at the equations from Theorem 8.6.2 of S. The group of projective automorphisms generated by the transformations which switch t_i to $-t_i$ realizes the subgroup 2^4.

Let G be the quotient of $\mathrm{Aut}(S)$ by the subgroup 2^4. The group $\mathrm{Aut}(S)$ acts on the pencil $|\mathcal{I}_S(2)|$ of quadrics containing S leaving invariant the subset of five singular quadrics. The kernel of this action is the subgroup 2^4. Thus G is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2)$ leaving a set of five points invariant. It follows from the classification of finite subgroups of $\mathrm{SL}(2)$ and their algebra of invariants that the only possible groups are the cyclic groups C_2, C_3, C_4, C_5, the dihedral group $D_6 \cong \mathfrak{S}_3$, and the dihedral group D_{10} of order 10. The corresponding binary forms are projectively equivalent to the following binary forms:

(i) $C_2 : u_1(u_1^2 - u_0^2)(u_1^2 - a^2u_0^2),\ a^2 \neq 0, 1;$
(ii) $C_4 : u_1(u_1^2 - u_0^2)(u_1^2 + u_0^2);$
(iii) $C_3, D_6 : u_1(u_1 - u_0)(2u_1 - u_0)(u_1 + \eta u_0)(u_1 + \eta^2 u_0^2);$
(iv) $C_5, D_{10} : (u_1 - u_0)(u_1 - \epsilon u_0)(u_1 - \epsilon^2 u_0)(u_1 - \epsilon^3 u_0)(u_1 - \epsilon^4 u_0),$

where $\eta = e^{2\pi i/3}, \epsilon = e^{2\pi i/5}$. In case (iii) the zeros of the binary form are $[1, 0], [1, 1],\ [1, 2], [1, -\eta], [1, -\eta^2]$ are projectively equivalent to the set $[1, 0], [0, 1], [1, 1],\ [1, \eta], [1, \eta^2]$. In all cases the symmetry becomes obvious; it consists of multiplication of the affine coordinate u_1/u_0 by some roots of unity, and, in cases (iii) and (iv), the additional symmetry $[u_0, u_1] \mapsto [u_1, u_0]$.

Using the equations of S from Theorem 8.6.2, we find that the singular quadrics in the pencil of quadrics

$$u_0 \sum_{i=0}^{4} a_i t_i^2 - u_1 \sum_{i=0}^{4} t_i^2 = 0$$

correspond to the parameters $[u_0, u_1] = [1, a_i]$. The corresponding surfaces are projectively equivalent to the following surfaces:

(i) $C_2 : t_0^2 + t_2^2 + t_1^2 + t_3^2 + t_4^2 = t_1^2 - t_2^2 + at_1^2 - at_3^2 = 0,\ a \neq 0, \pm 1;$
(ii) $C_4 : t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 = t_1^2 - t_2^2 + it_3^2 - it_4^2 = 0;$
(iii) $\mathfrak{S}_3 : t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 = t_1^2 + \frac{1}{2}t_2^2 + \eta t_3^2 + \eta^2 t_4^2 = 0;$
(iv) $D_{10} : t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 = t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 + \epsilon^3 t_3^2 + \epsilon^4 t_4^2 = 0.$

□

Remark 8.6.9 In 1894 G. Humbert [305] discovered a plane sextic Γ with five cusps that has an automorphism group isomorphic to 2^4. Its proper transform

on the blow-up of the five cusps is a nonsingular curve Γ' of genus 5 on a del Pezzo quartic surface S. It is embedded in $\mathbb{P}^5$ by its canonical linear system. The curve Γ' is cut out by a quadric $V(\sum a_i^2 t_i^2)$, where we consider S given by the equations from Theorem 8.6.2 (see [202]). The curve is tangent to all 16 lines on S. The double cover of S branched along this curve is a K3 surface isomorphic to a nonsingular model of a Kummer quartic surface. The following equation of Γ was found by W. Edge [205]

$$9t_2^2(t_2^2 - t_0^2)(t_2^2 - t_1^2) + (t_1^2 + 3t_2^2 - 4t_0^2)(t_1 + 2t_0)^2(t_1^2 - t_0^2) = 0.$$

The curve has peculiar properties: the residual points of each line containing two cusps coincide, and the two contact points are on a line passing through a cusp; the residual points of the conic through the five cusps coincide and all cuspidal tangents pass through the contact point (see loc.cit.). The five maps $S \to \mathbb{P}^1$ defined by the pencils of conics, restricted to Γ', define five g_4^1's on Γ'. The quotient by the involution defined by the negation of one of the coordinates t_i is an elliptic curve. This makes the 5-dimensional Jacobian variety of Γ' isogenous to the product of five elliptic curves (this is how it was found by Humbert). The quotient of Γ' by the involution defined by the negation of two coordinates t_i is a curve of genus 3. It is isomorphic to the quartic curve with automorphism group isomorphic to 2^3.

By taking special del Pezzo surfaces with isomorphism groups $2^4 \rtimes D_6$ and $2^4 \rtimes D_{10}$ we obtain curves of genus 5 with automorphism groups of order 96 and 160 (see [202]).

Let $p_1, \ldots, p_6$ be six points in $\mathbb{P}^3$ in general linear position. A *Humbert curve* can be also defined as the locus of tangency points of lines passing through p_6 with rational normal cubics passing through $p_1, \ldots, p_5$ (see [21], vol. 6, p. 24). It is also characterized by the property that it has ten effective even theta characteristics (see [585]).

The double cover of S ramified over Γ' is a K3 surface isomorphic to a nonsingular model of a Kummer quartic surface with 16 nodes. The preimages of the 16 lines on S split into 32 curves, the images of a subset of 16 of them on the Kummer surface are 16 nodes, and the images of the remaining 16 curves are the 16 conics cut by 16 tropes of the surface. The surface S admits a nonsingular model as a surface in the Grassmannian $G_1(\mathbb{P}^3)$ of degree and class equal to 2. It is one of irreducible components of the surface of bitangents of a Kummer quartic surface (see [319]).

8.7 Del Pezzo surfaces of degree 2

8.7.1 Singularities

Let S be a weak del Pezzo surface of degree 2. Recall that the anticanonical linear system defines a birational morphism $\phi' : S \to X$, where X is the anticanonical model of S isomorphic to the double cover of $\mathbb{P}^2$ branched along a plane quartic curve C with at most simple singularities (see Section 6.3.3). We have already discussed nonsingular del Pezzo surfaces of degree 2 in Chapter 6, in particularly the geometry associated with seven points in the plane in general position. A nonsingular del Pezzo surface is isomorphic to the double cover of the projective plane ramified over a nonsingular plane quartic. It has 56 lines corresponding to 28 bitangents of the branch curve.

Let $\phi : S \to \mathbb{P}^2$ be the composition of ϕ and the double cover map $\sigma : X \to \mathbb{P}^2$. The restriction of ϕ to a (-1)-curve E is a map of degree $-K_S \cdot E = 1$. Its image in the plane is a line ℓ. The preimage of ℓ is the union of E and a divisor $D \in |-K_S - E|$. Since $-K_S \cdot D = 1$, the divisor D is equal to $E' + R$, where E' is a (-1)-curve and R is the union of (-2)-curves. Also we immediately find that $E \cdot D = 2, D^2 = -1$. There are three possible cases:

(i) $E \neq E',\ E \cdot E' = 2$;
(ii) $E \neq E', E \cdot E' = 1$;
(iii) $E \neq E', E = E'$.

In the first case, the image of E is a line ℓ tangent to C at two nonsingular points. The image of $D - E'$ is a singular point of C. By Bezout's Theorem, ℓ cannot pass through the singular point. Hence $D = E'$ and ℓ is a bitangent of C.

In the second case, $E \cdot (D - E') = 1$. The line ℓ passes through the singular point $\phi(D - E')$ and is tangent to C at a nonsingular point.

Finally, in the third case, ℓ is a component of C.

Of course, when S is a del Pezzo surface, the quartic C is nonsingular, and we have 56 lines paired into 28 pairs corresponding to 28 bitangents of C. Let $\pi : S \to \mathbb{P}^2$ be the blow-up of seven points $x_1, \dots, x_7$ in general position. Then 28 pairs of lines are the proper inverse transforms of the isolated pairs of curves:

21 pairs: a line through x_i, x_j and the conic through the complementary five points;

7 pairs: a cubic with a double point at x_i and passing through other points plus the exceptional curve $\pi^{-1}(x_i)$.

We use the procedure of Borel-de Siebenthal-Dynkin to compile the list of

root bases in $\mathbf{E}_7$. It is convenient first to compile the list of maximal (by inclusions) root bases of type A, D, E (see [326], §12).

Type	rank $n-1$	rank n
A_n	$A_k + A_{n-k-1}$	
D_n	A_{n-1}, D_{n-1}	$D_k + D_{n-k}, k \geq 2$
E_6	D_5	$A_1 + A_5, A_2 + A_2 + A_2$
E_7	E_6	$A_1 + D_6, A_7, A_2 + A_5$
E_8		$D_8, A_1 + E_7, A_8, A_2 + E_6, A_4 + A_4$

Table 8.7 *Maximal root bases*

Here $D_2 = A_1 + A_1$ and $D_3 = A_3$.

From this we easily find the following table of root bases in $\mathbf{E}_7$. Note that

r	Types
7	$E_7, A_1 + D_6, A_7, 3A_1 + D_4, A_1 + 2A_3, A_5 + A_2, 7A_1$
6	$E_6, D_5 + A_1, D_6, A_6, A_1 + A_5, 3A_2, 2A_1 + D_4, 2A_3,$ $3A_1 + A_3, 6A_1, A_1 + A_2 + A_3, A_2 + A_4$
5	$D_5, A_5, A_1 + D_4, A_1 + A_4, A_1 + 2A_2, 2A_1 + A_3,$ $3A_1 + A_2, A_2 + A_3, 5A_1$
≤ 4	$D_4, A_{i_1} + \cdots + A_{i_k}, i_1 + \cdots + i_k \leq 4$

Table 8.8 *Root bases in the $\mathbf{E}_7$-lattice*

there are two root bases of types $A_1 + A_5$, $A_2 + 2A_1$, $3A_1$, $A_1 + A_3$ and $4A_1$ which are not equivalent with respect to the Weyl group.

The simple singularities of plane quartics were classified by P. Du Val [196], Part III.

A_1: one node;

$2A_1$: two nodes;

A_2: one cusp;

$3A_1$: irreducible quartic with three nodes;

$3A_1$: a cubic and a line;

$A_1 + A_2$: one node and one cusp;

A_3: one tacnode (two infinitely near ordinary double points);

$4A_1$: a nodal cubic and a line;

$4A_1$: two conics intersecting at 4 points;

$2A_1 + A_2$: two nodes and one cusp;

$A_1 + A_3$: a node and a tacnode;

$A_1 + A_3$: cubic and a tangent line;

A_4: one rhamphoid cusp (two infinitely near cusps);
$2A_2$: two cusps;
D_4: an ordinary triple point;
$5A_1$: a conic and two lines;
$3A_1 + A_2$: a cuspidal cubic and a line;
$2A_1 + A_3$: two conics tangent at one point;
$2A_1 + A_3$: a nodal cubic and its tangent line;
$A_1 + A_4$: a rhamphoid cusp and a node;
$A_1 + 2A_2$: a cusp and two nodes;
$A_2 + A_3$: a cusp and a tacnode;
A_5: one oscnode (two infinitely near cusps);
A_5: a cubic and its inflection tangent;
D_5: nodal cubic and a line tangent at one branch;
$A_1 + D_4$: a nodal cubic and line through the node;
E_6: an irreducible quartic with one e_6-singularity;
D_6: triple point with one cuspidal branch;
$A_1 + A_5$: two conics intersecting at two points with multiplicities 3 and 1;
$A_1 + A_5$: a nodal cubic and its inflection tangent;
$6A_1$: four lines in general position;
$3A_2$: a three-cuspidal quartic;
$2A_1 + D_4$: two lines and conic through their intersection point;
$D_5 + A_1$: cuspidal cubic and a line through the cusp;
$2A_3$: two conics intersecting at two points with multiplicities 2;
$3A_1 + A_3$: a conic plus its tangent line plus another line;
$A_1 + A_2 + A_3$: cuspidal cubic and its tangent;
A_6: one oscular rhamphoid cusp (three infinitely near $x_1 \succ x_2 \succ x_1$ cusps);
$A_2 + A_4$: one rhamphoid cusp and a cusp;
E_7: cuspidal cubic and its cuspidal tangent;
$A_1 + D_6$: conic plus tangent line and another line through point of contact;
$D_4 + 3A_1$: four lines with three concurrent;
A_7: two irreducible conics intersecting at one point;
$A_5 + A_2$: cuspidal cubic and an inflection tangent;
$2A_3 + A_1$: conic and two tangent lines.

Note that all possible root bases are realized except $7A_1$ (this can be realized in characteristic 2). One can compute the number of lines but this rather tedious. For example, in the case A_1 we have 44 lines and a one-nodal quartic C has 22 proper bitangents (i.e. lines with two nonsingular points of tangency) and six bitangents passing through the node.

The Gosset polytope $\Sigma_7 = 3_{21}$ has 576 facets of type α and 126 facets of type β. They correspond to contractions of seven disjoint (-1)-curves and pencils of conics arising from seven pencils of lines through one of the seven points in the plane, 35 pencils of conics through four points, 42 pencils of cubic curves through six points with a node at one of these points, 35 pencils of 3-nodal quartics through the seven points, and seven pencils of quintics through the seven points with six double points.

8.7.2 Geiser involution

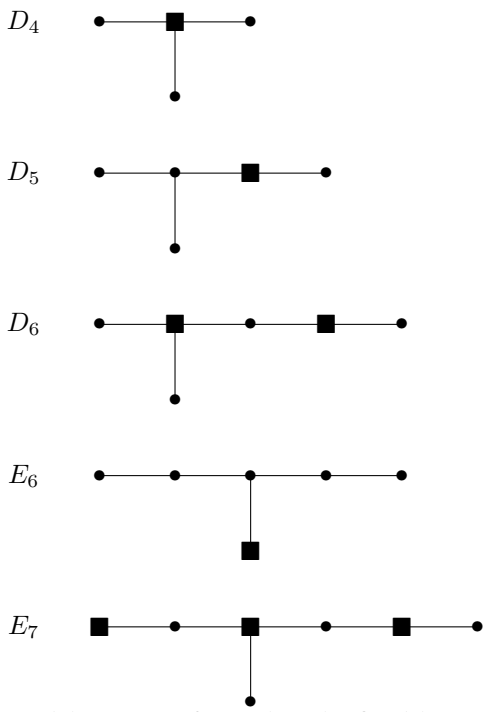
Let S be a weak del Pezzo surface of degree 2. Consider the degree 2 regular map $\phi : S \to \mathbb{P}^2$ defined by the linear system $|-K_S|$. In the blow-up model of S, the linear system $|-K_S|$ is represented by the net of cubic curves $\mathcal{N}$ with seven base bubble points $x_1, \dots, x_7$ in $\mathbb{P}^2$. It is an example of a Laguerre net considered in Subsection 7.3.3. Thus we can view S as the blow-up of 7 points in the plane $\mathbb{P}^2$ which is canonically identified with $|-K_S|$. The target plane $\mathbb{P}^2$ can be identified with the dual plane $|-K_S|^\vee$ of $|-K_S|$. The plane quartic curve C belongs to $|-K_S|^\vee$.

If S is a del Pezzo surface, then ϕ is a finite map of degree 2 and any subpencil of $|-K_S|$ has no fixed component. Any pencil contained in $\mathcal{N}$ has no fixed components and has two points outside the base points of the net. Assigning the line through these points, we will be able to identify the plane $\mathbb{P}^2$ with the net $\mathcal{N}$, or with $|-K_S|$. This is the property of a Laguerre net. The inverse map is defined by using the coresidual points of Sylvester. For every nonsingular member $D \in \mathcal{N}$, the restriction of $|-K_S|$ to D defines a g^1_2 realized by the projection from the coresidual point on D. This map extends to an isomorphism $\mathcal{N} \to \mathbb{P}^2$.

Let $X \subset \mathbb{P}(1,1,1,2)$ be an anticanonical model of S. The map ϕ factors through a birational map $\sigma : S \to X$ that blows down the Dynkin curves and a degree 2 finite map $\bar{\phi} : X \to \mathbb{P}^2$ ramified along a plane quartic curve C with simple singularities. The deck transformation γ of the cover $\bar{\phi}$ is a birational automorphism of S called the *Geiser involution*. In fact, the Geiser involution is a biregular automorphism of S. Since σ is a minimal resolution of singularities of X, this follows from the existence of an equivariant minimal

resolution of singularities of surfaces [363] and the uniqueness of a minimal resolution of surfaces.

Proposition 8.7.1 *The Geiser involution γ has no isolated fixed points. Its locus of fixed points is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \ldots, R_k$ are among irreducible components of Dynkin curves. The curve W is the normalization of the branch curve of the double cover $\phi : S \to \mathbb{P}^2$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of type D_4, D_5, D_6, E_6, E_7 have fixed components marked by square on their Coxeter-Dynkin diagrams.*



Assume that S is a del Pezzo surface. Then the fixed locus of the Geiser involution is a smooth irreducible curve W isomorphic to the branch curve of the cover. It belongs to the linear system $|-2K_S|$ and hence its image in the plane is a curve of degree 6 with double points at $x_1, \ldots, x_7$. It is equal to the Jacobian curve of the net of cubics, i.e. the locus of singular points of singular cubics from the set. It follows from the Lefschetz Fixed-Point-Formula that the trace of γ in $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is equal to $e(W) - 2 = -6$. This implies

that the trace of σ on $Q_S = (K_S)^\perp$ is equal to -7. Since rank $Q_S = 7$ this implies that γ acts as the minus identity on Q_S. It follows from the theory of finite reflection groups that the minus identity isogeny of the lattice $\mathbf{E}_7$ is represented by the element w_0 in $W(\mathbf{E}_7)$ of maximal length as a word in simple reflections. It generates the center of $W(\mathbf{E}_7)$.

We can also consider the Geiser involution as a Cremona involution of the plane. It coincides with the Geiser involution described in Chapter 7. The characteristic matrix of a Geiser involution with respect to the bases $e_0, \dots, e_7$ and $\sigma^*(e_0), \dots, \sigma^*(e_7)$ is the following matrix:

$$\begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix}. \tag{8.23}$$

The element w_0 acts on the Gosset polytope 3_{21} as the reflection with respect to the center defined by the vector $\frac{1}{2}\mathbf{k}_7 = -\frac{1}{56}\sum v_i$, where v_i are the exceptional vectors. The 28 orbits on the set of vertices correspond to 28 bitangents of a nonsingular plane quartic.

8.7.3 Automorphisms of del Pezzo surfaces of degree 2

Let S be a del Pezzo surface of degree 2. We know that the natural homomorphism

$$\mathrm{Aut}(S) \to W(S) \cong W(E_7)$$

is injective. The Geiser involution γ belongs to the center of $W(S)$. The quotient group $\mathrm{Aut}(S)/\langle\gamma\rangle$ is the group of projective automorphisms that leaves the branch curve C of the map invariant $\phi : S \to \mathbb{P}^2$. We use the classification of automorphisms of plane quartic curves from Chapter 6. Let G' be a group of automorphisms of the branch curve $C = V(f)$. Let $\chi : G' \to \mathbb{C}^*$ be the character of G' defined by $\sigma^*(f) = \chi(\sigma)f$. Let

$$G = \{(g', \alpha) \in G' \times \mathbb{C}^* : \chi(g') = \alpha^2\}.$$

This is a subgroup of the group $G' \times \mathbb{C}^*$. The projection to G' defines an isomorphism $G \cong 2.G'$. The extension splits if and only if χ is equal to the

square of some character of G'. In this case $G \cong G' \times 2$. The group G acts on S as given by Equation (8.16) by

$$(\sigma', \alpha) : [t_0, t_1, t_2, t_3] \mapsto [\sigma'^*(t_0), \sigma'^*(t_1), \sigma'^*(t_2), \alpha t_3].$$

Any group of automorphisms of S is equal to a group G as above. This easily gives the classification of possible automorphism groups of del Pezzo surfaces of degree 2 (see Table 8.9).

Type	Order	Structure	Equation
I	336	$2 \times L_2(7)$	$t_3^2 + t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0$
II	192	$2 \times (4^2 : \mathfrak{S}_3)$	$t_3^2 + t_0^4 + t_1^4 + t_2^4$
III	96	$2 \times 4\mathfrak{A}_4$	$t_3^2 + t_2^4 + t_0^4 + \alpha t_0^2 t_1^2 + t_1^4$
IV	48	$2 \times \mathfrak{S}_4$	$t_3^2 + t_2^4 + t_1^4 + t_0^4 + +a(t_0^2 t_1^2 + t_0^2 t_2^2 + t_1^2 t_2^2)$
V	32	2×4.2^2	$t_3^2 + t_2^4 + t_0^4 + a t_0^2 t_1^2 + t_1^4$
VI	18	18	$t_3^2 + t_0^4 + t_0 t_1^3 + t_1 t_2^3$
VII	16	$2 \times D_8$	$t_3^2 + t_2^4 + t_0^4 + t_1^4 + a t_0^2 t_1^2 + b t_2^2 t_0 t_1$
VIII	12	2×6	$t_3^2 + t_2^3 t_0 + t_0^4 + t_1^4 + a t_0^2 t_1^2$
IX	12	$2 \times \mathfrak{S}_3$	$t_3^2 + t_2^4 + a t_2^2 t_0 t_1 + t_2(t_0^3 + t_1^3) + b t_0^2 t_1^2$
X	8	2^3	$t_3^2 + t_2^4 + t_1^4 + t_0^4 + a t_2^2 t_0^2 + b t_1^2 t_2^2 + c t_0^2 t_1^2$
XI	6	6	$t_3^2 + t_2^3 t_0 + f_4(t_0, t_1)$
XII	4	2^2	$t_3^2 + t_2^4 + t_2^2 f_2(t_0, t_1) + f_4(t_0, t_1)$
XIII	2	2	$t_3^2 + f_4(t_0, t_1, t_2)$

Table 8.9 *Groups of automorphisms of del Pezzo surfaces of degree 2*

We leave it to a curious reader the task of classifying automorphism groups of weak del Pezzo surfaces. Notice that in the action of $\mathrm{Aut}(S)$ in the Picard group they correspond to certain subgroups of the group $\mathrm{Cris}(S)$. Also the action is not necessarily faithful; for example, the Geiser involution acts trivially on $\mathrm{Pic}(S)$ in the case of a weak del Pezzo surface with singularity of type E_7.

8.8 Del Pezzo surfaces of degree 1

8.8.1 Singularities

Let S be a weak del Pezzo surface of degree 1. It is isomorphic to the blow-up of a bubble cycle of eight points in almost general position. The anticanonical model X of S is a finite cover of degree 2 of a quadratic cone Q ramified over a curve B in the linear system $|\mathcal{O}_Q(3)|$. It is nonsingular or has simple singularities. The list of types of possible Dynkin curves is easy to compile. First we observe that all diagrams listed for the case of the $\mathbf{E}_7$-lattice are included

in the list. Also all the diagrams $A_1 + T$, where T is from the previous list are included. We give only the new types.

r	Types
8	$E_8, A_8, D_8, 2A_4, A_1 + A_2 + A_5, A_3 + D_5, 2D_4,$ $A_2 + E_6, A_3 + D_5, 4A_2$
7	$D_7, A_2 + D_5, A_3 + A_4, A_3 + D_4$
6	$A_2 + D_4$

Table 8.10 *Root bases in the* $\mathbf{E}_8$*-lattice*

Note that there are two root bases of types A_7, $2A_3$, $A_1 + A_5$, $2A_1 + A_3$ and $4A_1$, which are not equivalent with respect to the Weyl group.

The following result of P. Du Val [196] will be left without proof. Note that Du Val uses the following notation:

$$A_1 = [\,],\ A_n = [3^{n-1}],\ n \geq 2,\ D_n = [3^{n-3,1,1}],\ n \geq 4,$$

$$E_6 = [3^{3,2,1}],\ E_7 = [3^{4,2,1}],\ E_8 = [3^{5,2,1}].$$

Theorem 8.8.1 *All types of root bases in* $\mathbf{E}_8$ *can be realized by Dynkin curves except the cases* $7A_1, 8A_1, D_4 + 4A_1$.

In fact, Du Val describes explicitly the singularities of the branch sextic similarly to the case of weak del Pezzo surfaces of degree 2 (see also Table 8.10).

The number of lines on a del Pezzo surface of degree 1 is equal to 240. Note the coincidence with the number of roots. The reason is simple, for any root $\alpha \in \mathbf{E}_8$, the sum $-k_8 + \alpha$ is an exceptional vector. The image of a line under the cover $\phi : S \to Q$ is a conic. The plane spanning the conic is a *tritangent plane*, i.e. a plane touching the branch sextic W at three points. There are 120 tritangent planes, each cut out a conic in Q which splits under the cover in the union of two lines intersecting at three points. Note that the effective divisor D of degree 3 on W such that $2D$ is cut out by a tritangent plane, is an odd theta characteristic on W. This gives another explanation of the number $120 = 2^3(2^4 - 1)$.

The Gosset polytope $\Sigma_8 = 4_{21}$ has 17280 facets of type α corresponding to contractions of sets of eight disjoint (-1)-curves, and 2160 facets of type β corresponding to conic bundle structures arising from the pencils of conics $|de_0 - m_1e_1 - \cdots - m_8|$ in the plane which we denote by $(d; m_1, \ldots, m_8)$:

- 8 of type $(1; 1, 0^7)$,
- 70 of type $(2; 1^4, 0^5)$,

- 168 of type $(3; 2, 1^5, 0^2)$,
- 280 of type $(4; 2^3, 1^4, 0)$,
- 8 of type $(4; 3, 1^7)$,
- 56 of type $(5; 2^6, 1, 0)$,
- 280 of type $(5; 3, 2^3, 1^4)$,
- 420 of type $(6; 3^2, 2^4, 1^2)$,
- 280 of type $(7; 3^4, 2^3, 1)$,
- 56 of type $(7; 4, 3, 2^6)$,
- 8 of type $(8; 3^7, 1)$,
- 280 of type $(8; 4, 3^4, 2^3)$,
- 168 of type $(9; 4^2, 3^5, 2)$,
- 70 of type $(10; 4^4, 3^4)$,
- 8 of type $(11; 4^7, 3)$,

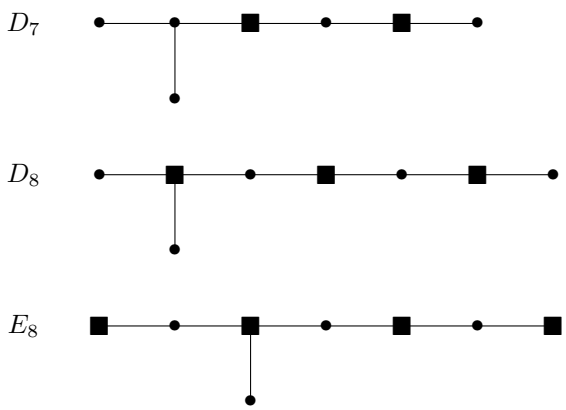
Observe the symmetry $(d; m_1, \dots, m_8) \mapsto -4\mathbf{k}_8 - (d; m_1, \dots, m_8)$.

8.8.2 Bertini involution

Let S be a weak del Pezzo surface of degree 1. Consider the degree 2 regular map $\phi : S \to Q$ defined by the linear system $|-2K_S|$. In the blow-up model of S, the linear system $|-2K_S|$ is represented by the web $\mathcal{W}$ of sextic curves with eight base bubble points $x_1, \dots, x_8$ in $\mathbb{P}^2$. If S is a del Pezzo surface, then ϕ is a finite map of degree 2.

Let $X \subset \mathbb{P}(1, 1, 2, 3)$ be the anticanonical model of S. The map ϕ factors through the birational map $\sigma : S \to X$ that blows down the Dynkin curves and a degree 2 finite map $\bar{\phi} : X \to Q$ ramified along a curve of degree 6 cut out by a cubic surface. The deck transformation β of the cover $\bar{\phi}$ is a birational automorphism of S called the *Bertini involution*. As in the case of the Geiser involution, we prove that the Bertini involution is a biregular automorphism of S.

Proposition 8.8.2 *The Bertini involution β has one isolated fixed point, the base point of $|-K_S|$. The 1-dimensional part of the locus of fixed points is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \dots, R_k$ are among irreducible components of Dynkin curves. The curve W is the normalization of the branch curve of the double cover $\phi : S \to Q$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of types D_4, D_7, D_8, E_8 have fixed components marked by square on their Coxeter-Dynkin diagrams. The fixed components of Dynkin curves of other types given in the diagrams from Proposition 8.7.1.*



Assume that S is a del Pezzo surface. Then the fixed locus of the Bertini involution is a smooth irreducible curve W of genus 4 isomorphic to the branch curve of the cover and the base point of $|-K_S|$. It belongs to the linear system $|-3K_S|$ and hence its image in the plane is a curve of degree 9 with triple points at $x_1, \ldots, x_8$. It follows from the Lefschetz fixed-point-formula that the trace of β in $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is equal to $1 + e(W) - 2 = -7$. This implies that the trace of σ on $Q_S = (K_S)^{\perp}$ is equal to -8. Since rank $Q_S = 8$ this implies that γ acts as the minus identity on Q_S. It follows from the theory of finite reflection groups that the minus identity isogeny of the lattice $\mathbf{E}_8$ is represented by the element w_0 in $W(\mathbf{E}_8)$ of maximal length as a word in simple reflections. It generates the center of $W(\mathbf{E}_8)$.

We can also consider the Bertini involution as a Cremona involution of the plane. It coincides with a Bertini involution described in Chapter 7. The characteristic matrix of a Bertini involution with respect to the bases $e_0, \ldots, e_8$ and $\sigma^*(e_0), \ldots, \sigma^*(e_8)$ is the following matrix:

$$\begin{pmatrix} 17 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -3 \end{pmatrix}.$$

We can consider this matrix as the matrix of the element $w_0 \in \mathrm{O}(I^{1,8})$ in the basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_8$. It is immediately checked that its restriction to $\mathbf{E}_8$ is equal to the minus identity transformation. As an element of the Weyl group

$W(\mathbf{E}_8)$, it is usually denoted by w_0. This is element of maximal length as a word in simple reflections. The group $\langle w_0 \rangle$ is equal to the center of $W(\mathbf{E}_8)$.

The element w_0 acts on the Gosset polytope 4_{21} as the reflection with respect to the center defined by the vector $\mathbf{k}_8 = -\frac{1}{240}\sum v_i$, where v_i are the exceptional vectors. The 120 orbits on the set of vertices correspond to 120 tritangent planes of the branch curve of the Bertini involution.

8.8.3 Rational elliptic surfaces

We know that the linear system $|-K_S|$ is an irreducible pencil with one base point x_0. Let $\tau : F \to S$ be its blow-up. The proper inverse transform of $|-K_S|$ in F is a base-point-free pencil of curves of arithmetic genus 1. It defines an elliptic fibration $\varphi : F \to \mathbb{P}^1$. The exceptional curve $E = \tau^{-1}(x_0)$ is a section of the fibration. Conversely, let $\varphi : F \to \mathbb{P}^1$ be an elliptic fibration on a rational surface F which admits a section E and is relative minimal in the sense that no fiber contains a (-1)-curve. It follows from the theory of elliptic surfaces that $-K_F$ is the divisor class of a fiber and E is a (-1)-curve. Blowing down E, we obtain a rational surface S with $K_S^2 = 1$. Since K_F is obviously nef, we obtain that K_S is nef, so S is a weak del Pezzo surface of degree 1.

Let $\varphi : F \to \mathbb{P}^1$ be a rational elliptic surface with a section E. The section E defines a rational point e on a generic fiber F_η, considered as a curve over the functional field K of the base of the fibration. It is a smooth curve of genus 1, so it admits a group law with the zero equal to the point e. It follows from the theory of relative minimal models of surfaces that any automorphism of F_η over K extends to a biregular automorphism of F over $\mathbb{P}^1$. In particular, the negation automorphism $x \to -x$ extends to an automorphism of F fixing the curve E. Its descent to the blowing down of E is the Bertini involution.

Let D be a Dynkin curve on S. The point x_0 cannot lie on D. In fact, otherwise the proper transform R' of a component of D that contains t_0 is a (-3)-curve on F. However, $-K_F$ is nef on F, hence $K_F \cdot R' \leq 0$ contradicting the adjunction formula. This implies that the preimage $\tau^*(D)$ of D on F is a Dynkin curve contained in a fibre. The whole fiber is equal to the union of $\tau^*(D) + R$, where R is a (-2)-curve intersecting the zero section E. Kodaira's classification of fibres of elliptic fibrations shows that the intersection graph of the irreducible components of each reducible fiber is equal to one of the extended Coxeter-Dynkin diagrams.

The classification of Dynkin curves on a weak del Pezzo surfaces of degree 1 gives the classification of all possible collections of reducible fibres on a rational elliptic surface with a section. The equation of the anticanonical model

in $\mathbb{P}(1,1,2,3)$

$$t_3^2 + t_2^3 + f_4(t_0,t_1)t_2 + f_6(t_0,t_1) = 0, \tag{8.24}$$

after dehomogenization $t = t_1/t_0, x = t_2/t_0^2, y = t_3/t_0^3$, becomes the *Weierstrass equation* of the elliptic surface

$$y^2 + x^3 + a(t)x + b(t) = 0.$$

The classification of all possible singular fibres of rational elliptic surfaces (not necessarily reducible) in terms of the Weierstrass equation was done by several people, e.g. [435].

8.8.4 Automorphisms of del Pezzo surfaces of degree 1

Let S be a nonsingular del Pezzo surface of degree 1. We identify it with its anticanonical model (8.24). The vertex of Q has coordinates $[0,0,1]$ and its preimage in the cover consists of one point $[0,0,1,a]$, where $a^2+1=0$ (note that $[0,0,1,a]$ and $[0,0,1,-a]$ represent the same point in $\mathbb{P}(1,1,2,3)$). This is the base point of $|-K_S|$. The members of $|-K_S|$ are isomorphic to genus 1 curves with equations $y^2+x^3+f_4(t_0,t_1)x+f_6(t_0,t_1)=0$. Our group $\bar{G}$ acts on $\mathbb{P}^1$ via a linear action on (t_0,t_1). The locus of zeros of $\Delta = 4f_4^3 + 27f_6^2$ is the set of points in $\mathbb{P}^1$ such that the corresponding genus 1 curve is singular. It consists of a simple roots and b double roots. The zeros of f_4 are either common zeros with f_6 and Δ, or represent nonsingular equianharmonic elliptic curves. The zeros of f_6 are either common zeros with f_4 and Δ, or represent nonsingular harmonic elliptic curves. The group $\bar{G}$ leaves both sets invariant.

Recall that $\bar{G}$ is determined up to conjugacy by its set of points in $\mathbb{P}^1$ with nontrivial stabilizers. If $\bar{G}$ is not cyclic, then there are three orbits in this set of cardinalities $n/e_1, n/e_2, n/e_3$, where $n = \#\bar{G}$ and (e_1,e_2,e_3) are the orders of the stabilizers. Let Γ be a finite noncyclic subgroup of $\mathrm{PGL}(2)$. We have the following possibilities:

(i) $\Gamma = D_{2k}, n = 2k, (e_1,e_2,e_3) = (2,2,k)$;
(ii) $\Gamma = \mathfrak{A}_4, n = 12, (e_1,e_2,e_3) = (2,3,3)$;
(iii) $\Gamma = \mathfrak{S}_4, n = 24, (e_1,e_2,e_3) = (2,3,4)$;
(iv) $\Gamma = \mathfrak{A}_5, n = 60, (e_1,e_2,e_3) = (2,3,5)$.

If $\bar{\Gamma}$ is a cyclic group of order n, there are 2 orbits of cardinality 1.

The polynomials f_4 and f_6 are projective invariants of $\bar{G}$ on $\mathbb{P}^1$, i.e. their sets of zeros are invariant with respect to the group action. Each orbit defines a binary form (the orbital form) with the set of zeros equal to the orbit. One

can show that any projective invariant is a polynomial in orbital forms. This immediately implies that $\bar{G} \ncong \mathfrak{A}_5$ and if $\bar{G} \cong \mathfrak{S}_4$, then $f_4 = 0$.

We choose to represent $\bar{G}$ by elements of $\mathrm{SL}(2)$, i.e. we consider $\bar{G}$ as a quotient of a binary polyhedral subgroup $G \subset \mathrm{SL}(2)$ by its intersection with the center of $\mathrm{SL}(2)$. A projective invariant of $\bar{G}$ becomes a relative invariant of G, i.e. elements of G which leave the line spanned by the form invariant. Each relative invariant defines a character of G defined by

$$\sigma^*(f) = \chi(\sigma) f.$$

We use the description of relative invariants and the corresponding characters of G from [539]. This allows us to list all possible polynomials f_4 and f_6.

The following is the list of generators of the groups $\bar{G}$, possible relative invariants f_4, f_6 and the corresponding character.

We use the fact that a multiple root of f_6 is not a root of f_4 (otherwise the surface is singular). In the following ϵ_k will denote a primitive k-th root of unity.

Case 1: $\bar{G}$ is cyclic of order n. If n is odd (even), we choose a generator σ given by the diagonal matrix $\mathrm{diag}[\epsilon_n, \epsilon_n^{-1}]$ ($\mathrm{diag}[\epsilon_{2n}, \epsilon_{2n}^{-1}]$). Any monomial $t_0^i t_1^j$ is a relative invariant with $\chi(\sigma) = \epsilon_n^{i-j}$ if n is odd and $\chi(\sigma) = \epsilon_{2n}^{i-j}$ if n is even. In Table 8. 11 we list relative invariants which are not monomials.

n	f_4	$\chi(\sigma)$	f_6	$\chi(\sigma)$
2	$at_0^4 + bt_0^2t_1^2 + ct_1^4$	1	$at_0^6 + t_0^2t_1^2(bt_0^2 + ct_1^2) + dt_1^6$	-1
	$t_0t_1(at_0^2 + bt_1^2)$	-1	$t_0t_1(at_0^4 + bt_0^2t_1^2 + ct_1^4)$	1
3	$t_0(at_0^3 + bt_1^3)$	ϵ_3	$at_0^6 + bt_0^3t_1^3 + ct_1^6$	1
	$t_1(at_0^3 + bt_1^3)$	ϵ_3^2	$t_0t_1^2(at_0^3 + bt_1^3)$	ϵ_3^2
			$t_0^2t_1(at_0^3 + bt_1^3)$	ϵ_3
4	$at_0^4 + bt_1^4$	-1	$t_0^2(at_0^4 + bt_1^4)$	-i
			$t_0t_1(at_0^4 + bt_1^4)$	-1
			$t_1^2(at_0^4 + bt_1^4)$	i
5			$t_0(at_0^5 + bt_1^5)$	ϵ_5
			$t_1(at_0^5 + bt_1^5)$	ϵ_5^4
6			$at_0^6 + bt_1^6$	-1

Table 8.11 *Relative invariants:cyclic group*

Case 2: $\bar{G} = D_n$ is a dihedral group of order $n = 2k$. It is generated by two matrices

$$\sigma_1 = \begin{pmatrix} \epsilon_{2k} & 0 \\ 0 & \epsilon_{2k}^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

k	f_4	$\chi(\sigma_1)$	$\chi(\sigma_2)$
2	$a(t_0^4+t_1^4)+bt_0^2t_1^2$	1	1
	$t_0t_1(t_0^2-t_1^2)$	-1	-1
	$t_0^4-t_1^4$	1	-1
	$t_0t_1(t_0^2+t_1^2)$	-1	1
3	$t_0^2t_1^2$	1	1
4	$t_0^4\pm t_1^4$	-1	± 1
6	$t_0^2t_1^2$	1	1

Table 8.12 *Relative invariants of degree 4:dihedral group*

k	f_6	$\chi(\sigma_1)$	$\chi(\sigma_2)$
2	$t_0t_1\big(a(t_0^4+t_1^4)+bt_0^2t_1^2\big)$	1	-1
	$a(t_0^6+t_1^6)+bt_0^2t_1^2(t_0^2+t_1^2)$	-1	-1
	$a(t_0^6-t_1^6)+bt_0^2t_1^2(t_0^2-t_1^2)$	-1	1
	$t_0t_1(t_0^4-t_1^4)$	1	1
3	$t_0^6+t_1^6+at_0^3t_1^3$	1	-1
	$t_0^6-t_1^6$	1	1
4	$t_0t_1(t_0^4\pm t_1^4)$	-1	∓ 1
6	$t_0^6\pm t_1^6$	-1	∓ 1

Table 8.13 *Relative invariants of degree 6:dihedral group*

Case 3: $\bar{G}=\mathfrak{A}_4$. It is generated by matrices

$$\sigma_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},\quad \sigma_3=\frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1}\\ \epsilon_8^{5} & \epsilon_8\end{pmatrix}.$$

Up to the variable change $t_0\to it_0, t_1\to t_1$, we have only one case

$$f_4=t_0^4+2\sqrt{-3}t_0^2t_1^2+t_1^4,\ (\chi(\sigma_1),\chi(\sigma_2),\chi(\sigma_3)=(1,1,\epsilon_3), \quad (8.25)$$

$$f_6=t_0t_1(t_0^4-t_1^4),\ (\chi(\sigma_1),\chi(\sigma_2),\chi(\sigma_3)=(1,1,1). \quad (8.26)$$

Case 4: $\bar{G}=\mathfrak{S}_4$. It is generated by matrices

$$\sigma_1=\begin{pmatrix} \epsilon_8 & 0\\ 0 & \epsilon_8^{-1}\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},\quad \sigma_3=\frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1}\\ \epsilon_8^{5} & \epsilon_8\end{pmatrix}.$$

There is only one, up to a change of variables, orbital polynomial of degree ≤ 6. It is

$$f_6=t_0t_1(t_0^4-t_1^4).$$

It is an invariant of $\bar{G}$. In this case $f_4=0$.

In the next Theorem we list all possible groups $G'=\mathrm{Aut}(S)/\langle\beta\rangle$ and their lifts G to subgroups of $\mathrm{Aut}(S)$. We extend the action of $\bar{G}$ on the coordinates

t_0, t_1 to an action on the coordinates t_0, t_1, t_2. Note that not all combinations of (f_4, f_6) admit such an extension. For example, a common root of f_4 and f_6 must be a simple root of f_6 since otherwise the surface S is singular.

In the following list, the vector $\mathbf{a} = (a_0, a_1, a_2, a_3)$ will denote the transformation $[t_0, t_1, t_2, t_3] \mapsto [a_0 t_0, a_1 t_1, a_2 t_2, a_3 t_3]$. The Bertini transformation β corresponds to the vector $(1, 1, 1, -1)$.

1. *Cyclic groups G'*

(i) $G' = 2, G = \langle (1, -1, 1, 1), \beta \rangle \cong 2^2$,

$$f_4 = at_0^4 + bt_0^2 t_1^2 + ct_1^4, \quad f_6 = dt_0^6 + et_0^4 t_1^2 + ft_0^2 t_1^4 + gt_1^6.$$

(ii) $G' = 2, G = \langle (1, -1, -1, i) \rangle$,

$$f_4 = at_0^4 + bt_0^2 t_1^2 + ct_1^4, \quad f_6 = t_0 t_1 (dt_0^4 + et_0^2 t_1^2 + ft_1^4).$$

(iii) $G' = 3, G = \langle (1, \epsilon_3, 1, -1) \rangle \cong 6$,

$$f_4 = t_0(at_0^3 + bt_1^3), \quad f_6 = at_0^6 + bt_0^3 t_1^3 + ct_1^6.$$

(iv) $G' = 3, G = \langle (1, \epsilon_3, \epsilon_3, -1) \rangle$,

$$f_4 = t_0^2 t_1^2, \quad at_0^6 + bt_0^3 t_1^3 + ct_1^6.$$

(v) $G' = 3, G = 6, \mathbf{a} = (1, 1, \epsilon_3, -1)$,

$$f_4 = 0.$$

(vi) $G' = 4, G = \langle (i, 1, -1, i), \beta \rangle \cong 4 \times 2$,

$$f_4 = at_0^4 + bt_1^4, \quad f_6 = t_0^2 (ct_0^4 + dt_1^4).$$

(vii) $G' = 4, G = \langle (i, 1, -i, -\epsilon_8) \rangle \cong 8$,

$$f_4 = at_0^2 t_1^2, \quad f_6 = t_0 t_1 (ct_0^4 + dt_1^4),$$

(viii) $G' = 5, G = \langle (1, \epsilon_5, 1, -1) \rangle \cong 10$,

$$f_4 = at_0^4, \quad f_6 = t_0(bt_0^5 + t_1^5).$$

(ix) $G' = 6, G = \langle (1, \epsilon_6, 1, 1), \beta \rangle \cong 2 \times 6$.

$$f_4 = t_0^4, \quad f_6 = at_0^6 + bt_1^6.$$

(x) $G' = 6, G = \langle (\epsilon_6, 1, \epsilon_3^2, 1), \beta \rangle \cong 2 \times 6$,

$$f_4 = t_0^2 t_1^2, \quad f_6 = at_0^6 + bt_1^6.$$

(xi) $G' = 6, G = \langle (-1, 1, \epsilon_3, 1), \beta \rangle \cong 2 \times 6$,

$$f_4 = 0, \quad f_6 = dt_0^6 + et_0^4 t_1^2 + ft_0^2 t_1^4 + gt_1^6,$$

(xii) $G' = 10, G = \langle (1, \epsilon_{10}, -1, i) \rangle \cong 20$,

$$f_4 = at_0^4, \quad f_6 = t_0 t_1^5.$$

(xiii) $G' = 12, G = \langle (\epsilon_{12}, 1, \epsilon_3^2, -1), \beta \rangle \cong 2 \times 12$,

$$f_4 = at_0^4, \quad f_6 = t_1^6.$$

(xiv) $G' = 12, G = \langle (i, 1, \epsilon_{12}, \epsilon_8) \rangle \cong 24$,

$$f_4 = 0, \quad f_6 = t_0 t_1 (t_0^4 + bt_1^4).$$

(xv) $G' = 15, G = \langle (1, \epsilon_5, \epsilon_3, \epsilon_{30}) \rangle \cong 30$,

$$f_4 = 0, \quad f_6 = t_0 (t_0^5 + t_1^5).$$

2. *Dihedral groups*

(i) $G' = 2^2, G = D_8$,

$$f_4 = a(t_0^4 + t_1^4) + bt_0^2 t_1^2, \quad f_6 = t_0 t_1 [c(t_0^4 + t_1^4) + dt_0^2 t_1^2],$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_1, -t_0, t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^4 = \sigma_2^2 = 1, \sigma_1^2 = \beta, \sigma_2 \sigma_1 \sigma_2^{-1} = \sigma_2^{-1}.$$

(ii) $G' = 2^2, G = 2.D_4$,

$$f_4 = a(t_0^4 + t_1^4) + bt_0^2 t_1^2, \quad f_6 = t_0 t_1 (t_0^4 - t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, -t_1, -t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, -t_2, it_3],$$

$$\sigma_1^2 = \sigma^2 = (\sigma_1 \sigma_2)^2 = \beta.$$

(iii) $G' = D_6, G = D_{12}$,

$$f_4 = at_0^2 t_1^2, \quad f_6 = t_0^6 + t_1^6 + bt_0^3 t_1^3,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3 t_1, \epsilon_3 t_2, -t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^3 = \beta, \sigma_2^2 = 1, \ \sigma_2 \sigma_3 \sigma_2^{-1} = \sigma_1^{-1}.$$

(v) $G' = D_8, G = D_{16}$,

$$f_4 = at_0^2t_1^2, \quad f_6 = t_0t_1(t_0^4 + t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^4 = \beta, \sigma_2^2 = 1, \ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}.$$

(vi) $G' = D_{12}, G = 2.D_{12}$,

$$f_4 = at_0^2t_1^2, \quad f_6 = t_0^6 + t_1^6,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, \epsilon_3^2 t_2, t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3], \sigma_3 = \beta.$$

We have

$$\sigma_1^6 = \sigma_2^2 = \sigma_3^3 = 1, \ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}\sigma_3.$$

3. *Other groups*

(i) $G' = \mathfrak{A}_4, G = 2.\mathfrak{A}_4$,

$$f_4 = t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_2^4, \quad f_6 = t_0t_1(t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0 \\ \epsilon_8^5 & \epsilon_8 & 0 & 0 \\ 0 & 0 & \sqrt{2}\epsilon_3 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

(ii) $G' = 3 \times D_4, G = 3 \times D_8$,

$$f_4 = 0, \quad f_6 = t_0t_1(t_0^4 + at_0^2t_1^2 + t_1^4).$$

(iii) $G' = 3 \times D_6, G = 6.D_6 \cong 2 \times 3.D_6$,

$$f_4 = 0, \quad f_6 = t_0^6 + at_0^3t_1^3 + t_1^6.$$

It is generated by

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3 t_1, t_2, t_3],$$
$$\sigma_3 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].$$

They satisfy $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1^4$.

(iv) $G' = 3 \times D_{12}, G = 6.D_{12}$,

$$f_4 = 0, \quad f_6 = t_0^6 + t_1^6.$$

It is generated by

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, t_2, t_3],$$
$$\sigma_3 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].$$

We have $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1$.

(v) $G' = 3 \times \mathfrak{S}_4, G = 3 \times 2.\mathfrak{S}_4$,

$$f_4 = 0, \quad f_6 = t_0 t_1 (t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} \epsilon_8 & 0 & 0 & 0 \\ 0 & \epsilon_8^{-1} & & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0 \\ \epsilon_8^5 & \epsilon_8 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following Table gives a list of the full automorphism groups of del Pezzo surfaces of degree 1.

Type	Order	Structure	f_4	f_6
I	144	$3 \times (\overline{T} : 2)$	0	$t_0t_1(t_0^4 - t_1^4)$
II	72	$3 \times 2D_{12}$	0	$t_0^6 + t_1^6$
III	36	$6 \times D_6$	0	$t_0^6 + at_0^3t_1^3 + t_1^6$
IV	30	30	0	$t_0(t_0^5 + t_1^5)$
V	24	$\overline{T}$	$a(t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_1^4)$	$t_0t_1(t_0^4 - t_1^4)$
VI	24	$2D_{12}$	$at_0^2t_1^2$	$t_0^6 + t_1^6$
VII	24	2×12	t_0^4	t_1^6
VIII	20	20	t_0^4	$t_0t_1^5$
IX	16	D_{16}	$at_0^2t_1^2$	$t_0t_1(t_0^4 + t_1^4)$
X	12	D_{12}	$t_0^2t_1^2$	$t_0^6 + at_0^3t_1^3 + t_1^6$
XI	12	2×6	0	$g_3(t_0^2, t_1^2)$
XII	12	2×6	t_0^4	$at_0^6 + t_1^6$
XIII	10	10	t_0^4	$t_0(at_0^5 + t_1^5)$
XIV	8	Q_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$bt_0t_1(t_0^4 - t_1^4)$
XV	8	2×4	$at_0^4 + t_1^4$	$t_0^2(bt_0^4 + ct_1^4)$
XVI	8	D_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$t_0t_1(b(t_0^4 + t_1^4) + ct_0^2t_1^2)$
XVII	6	6	0	$f_6(t_0, t_1)$
XVIII	6	6	$t_0(at_0^3 + bt_1^3)$	$ct_0^6 + dt_0^3t_1^3 + t_1^6$
XIX	4	4	$g_2(t_0^2, t_1^2)$	$t_0t_1f_2(t_0^2, t_1^2)$
XX	4	2^2	$g_2(t_0^2, t_1^2)$	$g_3(t_0^2, t_1^2)$
XXI	2	2	$f_4(t_0, t_1)$	$f_6(t_0, t_1)$

Table 8.14 *Groups of automorphisms of del Pezzo surfaces of degree 1*

The parameters here satisfy some conditions in order the different tips do not overlap.

Exercises

8.1 Show that a del Pezzo surface of degree 8 in $\mathbb{P}^8$ isomorphic to a quadric is projectively isomorphic to the image of $\mathbb{P}^2$ defined by the linear system of plane quartic curves with two fixed double points.

8.2 Let S be a weak del Pezzo surface of degree 6. Show that its anticanonical model is isomorphic to a hyperplane section of the Segre variety $\mathsf{s}_{1,1,1}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$.

8.3 Show that a general point in $\mathbb{P}^6$ is contained in three secants of a del Pezzo surface of degree 6.

8.4 Prove that a del Pezzo surface of degree 6 in $\mathbb{P}^6$ has the property that all hyperplanes intersecting the surface along a curve with a singular point of multiplicity ≥ 3 have a common point in $\mathbb{P}^6$. According to [576] this distinguishes this surface among all other smooth projections of the Veronese surface $\mathsf{V}_3^2 \subset \mathbb{P}^9$ to $\mathbb{P}^6$ (see [583]).

8.5 Describe all weak del Pezzo surfaces which are toric varieties (i.e. contain an open Zariski subset isomorphic to the torus $(\mathbb{C}^*)^2$ such that each translation of the torus extends to an automorphism of the surface).

8.6 Show that a del Pezzo surface of degree 5 embeds into $\mathbb{P}^1 \times \mathbb{P}^2$ as a hypersurface of bidegree $(1, 2)$.

8.7 Show that a canonical curve of genus 6 in $\mathbb{P}^5$ lies on a unique del Pezzo quintic surface [29], [532].

8.8 Consider a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$ as the variety of lines intersecting five planes spanning a 3-dimensional space in the Plücker space. Prove that the pencil of hyperplanes through each of the planes cuts out on S a pencil of conics.

8.9 Show that the Petersen graph of ten lines on a del Pezzo quintic surface contains 12 pentagons and each pentagon represents five lines contained in a hyperplane.

8.10 Show that the union of tangent planes to a nonsingular del Pezzo surface S of degree $d \geq 5$ in $\mathbb{P}^d$ not isomorphic to a quadric is a hypersurface of degree $4(d-3)$ which is singular along S with multiplicity 4 [204],[21], vol. 6, p.275.

8.11 Show that the quotient of a nonsingular quadric by an involution with four isolated fixed points is isomorphic to a quartic del Pezzo surface with four nodes.

8.12 A *Dupont cyclide surface* is a quartic cyclide surface with 4 isolated singular points. Find an equation of such a surface.

8.13 Let S be a del Pezzo surface of degree 4 obtained by blowing up five points in the plane. Show that there exists a projective isomorphism from the conic containing the five points and the pencil of quadrics whose base locus is an anticanonical model of S such that the points are sent to singular quadrics.

8.14 Show that the Wiman pencil of 4-nodal plane sextics contains two 10-nodal rational curves [203].

8.15 Show that the linear system of quadrics in $\mathbb{P}^3$ with $8-d$ base points in general position maps $\mathbb{P}^3$ onto a 3-fold in $\mathbb{P}^{d+1}$ of degree d. Show that a del Pezzo surface of degree $d \leq 8$ in $\mathbb{P}^d$ is projectively equivalent to a hyperplane section of this threefold.

8.16 Show that the projection of a del Pezzo surface of degree d in $\mathbb{P}^d$ from a general point in the space is a surface of degree d in $\mathbb{P}^{d-1}$ with the double curve of degree $d(d-3)/2$.

8.17 Compute the number of (-1)-curves on a weak del Pezzo surfaces of degree 1 or 2.

8.18 Let X be a Bordiga surface obtained by the blow-up of ten general points in the plane and embedded in $\mathbb{P}^4$ by the linear system of quartic curves passing through the ten points. Show that X is a OADP surface.

8.19 Let X be a rational elliptic surface. Show that any pair of two disjoint sections defines an involution on X whose fixed locus is a nonsingular curve of genus 3 and the quotient by the involution is isomorphic to the ruled surface $\mathbf{F}_1$.

Historical Notes

As the name suggests, P. del Pezzo was the first who laid the foundation of the theory. In his paper of 1887 [162] he proves that a non-ruled nondegenerate surface of degree d in $\mathbb{P}^d$ can be birationally projected to a cubic surface in $\mathbb{P}^3$ from $d-3$ general points on it. He showed that the images of the tangent planes at the points are skew lines on the cubic surface and deduced from

this that $d \leq 9$. He also gave a blow-up model of del Pezzo surfaces of degree $d \geq 3$, found the number of lines and studied some singular surfaces. A realization of a del Pezzo surface of degree 5 as the variety of planes in $\mathbb{P}^4$ intersecting five planes is due to C. Segre [520]. He called the five planes the associated planes. The quartic cyclides in $\mathbb{P}^3$ with a nodal conic were first studied in 1864 by G. Darboux [152] and M. Moutard [395] and a year later by E. Kummer [351]. The detailed exposition of Darboux's work can be found in [154], [155]. Some special types of these surfaces were considered much earlier by Ch. Dupin [193]. Kummer was the first to observe the existence of five quadratic cones whose tangent planes cut out two conics on the surface (the *Kummer cones*). They correspond to the five singular quadrics in the pencil defining the corresponding quartic surface in $\mathbb{P}^4$. A. Clebsch finds a plane representation of a quartic cyclide by considering a web of cubics through five points in the plane [109]. He also finds in this way the configuration of 16 lines previously discovered by Darboux and proves that the Galois group of the equation for the 16 lines is isomorphic to $2^4 \rtimes \mathfrak{S}_5$. An 'epoch-making memoir' (see [526], p. 141) of C. Segre [517] finishes the classification of quartic cyclides by considering them as projections of a quartic surface in $\mathbb{P}^4$. Jessop's book [320] contains a good exposition of the theory of singular quartic surfaces including cyclides. At the same time Segre classified the anticanonical models of singular del Pezzo surfaces of degree 4 in terms of the pencil of quadrics they are defined by. The Segre symbol describing a pencil of quadratic forms was introduced earlier by A. Weiler [598]. The theory of canonical forms of pencils of quadrics was developed by K. Weierstrass [597] based an earlier work of J. Sylvester [553]. J. Steiner was probably the first who related seven points in the plane with curves of genus 3 by proving that the locus of singular points of the net of cubic curves is a plane sextic with nodes at the seven points [543]. A. Clebsch should be considered as a founder of the theory of del Pezzo surfaces of degree 2. In his memoir [111] on rational double planes he considers a special case of double planes branched along a plane quartic curve. He shows that the preimages of lines are cubic curves passing through a fixed set of seven points. He identifies the branch curve with the Steiner sextic and relates the Aronhold set of seven bitangents with the seven base points. Although C. Geiser was the first to discover the involution defined by the double cover, he failed to see the double plane construction.

E. Bertini, in [40], while describing his birational involution of the plane, proves that the linear system of curves of degree 6 with eight double base points has the property that any curve from the linear system passing through a general point x must also pass through a unique point x' (which are in the Bertini involution). He mentions that the same result was proved independently

by L. Cremona. This can be interpreted by saying that the linear system defines a rational map of degree 2 onto a quadric surface. Bertini also shows that the set of fixed points of the involution is a curve of degree 9 with triple points at the base points.

The classification of double singular points on algebraic surfaces in $\mathbb{P}^3$ started from the work of G. Salmon [485] who introduced the following notation C_2 for an ordinary node, B_k for *binode* (the tangent cone is the union of two different planes), which depend on how the intersection of the planes intersect the surface, and *unode* U_k with the tangent cone being a double plane. The indices here indicates the difference k between the degree of the dual surface and the dual of the nonsingular surface of the same degree. This nomenclature can be applied to surfaces in spaces of arbitrary dimension if the singularity is locally isomorphic to the above singularities. For del Pezzo surfaces the defect k cannot exceed 8 and all corresponding singularities must be rational double points of types $A_1 = C_2$, $A_{k-1} = B_k$, $D_{k-2} = U_k, k = 6, 7$, $E_6 = U_8$. Much later, P. Du Val [196] characterized these singularities as ones which do not affect the conditions on adjunctions, the conditions which can be applied to any normal surface. He showed that each RDP is locally isomorphic to either a node C_2, or binode B_k, or unode U_k, or other unodes $U_8^* = E_6, U_9^* = E_7$ and $U_{10}^* = E_8$ (he renamed U_8 with U_8^*). A modern treatment of RDP singularities was given by M. Artin [16].

In the same series of papers, P. Du Val classifies all possible singularities of anticanonical models of weak del Pezzo surfaces of any degree and relates them to Coxeter's classification of finite reflection groups. The relationship of this classification to the study of the singular fibres of a versal deformation of a simple elliptic singularities was found by J. Mérindol [384], H. Pinkham [439], [581], and E. Looijenga (unpublished).

In a fundamental paper by G. Timms [565] one can find a detailed study of the hierarchy of del Pezzo surfaces obtained by projections from a Veronese surface of degree 9. In this way he finds all possible configurations of lines and singularities. Possible projections of a nonsingular del Pezzo surface from a point outside the surface were studied by H. Baker [21], vol. 6, p. 275.

The Weyl group $W(\mathbf{E}_6)$ and $W(\mathbf{E}_7)$ as the Galois group of 27 lines on a cubic surface and the group of 28 bitangents on a plane quartic were first studied by C. Jordan [322]. These groups are discussed in many classical text-books on algebra (e.g. [596], B. II). S. Kantor [327] realized the Weyl groups $W(\mathbf{E}_n)$ as groups of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. A Coble [117], Part II, was the first who showed that the group is generated by the group of permutations and one additional involution. So we should credit him with the discovery of the Weyl groups

as reflection groups. Apparently independently of Coble, this fact was rediscovered by P. Du Val [195]. We refer to [52] for the history of Weyl groups, reflection groups and root systems. These parallel directions of study of Weyl groups have been reconciled only recently.

The Gosset polytopes were discovered in 1900 by T. Gosset [257]. The notation n_{21} belongs to him. They were later rediscovered by E. Elte and H. S. M. Coxeter (see [137]) but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on del Pezzo surfaces of degree $5 - n$ was found by Du Val [195]. In the case of $n = 2$, it goes back to [501]. The fundamental paper of Du Val is the origin of a modern approach to the study of del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [163], [377].

We refer to modern texts on del Pezzo surfaces [526], [377], [163], [344].

9

Cubic surfaces

9.1 Lines on a nonsingular cubic surface

9.1.1 More about the $\mathbf{E}_6$-lattice

Let us study the lattice $I^{1,6}$ and its sublattice $\mathbf{E}_6$ in more details.

Definition 9.1.1 *A* sixer *in $I^{1,6}$ is a set of six mutually orthogonal exceptional vectors in $I^{1,6}$.*

An example of a sixer is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$.

Lemma 9.1.2 *Let $\{v_1, \dots, v_6\}$ be a sixer. Then there exists a unique root α such that*

$$(v_i, \alpha) = 1, \quad i = 1, \dots, 6.$$

Moreover, $(w_1, \dots, w_6) = (r_\alpha(v_1), \dots, r_\alpha(v_6))$ is a sixer satisfying

$$(v_i, w_j) = 1 - \delta_{ij}.$$

The root associated to $(w_1, \dots, w_6)$ is equal to $-\alpha$.

Proof The uniqueness is obvious since $v_1, \dots, v_6$ are linearly independent, so no vector is orthogonal to all of them. Let

$$v_0 = \frac{1}{3}(-\mathbf{k}_6 + v_1 + \cdots + v_6) \in \mathbb{R}^{1,6}.$$

Let us show that $v_0 \in I^{1,6}$. Since $I^{1,6}$ is a unimodular lattice, it suffices to show that (v_0, v) is an integer for all $v \in I^{1,6}$. Consider the sublattice N of $I^{1,6}$ spanned by $v_1, \dots, v_6, \mathbf{k}_6$. We have $(v_0, v_i) = 0, i > 0$, and $(v_0, \mathbf{k}_6) = -3$. Thus $(v_0, I^{1,6}) \subset 3\mathbb{Z}$. By computing the discriminant of N, we find that it is equal to 9. By Lemma 8.2.1 N is a sublattice of index 3 of $I^{1,6}$. Hence for any

$x \in I^{1,6}$ we have $3x \in N$. This shows that

$$(v_0, x) = \frac{1}{3}(v_0, 3x) \in \mathbb{Z}.$$

Now let us set

$$\alpha = 2v_0 - v_1 - \cdots - v_6. \tag{9.1}$$

We check that α is a root, and $(\alpha, v_i) = 1, i = 1, \ldots, 6$.

Since r_α preserves the symmetric bilinear form, $\{w_1, \ldots, w_6\}$ is a sixer. We have

$$(v_i, w_j) = \big(v_i, r_\alpha(v_j)\big) = \big(v_i, v_j + (v_j, \alpha)\alpha\big) = (v_i, v_j) + (v_i, \alpha)(v_j, \alpha)$$

$$= (v_i, v_j) + 1 = 1 - \delta_{ij}.$$

Finally, we check that

$$(r_\alpha(v_i), -\alpha) = \big(r_\alpha^2(v_i), -r_\alpha(\alpha)\big) = -(v_i, \alpha) = 1.$$

□

The two sixes with opposite associated roots form a *double-six* of exceptional vectors.

We recall the list of exceptional vectors in $\mathbf{E}_6$ in terms of the standard orthonormal basis in $I^{1,6}$.

$$\mathbf{a}_i = \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.2}$$

$$\mathbf{b}_i = 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6 + \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.3}$$

$$\mathbf{c}_{ij} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, \quad 1 \le i < j \le 6. \tag{9.4}$$

Theorem 9.1.3 *The following is the list of 36 double-sixes with corresponding associated roots.*

1 of type D

$$\begin{array}{cccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & & \boldsymbol{\alpha}_{\max} \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 & \mathbf{b}_6 & & -\boldsymbol{\alpha}_{\max} \end{array},$$

15 of type D_{ij}

$$\begin{array}{cccccccc} \mathbf{a}_i & \mathbf{b}_i & \mathbf{c}_{jk} & \mathbf{c}_{jl} & \mathbf{c}_{jm} & \mathbf{c}_{jn} & & \boldsymbol{\alpha}_{ij} \\ \mathbf{a}_j & \mathbf{b}_j & \mathbf{c}_{ik} & \mathbf{c}_{il} & \mathbf{c}_{im} & \mathbf{c}_{in} & & -\boldsymbol{\alpha}_{ij} \end{array},$$

20 of type D_{ijk}

$$\begin{array}{cccccccc} \mathbf{a}_i & \mathbf{a}_j & \mathbf{a}_k & \mathbf{c}_{lm} & \mathbf{c}_{mn} & \mathbf{c}_{ln} & & \boldsymbol{\alpha}_{ijk} \\ \mathbf{c}_{jk} & \mathbf{c}_{ik} & \mathbf{c}_{ij} & \mathbf{b}_n & \mathbf{b}_l & \mathbf{b}_m & & -\boldsymbol{\alpha}_{ijk} \end{array}.$$

Here $\boldsymbol{\alpha}_{\max}$ is the maximal root of the root system $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_6$ equal to $2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6$. The reflection with respect to the associated root interchanges the rows, preserving the order.

Proof We have constructed a map from the set of sixes (resp. double-sixes) to the set of roots (resp. pairs of opposite roots). Let us show that no two sixes $\{v_1, \dots, v_6\}$ and $\{w_1, \dots, w_6\}$ can define the same root. Since $w_1, \dots, w_6, \mathbf{k}_6$ span a sublattice of finite index in $I^{1,6}$, we can write

$$v_i = \sum_{j=1}^{6} a_j w_j + a_0 \mathbf{k}_6 \tag{9.5}$$

with some $a_j \in \mathbb{Q}$. Assume that $v_i \neq w_j$ for all j. Taking the inner product of both sides with α, we get

$$1 = a_0 + \cdots + a_6. \tag{9.6}$$

Taking the inner product with $-\mathbf{k}_6$, we get $1 = a_1 + \cdots + a_6 - 3a_0$, hence $a_0 = 0$. Taking the inner product with w_j, we obtain $-a_j = (v_i, w_j)$. Applying Proposition 8.2.21, we get $a_j \leq -1$. This contradicts (9.6). Thus each v_i is equal to some w_j.

The verification of the last assertion is straightforward.

□

Proposition 9.1.4 *The group $W(\mathbf{E}_6)$ acts transitively on sixes and double-sixes. The stabilizer subgroup of a sixer (resp. double-six) is of order $6!$ ($2 \cdot 6!$).*

Proof We know that the Weyl group $W(\mathbf{E}_N)$ acts transitively on the set of roots and the number of sixes is equal to the number of roots. This shows that all sixes form one orbit. The stabilizer subgroup of the sixer $(\mathbf{a}_1, \dots, \mathbf{a}_6)$ (and hence of a root) is the group $\mathfrak{S}_6$. The stabilizer of the double-six D is the subgroup $\langle \mathfrak{S}_6, s_{\boldsymbol{\alpha}_0} \rangle$ of order $2.6!$. □

One can check that two different double-sixes can share either four or six exceptional vectors. More precisely, we have

$$\#D \cap D_{ij} = 4, \ \#D \cap D_{ijk} = 6,$$

$$\#D_{ij} \cap D_{kl} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 0, \\ 6 & \text{otherwise;} \end{cases}$$

$$\#D_{ij} \cap D_{klm} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l,m\} = 0, 2, \\ 6 & \text{otherwise;} \end{cases}$$

$$\#D_{ijk} \cap D_{lmn} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 1, \\ 6 & \text{otherwise.} \end{cases}$$

A pair of double-sixes is called a *syzygetic duad* (resp. *azygetic duad*) if they have four (resp. six) exceptional vectors in common.

The next Lemma is an easy computation.

Lemma 9.1.5 *Two double-sixes with associated roots α, β form a syzygetic duad if and only if $(\alpha, \beta) \in 2\mathbb{Z}$.*

This can be interpreted as follows. Consider the vector space

$$V = \mathbf{E}_6/2\mathbf{E}_6 \cong \mathbb{F}_2^6 \tag{9.7}$$

equipped with the quadratic form

$$q(x + 2\mathbf{E}_6) = \tfrac{1}{2}(x, x) \mod 2.$$

Notice that the lattice $\mathbf{E}_6$ is an even lattice. So, the definition makes sense. The associated symmetric bilinear form is the symplectic form

$$(x + 2\mathbf{E}_6, y + 2\mathbf{E}_6) = (x, y) \mod 2.$$

Each pair of opposite roots $\pm\alpha$ defines a vector v in V with $q(v) = 1$. It is easy to see that the quadratic form q has Arf-invariant equal to 1 and hence vanishes on 28 vectors. The remaining 36 vectors correspond to 36 pairs of opposite roots or, equivalently, double-sixes.

Note that we have a natural homomorphism of groups

$$W(\mathbf{E}_6) \cong O(6, \mathbb{F}_2)^- \tag{9.8}$$

obtained from the action of $W(\mathbf{E}_6)$ on V. It is an isomorphism. This is checked by verifying that the automorphism $v \mapsto -v$ of the lattice $\mathbf{E}_6$ does not belong to the Weyl group W and then comparing the known orders of the groups.

It follows from the above that a syzygetic pair of double-sixes corresponds to orthogonal vectors v, w. Since $q(v+w) = q(v)+q(w)+(v, w) = 0$, we see that each nonzero vector in the isotropic plane spanned by v, w comes from a double-six.

A triple of pairwise syzygetic double-sixes is called a *syzygetic triad* of double-sixes. They span an isotropic plane. Similarly, we see that a pair of azygetic double-sixes spans a non-isotropic plane in V with three nonzero vectors corresponding to a triple of double-sixes which are pairwise azygetic. It is called an *azygetic triad* of double-sixes.

We say that three azygetic triads form a *Steiner complex of triads of double-sixes* if the corresponding planes in V are mutually orthogonal. It is easy to

see that an azygetic triad contains 18 exceptional vectors and thus defines a set of nine exceptional vectors (the omitted ones). The set of 27 exceptional vectors omitted from three triads in a Steiner complex is equal to the set of 27 exceptional vectors in the lattice $I^{1,6}$. There are 40 Steiner complexes of triads:

10 of type

$$\Gamma_{ijk,lmn} = (D, D_{ijk}, D_{lmn}),\ (D_{ij}, D_{ik}, D_{jk}),\ (D_{lm}, D_{ln}, D_{mn}),$$

30 of type

$$\Gamma_{ij,kl,mn} = (D_{ij}, D_{ikl}, D_{jkl}),\ (D_{kl}, D_{kmn}, D_{lmn}),\ (D_{mn}, D_{mij}, D_{nij}).$$

Theorem 9.1.6 *The Weyl group $W(\mathbf{E}_6)$ acts transitively on the set of triads of azygetic double-sixes with stabilizer subgroup isomorphic to the group $\mathfrak{S}_3 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$ of order 432. It also acts transitively on Steiner complexes of triads of double-sixes. A stabilizer subgroup is a maximal subgroup of $W(\mathbf{E}_6)$ of order 1296 isomorphic to the wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_3$.*

Proof We know that a triad of azygetic double-sixes corresponds to a pair of roots (up to replacing the root with its negative) α, β with $(\alpha, \beta) = \pm 1$. This pair spans a root sublattice Q of $\mathbf{E}_6$ of type A_2. Fix a root basis. Since the Weyl group acts transitively on the set of roots, we find $w \in W$ such that $w(\alpha) = \boldsymbol{\alpha}_{\max}$. Since $(w(\beta), \boldsymbol{\alpha}_{\max}) = (\beta, \alpha) = 1$, we see that $w(\beta) = \pm\boldsymbol{\alpha}_{ijk}$ for some i, j, k. Applying elements from $\mathfrak{S}_6$, we may assume that $w(\beta) = -\boldsymbol{\alpha}_{123}$. Obviously, the roots $\boldsymbol{\alpha}_{12}, \boldsymbol{\alpha}_{23}, \boldsymbol{\alpha}_{45}, \boldsymbol{\alpha}_{56}$ are orthogonal to $w(\alpha)$ and $w(\beta)$. These roots span a root sublattice of type $2A_2$. Thus we obtain that the orthogonal complement of Q in $\mathbf{E}_6$ contains a sublattice of type $2A_2 \perp A_2$. Since $|\mathrm{disc}(A_2)| = 3$, it follows easily from Lemma 8.2.1 that $Q^\perp$ is a root lattice of type $A_2 + A_2$ ($2A_2$, for short). Obviously, any automorphism Wthe two roots α, β invariant leaves invariant the sublattice Q and its orthogonal complement $Q^\perp$. Thus the stabilizer contains a subgroup isomorphic to $W(A_2) \times W(A_2) \times W(A_2)$ and the permutation of order 2 which switches the two copies of A_2 in $Q^\perp$. Since $W(A_2) \cong \mathfrak{S}_3$ we obtain that a stabilizer subgroup contains a subgroup of order $2 \cdot 6^3 = 432$. Since its index is equal to 120, it must coincide with the stabilizer group.

It follows from the above that a Steiner complex corresponds to a root sublattice of type $3A_2$ contained in $\mathbf{E}_6$. The group $W(A_2) \wr \mathfrak{S}_3$ of order $3 \cdot 432$ is contained in the stabilizer. Since its index is equal to 40, it coincides with the stabilizer. □

Remark 9.1.7 The notions of syzygetic (azygetic) pairs, triads and a Steiner complex of triads of double-sixes is analogous to the notions of syzygetic

(azygetic) pairs, triads, and a Steiner complex of bitangents of a plane quartic (see Chapter 6). In both cases we deal with a 6-dimensional quadratic space $\mathbb{F}_2^6$. However, they have different Arf invariants.

A triple v_1, v_2, v_3 of exceptional vectors is called a *tritangent trio* if

$$v_1 + v_2 + v_3 = -\mathbf{k}_6.$$

If we view exceptional vectors as cosets in $I^{1,6}/\mathbb{Z}\mathbf{k}_6$, this is equivalent to saying that the cosets add up to zero.

It is easy to list all tritangent trios.

Lemma 9.1.8 *There are 45 tritangent trios:*
30 of type

$$\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij}, \quad i \neq j,$$

15 of type

$$\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn}, \quad \{i,j\} \cup \{k,l\} \cup \{m,n\} = \{1,2,3,4,5,6\}.$$

Theorem 9.1.9 *The Weyl group acts transitively on the set of tritangent trios.*

Proof We know that the permutation subgroup $\mathfrak{S}_6$ of the Weyl group acts on tritangent trios by permuting the indices. Thus it acts transitively on the set of tritangent trios of the same type. Now consider the reflection w with respect to the root $\boldsymbol{\alpha}_{123}$. We have

$$r_{\boldsymbol{\alpha}_{123}}(\mathbf{a}_1) = \mathbf{e}_1 + \boldsymbol{\alpha}_{123} = \mathbf{e}_0 - \mathbf{e}_3 - \mathbf{e}_4 = \mathbf{c}_{34},$$
$$r_{\boldsymbol{\alpha}_{123}}(\mathbf{b}_2) = (2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6) - \boldsymbol{\alpha}_{123} = \mathbf{e}_0 - \mathbf{e}_5 - \mathbf{e}_6 = \mathbf{c}_{56},$$
$$r_{\boldsymbol{\alpha}_{123}}(\mathbf{c}_{12}) = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{c}_{12}.$$

Thus $w(\mathbf{a}_1, \mathbf{b}_2, \mathbf{c}_{12}) = (\mathbf{c}_{34}, \mathbf{c}_{56}, \mathbf{c}_{12})$. This proves the assertion. □

Remark 9.1.10 The stabilizer subgroup of a tritangent trio is a maximal subgroup of $W(\mathbf{E}_6)$ of index 45 isomorphic to the Weyl group of the root system of type F_4.

Let $\Pi_1 = \{v_1, v_2, v_3\}$ and $\Pi_2 = \{w_1, w_2, w_3\}$ be two tritangent trios with no common elements. We have

$$(v_i, w_1 + w_2 + w_3) = -(v_i, \mathbf{k}_6) = 1$$

and, by Proposition 8.2.21, $(v_i, w_j) \geq 0$. This implies that there exists a unique j such that $(v_i, w_j) = 1$. After reordering, we may assume $j = i$. Let $u_i =$

$-k_6 - v_i - w_i$. Since $u_i^2 = -1, (u_i, k_6) = -1$, the vector u_i is an exceptional vector. Since

$$u_1 + u_2 + u_3 = \sum_{i=1}^{3}(-k_6 - v_i - w_i) = -3k_6 - \sum_{i=1}^{3} v_i - \sum_{i=1}^{3} w_i = -\mathbf{k}_6,$$

we get a new tritangent trio $\Pi_3 = (u_1, u_2, u_3)$. The union $\Pi_1 \cup \Pi_2 \cup \Pi_3$ contains nine lines $v_i, w_i, u_i, i = 1, 2, 3$. There is a unique triple of tritangent trios that consists of the same nine lines. It is formed by tritangent trios $\Pi_i' = (v_i, w_i, u_i), i = 1, 2, 3$. Any pair of triples of tritangent trios that consists of the same set of nine lines is obtained in this way. Such a pair of triples of tritangent trios is called a pair of *conjugate triads of tritangent trios*.

We can list all conjugate pairs of triads of tritangent trios:

$$\begin{array}{rccc} & \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ (I) & \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ & \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{array}, \quad \begin{array}{rccc} & \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ (II) & \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ & \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{array}, \quad \begin{array}{rccc} & \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ (III) & \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ & \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{array}. \tag{9.9}$$

Here a triad is represented by the columns of the matrix and its conjugate triad by the rows of the same matrix. Altogether we have $20 + 10 + 90 = 120$ different conjugate pairs of triads.

There is a bijection from the set of pairs of conjugate triads to the set of azygetic triads of double-sixes. The 18 exceptional vectors contained in the union of the latter is the complementary set of the set of nine exceptional vectors defined by a triad in the pair. Here is the explicit bijection.

$$\begin{array}{ccc} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{array} \leftrightarrow D_{ij}, D_{ik}, D_{jk};$$

$$\begin{array}{ccc} \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{array} \leftrightarrow D, D_{ikn}, D_{jlm};$$

$$\begin{array}{ccc} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{array} \leftrightarrow D_{mn}, D_{jkm}, D_{jkn}.$$

Recall that the set of exceptional vectors omitted from each triad entering in a Steiner complex of triads of azygetic double-sixes is the set of 27 exceptional vectors. Thus a Steiner complex defines three pairs of conjugate triads of tritangent trios which contains all 27 exceptional vectors. We have 40 such triples of conjugate pairs.

Theorem 9.1.11 *The Weyl group acts transitively on the set of 120 conjugate pairs of triads of tritangent trios. A stabilizer subgroup H is contained in the maximal subgroup of $W(\mathbf{E}_6)$ of index 40 realized as a stabilizer of a Steiner complex. The quotient group is a cyclic group of order 3.*

Proof This follows from the established bijection between pairs of conjugate triads and triads of azygetic double-sixes and Theorem 9.1.6. In fact it is easy to see directly the transitivity of the action. It is clear that the permutation subgroup $\mathfrak{S}_6$ acts transitively on the set of pairs of conjugate triads of the same type. Since the Weyl group acts transitively on the set of tritangent trios, we can send a tritangent trio $(\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn})$ to a tritangent trio $(\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij})$. By inspection, this sends a conjugate pair of type III to a pair of conjugate triads of type I. Also it sends a conjugate pair of type II to type I or III. Thus all pairs are W-equivalent. □

Remark 9.1.12 Note that each monomial entering into the expression of the determinant of the matrix (9.9) expressing a conjugate pair of triads represents three orthogonal exceptional vectors. If we take only monomials corresponding to even (resp. odd) permutations we get a partition of the set of nine exceptional vectors into the union of three triples of orthogonal exceptional vectors such that each exceptional vector from one triple has a nonzero intersection with two exceptional vectors from any other triple.

9.1.2 Lines and tritangent planes

Let S be an nonsingular cubic surface in $\mathbb{P}^3$. Fix a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$. We can transfer all the notions and the statements from the previous Subsection to the Picard lattice $\mathrm{Pic}(S)$. The image of an exceptional vector is the divisor class of a line on S. So, we will identify exceptional vectors with lines on S. We have 27 lines. A tritangent trio of exceptional vectors defines a set of three coplanar lines. The plane containing them is called a *tritangent plane*. We have 45 tritangent planes.

Thus we have 72 sixes of lines, 36 double-sixes and 40 Steiner complexes of triads of double-sixes. If $e_0, e_1, \dots, e_6$ define a geometric marking, then we can identify the divisor classes e_i with the exceptional curves of the blow-up $S \to \mathbb{P}^2$ of six points $x_1, \dots, x_6$ in general position. They correspond to the exceptional vectors $\mathbf{a}_i$. We identify the proper transforms of the conic through the six points excluding the x_i with the exceptional vector $\mathbf{b}_i$. Finally, we identify the line through the points x_i and x_j with the exceptional vector $\mathbf{c}_{ij}$. Under the geometric marking the Weyl group $W(\mathbf{E}_6)$ becomes isomorphic to the index 2 subgroup of the isometry group of $\mathrm{Pic}(S)$ leaving the canonical class

invariant (see Corollary 8.2.16). It acts transitively on the set of lines, sixes, double-sixes, tritangent planes, and on the set of conjugate pairs of triples of tritangent planes.

An elementary geometric proof of the fact that any nonsingular cubic surface contains 27 lines can be found in [457]. The first proof of A. Cayley applies only to general nonsingular cubic surfaces. For completeness sake, let us reproduce the original proof of Cayley [74].

Theorem 9.1.13 *A general nonsingular cubic surface contains 27 lines and 45 tritangent planes.*

Proof First of all, let us show that any cubic surface contains a line. Consider the incidence variety

$$X = \{(S, \ell) \in |\mathcal{O}_{\mathbb{P}^3}(3)| \times G : \ell \subset S\}.$$

The assertion follows if we show that the first projection is surjective. It is easy to see that the fibres of the second projections are linear subspaces of codimension 4. Thus $\dim X = 4 + 15 = 19 = \dim |\mathcal{O}_{\mathbb{P}^3}(3)|$. To show the surjectivity of the first projection, it is enough to find a cubic surface with only finitely many lines on it. Let us consider the surface S given by the equation

$$t_1 t_2 t_3 - t_0^3 = 0.$$

Suppose a line ℓ lies on S. Let $[a_0, a_1, a_2, a_3] \in \ell$. If $a_0 \neq 0$, then $a_i \neq 0, i \neq 0$. On the other hand, every line hits the planes $t_i = 0$. This shows that ℓ is contained in the plane $t_0 = 0$. But there are only three lines on S contained in this plane: $t_i = t_0 = 0, i = 1, 2$ and 3. Therefore S contains only three lines. This proves the first assertion.

We already know that every cubic surface $S = V(f)$ has at least one line. Pick up such a line ℓ_0. Without loss of generality, we may assume that it is given by the equation:

$$t_2 = t_3 = 0.$$

Thus

$$f = t_2 q_0(t_0, t_1, t_2, t_3) + t_3 q_1(t_0, t_1, t_2, t_3) = 0, \tag{9.10}$$

where q_0 and q_1 are quadratic forms. The pencil of planes $\Pi_{\lambda,\mu} = V(\lambda t_2 - \mu t_3)$ through the line ℓ_0 cuts out a pencil of conics on S. The equation of the conic in the plane $\Pi_{\lambda,\mu}$ is

$$A_{00}(\lambda, \mu) t_0^2 + A_{11}(\lambda, \mu) t_1^2 + A_{22}(\lambda, \mu) t_2^2 +$$

$$2A_{01}(\lambda, \mu) t_0 t_1 + 2A_{12}(\lambda, \mu) t_1 t_2 + 2A_{02}(\lambda, \mu) t_0 t_2 = 0,$$

where A_{00}, A_{11}, A_{01} are binary forms of degree 1, A_{02}, A_{12} are binary forms of degree 2 and A_{22} is a binary form of degree 3. The discriminant equation of this conic is equal to

$$\begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{vmatrix} = 0.$$

This is a homogeneous equation of degree 5 in variables λ, μ. Thus we expect five roots of this equation which gives us five reducible conics. This is the tricky point because we do not know whether the equation has five distinct roots. First, we can exhibit a nonsingular cubic surface and a line on it and check that the equation indeed has five distinct roots. For example, let us consider the cubic surface

$$2t_0t_1t_2 + t_3(t_0^2 + t_1^2 + t_2^2 + t_3^2) = 0.$$

The equation becomes $\lambda(\lambda^4 - \mu^4) = 0$. It has five distinct roots. This implies that, for general nonsingular cubic surface, we have five reducible residual conics. Note that no conic is a double line since otherwise the cubic surface is singular.

Thus each solution of the quintic equation defines a tritangent plane Π_i of S consisting of three lines, one of them is ℓ_0. Thus we found 11 lines on X: the line ℓ_0 and five pairs of lines ℓ_i, ℓ_i' lying in the plane Π_i. Pick up some plane, say Π_1. We have 3 lines ℓ_0, ℓ_1, ℓ_2 in Π_1. Replacing ℓ_0 by ℓ_1, and then by ℓ_2, and repeating the construction, we obtain four planes through ℓ_1 and four planes through ℓ_2 not containing ℓ_0 and each containing a pair of additional lines. Altogether we found $3 + 8 + 8 + 8 = 27$ lines on S. To see that all lines are accounted for, we observe that any line intersecting either ℓ_0, or ℓ_1, or ℓ_2 lies in one of the planes we have considered before. So it has been accounted for. Now let ℓ be any line. We find a plane Π through ℓ that contains three lines ℓ, ℓ' and ℓ'' on S. This plane intersects the plane containing $\ell_{,0}\, \ell_1$, and ℓ_1' along a line. This line intersects S at some point on ℓ and on one of the lines ℓ_0, ℓ_1, ℓ_1'. Thus ℓ intersects one of the lines ℓ_0, ℓ_1, ℓ_1' and has been accounted for.

It remains for us to count tritangent planes. Each line belongs to five tritangent planes, each tritangent plane contains three lines. This easily gives the number of tritangent planes as being equal to 45. □

Remark 9.1.14 Reid's extension of Cayley's proof to any nonsingular surface uses some explicit computations. Instead, we may use that the number of singular conics in the pencil of conics residual to a line determines the topological Euler-Poincaré characteristic of the surface. Using the additivity of the

Euler-Poincaré characteristic of a CW-complex, we obtain the formula

$$\chi(X) = \chi(B)\chi(F) + \sum_{b\in B}(\chi(F_b) - \chi(F)), \tag{9.11}$$

where $f : X \to B$ is any regular map of an algebraic variety onto a curve B with general fiber F and fibres F_b over points $b \in B$. In our case $\chi(B) = \chi(F) = 2$ and $\chi(F_b) = 3$ for a singular fibre. This gives $\chi(S) = 4 + s$, where s is the number of singular conics. Since any two nonsingular surfaces are homeomorphic (they are parameterized by an open subset of a projective space), we obtain that s is the same for all nonsingular surfaces. We know that $s = 5$ for the example in above, hence $s = 5$ for all nonsingular surfaces. Also we obtain $\chi(S) = 9$, which of course agrees with the fact that S is the blow-up of six points in the plane.

The closure of the effective cone $\overline{\text{Eff}}(S)$ of a nonsingular cubic surface is isomorphic to the Gosset polytope $\Sigma_6 = 2_{21}$. It has 72 facets corresponding to sixes and 27 faces corresponding to conic bundles on S. In a geometric basis $e_0, e_1, \ldots, e_6$ they are expressed by the linear systems of types $|e_0 - e_1|, |2e_0 - e_1 - e_2 - e_3 - e_4|, |3e_0 - 2e_1 - e_2 - \cdots - e_6|$. The center of $\overline{\text{Eff}}(S)$ is equal to $O = -\frac{1}{3}K_S = (e_1 + \cdots + e_{27})/27$, where $e_1, \ldots, e_{27}$ are the divisor classes of lines. A double-six represents two opposite facets whose centers lie on a line passing through O. In fact, if we consider the double-six $(e_i, e_i' = 2e_0 - e_1 - \cdots - e_6 + e_i), i = 1, \ldots, 6$, then

$$\frac{1}{12}(\sum_{i=1}^{6} e_i) + \frac{1}{12}\sum_{i=1}^{6} e_i' = -\frac{1}{3}K_S = O.$$

The line joining the opposite face is perpendicular to the facets. It is spanned by the root corresponding to the double-six. The three lines e_i, e_j, e_k in a tritangent plane add up to $-K_S$. This can be interpreted by saying that the center of the triangle with vertices e_i, e_j, e_k is equal to the center O. This easily implies that the three lines joining the center O with e_i, e_j, e_k are coplanar.

Remark 9.1.15 Let a_i, b_i, c_{ij} denotes the set of 27 lines on a nonsingular cubic surface. Consider them as 27 unknowns. Let F be the cubic form in 27 variables equal to the sum of 45 monomials $a_i b_j c_{ij}, c_{ij}c_{kl}c_{mn}$ corresponding to tritangent planes. It was shown by E. Cartan that the group of projective automorphisms of the cubic hypersurface $V(F)$ in $\mathbb{P}^{26}$ is isomorphic to the simple complex Lie group of type E_6. We refer to [371] for integer models of this cubic.

9.1.3 Schur's quadrics

There are 36 double-sixes of lines on a nonsingular cubic surface S corresponding to 36 double-sixes of exceptional vectors in $I^{1,6}$. Let $(\ell_1,\dots,\ell_6)$, $(\ell'_1,\dots,\ell'_6)$ be one of them. Choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$ such that $\phi(\mathbf{e}_i) = e_i = [\ell_i], i = 1,\dots,6$. Then the linear system $|e_0|$ defines a birational map $\pi_1 : S \to {}_1\mathbb{P}^2 = |e_0|^\vee$ which blows down the lines ℓ_i to points $x_1,\dots,x_6$. The class of the line ℓ'_i is equal to $2e_0 - (e_1 + \cdots + e_6) + e_i$. Its image in the plane ${}_1\mathbb{P}^2$ is the conic C_i passing through all p_j except p_i. Let $\phi' : I^{1,6} \to \mathrm{Pic}(S)$ be the geometric marking such that $\phi'(\mathbf{e}_i) = \ell'_i$. It is obtained from ϕ by composing ϕ with the reflection $s = s_{\boldsymbol{\alpha}_{\max}} \in O(I^{1,6})$. We have

$$e'_0 = s(e_0) = e_0 + 2(2e_0 - e_1 - \cdots - e_6) = 5e_0 - 2e_1 - \cdots - 2e_6.$$

The linear system $|e'_0|$ defines a birational map $\pi' : S \to {}_2\mathbb{P}^2 = |e'_0|^\vee$ which blows down the lines ℓ'_i to points x'_i in ${}_2\mathbb{P}^2$. The Cremona transformation

$$T = \pi_2 \circ \pi_1^{-1} : {}_1\mathbb{P}^1 \dashrightarrow {}_2\mathbb{P}^2$$

is the symmetric Cremona transformation of degree 5. It is given by the homaloidal linear system $|\mathcal{I}^2_{x_1,\dots,x_6}(5)|$. The P-locus of T consists of the union of the conics C_i. Note that the ordered sets of points $(x_1,\dots,x_6)$ and $(x'_1,\dots,x'_6)$ are not projectively equivalent.

Consider the map

$$ {}_1\mathbb{P}^2 \times {}_2\mathbb{P}^2 = |e_0| \times |e'_0| \to |e_0 + e'_0| = |-2K_S| \cong |\mathcal{O}_{\mathbb{P}^3}(2)| \cong \mathbb{P}^9. \quad (9.12)$$

It is isomorphic to the Segre map $\mathsf{s}_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, and its image is a hyperplane H in the space of quadrics in $\mathbb{P}^3$. Let Q be the unique quadric in the dual space of quadrics which is apolar to H.

The following beautiful result belongs to F. Schur.

Theorem 9.1.16 (F. Schur) *The quadric Q is nonsingular. The polar of each line ℓ_i with respect to the dual quadric $Q^\vee$ is equal to ℓ'_i. The quadric $Q^\vee$ is uniquely determined by this property.*

Proof Let $(\ell_1,\dots,\ell_6)$ and $(\ell'_1,\dots,\ell'_6)$ form a double-six. We use the notations a_i, b_j, c_{ij} (resp. (a_i, b'_j, c'_{ij})) for lines defined by the geometric basis $(e_0,\dots,e_6)$ (resp. $(e'_0,\dots,e'_6)$). The divisor class of the sum of six lines a_i, a_j, c_{ij} and a'_i, b'_k, c'_{ik} is equal to

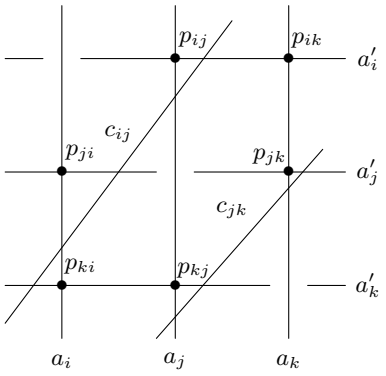
$$e_i+e_j+(e_0-e_i-e_j)+2e_0-(e_1+\cdots+e_6-e_j)+2e_0-(e_1+\cdots+e_6-e_k)+$$

$$(e_0 - e_j - e_k) = 6e_0 - 2(e_1 + \cdots + e_6) = -2K_S.$$

The corresponding quadric Q_{ijk} cuts out six lines distributed into two triples of coplanar lines a_i, a'_j, c_{jk} and a_j, a'_k, c'_{jk}. Thus Q_{ijk} consists of the union of two planes H_{ij} and H_{jk} (note that k here could be equal to i). This implies that, considered as points in the dual space, the polar plane $(H_{ij})^{\perp}_{Q}$ of H_{ij} with respect to Q contains H_{jk}. Let p_{ij} be the point in the original space $\mathbb{P}^3$ which corresponds to the hyperplane $(H_{ij})^{\perp}_{Q}$ in the dual projective space. Then $H_{ij} \subset (p_{ij})^{\perp}_{Q^\vee}$, or, $p_{ij} \in (H_{ij})^{\perp}_{Q^\vee}$. The inclusion $H_{ij} \in (H_{jk})^{\perp}_{Q}$ means that $p_{ij} \in H_{jk}$. Since $H_{bc} \in (H_{ab})^{\perp}_{Q}$ for any three indices a, b, c, we get

$$p_{ij} \in H_{jk} \cap H_{ji} \cap H_{ki} = \{a_j + a'_k + c_{jk}\} \cap \{a_j + a'_i + c_{ji}\} \cap \{a_k + a'_i + c_{ki}\}.$$

The point $a_j \cap a'_i$ belongs to the intersection. Since no three tritangent planes intersect along a line, we obtain that $p_{ij} = a_j \cap a'_i$, and, similarly, $p_{ji} = a_i \cap a'_j$. Now, we use that $p_{ji} \in (H_{ji})^{\perp}_{Q^\vee}$ and $p_{ij} \in (H_{ij})^{\perp}_{Q^\vee}$. Since $a_i \in H_{ij}, a'_i \subset H_{ji}$, we obtain that the points p_{ij} and p_{ji} are orthogonal with respect to $Q^\vee$. Similarly, we find that the pairs p_{ki}, p_{ik} and p_{jk}, p_{jk} are orthogonal. Since a_i contains p_{ji}, p_{ki}, and a'_i contains p_{ik}, p_{ij}, we see that the lines a_i and a'_i are orthogonal with respect to $Q^\vee$.



Let us show that Q is a nondegenerate quadric. Suppose Q is degenerate, then its set of singular points is a non-empty linear space L_0. Thus, for any subspace L of the dual space of $\mathbb{P}^3$, the polar subspace $L^{\perp}_{Q}$ contains L_0. Therefore, all the points p_{ij} lie in a proper subspace of $\mathbb{P}^3$. But this is obviously impossible, since some of these points lie on a pair of skew lines and span $\mathbb{P}^3$. Thus the dual quadric $Q^\vee$ is nonsingular and the lines ℓ_i, ℓ'_i are orthogonal with respect to $Q^\vee$.

Let us show the uniqueness of $Q^\vee$. Suppose we have two quadrics Q_1 and Q_2 such that $\ell'_i = (\ell_i)^{\perp}_{Q_i}, i = 1, \dots, 6$. Let Q be a singular quadric in the pencil spanned by Q_1 and Q_2. Let K be its space of singular points. Then K

is orthogonal to each subspace of $\mathbb{P}^3$. Hence, it is contained in ℓ_i' and ℓ_i. Since these lines are skew, we get a contradiction.

□

Definition 9.1.17 *Let $(\ell_1, \dots, \ell_6), (\ell_1', \dots, \ell_6')$ be a double-six of lines on a nonsingular cubic surface S. The unique quadric $\mathcal{Q}$ such that $(\ell_i)_{\mathcal{Q}}^{\perp} = \ell_i'$ is called the* Schur quadric *with respect to the double-six.*

Consider the bilinear map corresponding to the pairing (9.12)

$$H^0(S, \mathcal{O}_S(e_0)) \times H^0(S, \mathcal{O}_S(5e_0 - 2e_1 - \cdots - 2e_6))$$

$$\to H^0(S, \mathcal{O}_S(-2K_S)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)).$$

A choice of an equation of the dual of the Schur quadric defines a linear map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to \mathbb{C}$. Composing the pairing with this map, we obtain an isomorphism

$$H^0(S, \mathcal{O}_S(5e_0 - 2e_1 - \cdots - 2e_6)) \cong H^0(S, \mathcal{O}_S(e_0))^{\vee}.$$

This shows that the Schur quadric allows us to identify the plane ${}_1\mathbb{P}^2$ and ${}_2\mathbb{P}^2$ as the dual to each other. Under this identification, the linear system $|-2K_S - e_0|$ defines an involutive Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Fix six points $x_1, \dots, x_6 \in \mathbb{P}^2$ in general positions. The linear system $|6h - 2x_1 - \cdots - 2x_6|$ is equal to the preimage of the linear system of quadrics in $\mathbb{P}^3 = |3e_0 - x_1 - \cdots - x_6|^{\vee}$ under the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by the linear system $|3h - x_1 - \cdots - x_6|$. The preimage of the Schur quadric corresponding to the double-six $(e_1, \dots, e_6), (e_1', \dots, e_6')$ is a curve of degree 6 with double points at $x_1, \dots, x_6$. It is called the *Schur sextic* associated with six points. Note that it is defined uniquely by the choice of six points. The proper transform of the Schur sextic under the blow-up of the points is a nonsingular curve of arithmetic genus 4. In the anticanonical embedding, it is the intersection of the Schur quadric with the cubic surface.

Proposition 9.1.18 *The six double points of the Schur sextic are biflexes, i.e. the tangent line to each branch is tangent to the branch with multiplicity ≥ 3.*

Proof Let $\mathcal{Q}$ be the Schur quadric corresponding to the Schur sextic and ℓ_i be the lines on the cubic surface S corresponding to the points $x_1, \dots, x_6$. Let $\ell_i \cap \mathcal{Q} = \{a, b\}$ and $\ell_i' \cap \mathcal{Q} = \{a', b'\}$. We know that

$$P_a(\mathcal{Q}) \cap \mathcal{Q} = \{x \in Q : a \in \mathbb{T}_x(\mathcal{Q})\}.$$

Since $\ell_i' = (\ell_i)_{\mathcal{Q}}^{\perp}$, we have

$$\ell_i' \cap \mathcal{Q} = (P_a(\mathcal{Q}) \cap P_b(\mathcal{Q})) \cap \mathcal{Q} = \{a', b'\}.$$

This implies that $a', b' \in \mathbb{T}_a(\mathcal{Q})$ and hence the lines $\overline{aa'}, \overline{ab'}$ span $\mathbb{T}_a(\mathcal{Q})$. The tangent plane $\mathbb{T}_a(\mathcal{Q})$ contains the line ℓ_i' and hence intersects the cubic surface S along ℓ_i' and some conic $K(a)$. We have

$$\mathbb{T}_a(K(a)) = \mathbb{T}_a(S) \cap \mathbb{T}_a(\mathcal{Q}) = \mathbb{T}_a(\mathcal{Q} \cap S).$$

Thus the conic $K(a)$ and the curve $C = \mathcal{Q} \cap S$ are tangent at the point a. Since the line ℓ_i' is equal to the proper inverse transform of the conic C' in $\mathbb{P}^2$ passing through the points $x_j, j \neq i$, the conic $K(a)$ is the proper inverse transform of some line ℓ in the plane passing through x_i. The point a corresponds to the tangent direction at x_i defined by a branch of the Schur sextic at x_i. The fact that $K(a)$ is tangent to C at a means that the line ℓ is tangent to the branch with multiplicity ≥ 3. Since the same is true, when we replace a with b, we obtain that x_i is a biflex of the Schur sextic. □

Remark 9.1.19 A biflex is locally given by an equation whose Taylor expansion looks like $xy + xy(ax + by) + f_4(x, y) + \cdots$. This shows that one has to impose five conditions to get a biflex. To get six biflexes for a curve of degree 6 one has to satisfy 30 linear equations. The space of homogeneous polynomials of degree 6 in three variables has dimension 28. So, one does not expect that such sextics exist.

Also observe that the set of quadrics Q such that $\ell_Q^{\perp} = \ell'$ for a fixed pair of skew lines (ℓ, ℓ') is a linear (projective) subspace of codimension 4 of the 9-dimensional space of quadrics. So the existence of the Schur quadric is unexpected!

I do not know whether, for a given set of six points on $\mathbb{P}^2$ defining a nonsingular cubic surface, there exists a unique sextic with biflexes at these points. We refer to [179], where the Schur sextic is realized as the curve of jumping lines of the second kind of a rank 2 vector bundle on $\mathbb{P}^2$.

Example 9.1.20 Let S be the *Clebsch diagonal surface* given by two equations in $\mathbb{P}^4$:

$$\sum_{i=1}^{5} t_i = \sum_{i=1}^{5} t_i^3 = 0. \tag{9.13}$$

It exhibits an obvious symmetry defined by permutations of the coordinates. Let $a = \frac{1}{2}(1+\sqrt{5}), a' = \frac{1}{2}(1-\sqrt{5})$ be two roots of the equation $x^2-x-1 = 0$. One checks that the skew lines

$$\ell : t_1 + t_3 + at_2 = at_3 + t_2 + t_4 = at_2 + at_3 - t_5 = 0$$

and

$$\ell' : t_1 + t_2 + a't_4 = t_3 + a't_1 + t_4 = a't_1 + a't_4 - t_5 = 0$$

lie on S. Applying to each line even permutations we obtain a double-six. The Schur quadric is $\sum t_i^2 = \sum t_i = 0$.

Let $\pi_1 : S \to {}_1\mathbb{P}^2, \pi_2 : S \to {}_2\mathbb{P}^2$ be two birational maps defined by blowing down two sixes forming a double-six. We will see later in Section 9.3.2 that there exists a 3×3-matrix $A = (a_{ij})$ of linear forms such that $S = V(\det A)$. The map π_1 (resp. π_2) is given by the left (resp. right) kernel of A. In coordinates, it is given by a row (resp. column) of $\mathrm{adj}(A)$. The composition of the map $(\pi_1, \pi_2) : S \to {}_1\mathbb{P}^2 \times {}_2\mathbb{P}^2$ with the Segre map ${}_1\mathbb{P}^2 \times {}_2\mathbb{P}^2 \to \mathbb{P}^8$ is given by $x \mapsto [\mathrm{adj}(A)(x)]$. We immediately identify this map with the map (9.12). Thus the entries of $\mathrm{adj}(A)$ define the quadrics in the image of this map. They are apolar to the dual of the Schur quadrics.

Let $x = [z_0, \dots, z_3]$ be any point in $\mathbb{P}^3$. The polar quadric of S with center at x is given by the equation

$$\begin{vmatrix} Da_{11} & Da_{12} & Da_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ Da_{21} & Da_{22} & Da_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ Da_{31} & Da_{32} & Da_{33} \end{vmatrix} = 0,$$

where D is the linear differential operator $\sum z_i \frac{\partial}{\partial t_i}$. It is clear that the left-hand side this equation is a linear combination of the entries of $\mathrm{adj}(A)$. Thus all polar quadrics of S are apolar to the duals of all 36 Schur quadrics. This proves the following.

Proposition 9.1.21 *The duals of the 36 Schur quadrics belong to the 5-dimensional projective space of quadrics apolar to the 3-dimensional linear system of polar quadrics of S.*

This result was first mentioned by H. Baker in [20], its proof appears in his book [21], Vol. 3, p. 187. In the notation of Theorem 9.1.3, let Q_α is the Schur quadric corresponding to the double-six defined by the root α (see Theorem 9.1.3). Any three of type $Q_{\alpha_{\max}}, Q_{\alpha_{123}}, Q_{\alpha_{456}}$ are linearly dependent. Among $Q_{\alpha_{ij}}$'s at most five are linearly independent ([474]).

Remark 9.1.22 We refer to [179] for the relationship between Schur quadrics and rank 2 vector bundles on $\mathbb{P}^2$ with odd first Chern class. The case of cubic surfaces corresponds to vector bundles with $c_1 = -1$ and $c_2 = 4$. For higher n values the Schur quadrics define some polarity relation for a configuration of $\binom{n+1}{2}$ lines and $(n-2)$-dimensional subspaces in $\mathbb{P}^n$ defined by a *White surface* X, the blow-up of a set Z of $\binom{n+1}{2}$ points in the plane which do not lie on a curve of degree $n-1$ and no n points among them are collinear [599]. The case $n = 3$ corresponds to cubic surfaces and the case $n = 4$ to Bordiga surfaces. The linear system $|\mathcal{I}_Z(n)|$ embeds X in $\mathbb{P}^n$. The images

of the exceptional curves are lines, and the images of the curves through all points in Z except one (for each point, there is a unique such curve) spans a subspace of dimension $n-3$. The configuration generalizes a double-six on a cubic surface. The difference here is that, in the case $n>3$, the polarity of the configuration exists only for a non-general White surface.

9.1.4 Eckardt points

A point of intersection of three lines in a tritangent plane is called an *Eckardt point*. As we will see later, the locus of nonsingular cubic surfaces with an Eckardt point is of codimension 1 in the moduli space of cubic surfaces.

Recall that the fixed locus of an automorphism τ of order 2 of $\mathbb{P}^n$ is equal to the union of two subspaces of dimensions k and $n-k-1$. The number k determines the conjugacy class of τ in the group $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1)$. In the terminology of classical projective geometry, a projective automorphism with a hyperplane of fixed points is called a *homology*. A homology of order 2 was called a *harmonic homology*. The isolated fixed point is the *center* of the homology.

Proposition 9.1.23 *There is a bijective correspondence between the set of Eckardt points on a nonsingular cubic surface S and the set of harmonic homologies in $\mathbb{P}^3$ with center in S.*

Proof Let $x=\ell_1\cap\ell_2\cap\ell_3\in S$ be an Eckardt point. Choose coordinates such that $x=[1,0,0,0]$ and the equation of the tritangent plane is $t_1=0$. The equation of S is

$$t_0^2t_1+2t_0g_2+g_3=0, \tag{9.14}$$

where g_2,g_3 are homogeneous forms in t_1,t_2,t_3. The polar quadric $P_x(S)$ contains the three coplanar lines ℓ_i passing through one point. This implies that $P_x(S)$ is the union of two planes; one of them is $V(t_1)$. Since the equation of $P_x(S)$ is $t_0t_1+g_2=0$, we obtain that $g_2=t_1g_1(t_1,t_2,t_3)$. Making one more coordinate change $t_0\to t_0+g_1$, we reduce the equation to the form $t_0^2t_1+g_3'(t_1,t_2,t_3)$. The intersection $V(t_0)\cap S$ is isomorphic to the cubic curve $V(g_3)$. Now we define the homology

$$\tau:[t_0,t_1,t_2,t_3]\mapsto[-t_0,t_1,t_2,t_3]. \tag{9.15}$$

It obviously leaves S invariant and has x as its isolated fixed point. The other component of the fixed locus is the cubic curve $V(t_0)\cap V(S)$.

Conversely, assume S admits a projective automorphism τ of order 2 with one isolated fixed point p on S. Choose projective coordinates such that τ is

given by formula (9.15). Then S can be given by Equation (9.14). The surface is invariant with respect to τ if and only if $g_2 = 0$. The plane $V(t_0)$ is the tritangent plane with Eckardt point $[1, 0, 0, 0]$. □

It is clear that the automorphism τ is defined by the projection from the Eckardt point x. It extends to a biregular automorphism of the blow-up $\pi : S' \to S$ of the point x which fixes pointwisely the exceptional curve E of π. The surface S' is a weak del Pezzo surface of degree 2. It has three disjoint (-2)-curves R_i equal to the proper transforms of the lines ℓ_i containing x. The projection map $S' \to \mathbb{P}^2$ is equal to the composition of the birational morphism $S' \to X$ which blows down the curves R_i and a finite map of degree 2 $X \to \mathbb{P}^2$. The surface X is an anticanonical model of S' with three singular points of type A_1. The branch curve of $X \to \mathbb{P}^2$ is the union of a line and a nonsingular cubic intersecting the line transversally. The line is the image of the exceptional curve E.

Example 9.1.24 Consider a *cyclic cubic surface* S given by equation

$$f_3(t_0, t_1, t_2) + t_3^3 = 0,$$

where $C = V(f_3)$ is a nonsingular plane cubic in the plane with coordinates t_0, t_1, t_2. Let ℓ be an inflection tangent of C. We can choose coordinates such that $\ell = V(t_1)$ and the tangency point is $[1, 0, 0]$. The equation of S becomes

$$t_0^2 t_1 + t_0 t_1 g_1(t_1, t_2) + t_1 g_2(t_1, t_2) + t_2^3 + t_3^3 = 0.$$

The preimage of the line ℓ under the projection map $[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, t_2]$ splits into the union of three lines with equation $t_1 = t_2^3 + t_3^3 = 0$. The point $[1, 0, 0, 0]$ is an Eckardt point. Because there are nine inflection points on a nonsingular plane cubic, the surface contains nine Eckardt points. Note that the corresponding 9 tritangent planes contain all 27 lines.

Example 9.1.25 Consider a cubic surface given by equations

$$\sum_{i=0}^{4} a_i t_i^3 = \sum_{i=0}^{4} t_i = 0,$$

where $a_i \neq 0$. We will see later that a general cubic surface is projectively equivalent to such surface (but not the cyclic one). Assume $a_0 = a_1$. Then the point $p = [1, -1, 0, 0, 0]$ is an Eckardt point. In fact, the tangent plane at this point is $t_0 + t_1 = t_2 + t_3 + t_4 = 0$. It cuts out the surface along the union of three lines intersecting at the point p. Similarly, we have an Eckardt point whenever $a_i = a_j$ for some $i \neq j$. Thus we may have 1, 2, 3, 4, 6 or 10 Eckardt points dependent on whether we have just two equal coefficients, or two pairs of equal coefficients, or three equal coefficients, or a pair and a triple

of equal coefficients, or four equal coefficients, or five equal coefficients. The other possibilities for the number of Eckardt points are nine as in the previous example, or 18 in the case when the surface is isomorphic to a Fermat cubic surface. We will prove later that no other case occurs.

Let us prove the following, as it will be needed in the future.

Proposition 9.1.26 *Let x and y be two Eckardt points on S such that the line $\ell = \overline{xy}$ is not contained in S. Then ℓ intersects S in a third Eckardt point. Moreover, no three Eckardt points lie on a line contained in the surface.*

Proof Let τ be the harmonic homology involution of S defined by the Eckardt point x. Then ℓ intersects S at the point $z = \tau(y)$. The points y and z are on the line $\overline{xy}$. If ℓ is not contained in S, then it is not contained in the polar quadric $P_x(S)$, and hence does not intersect the 1-dimensional component F of the fixed locus of τ. This shows that $y \neq z$. On the other hand, if ℓ is contained in S, then it is one of the three lines in the tritangent plane containing x. Suppose that we have two more Eckardt points on ℓ. Then the plane that cuts out F intersects the corresponding tritangent planes of the two new Eckardt points. This implies that ℓ contains three fixed points of the involution τ, a contradiction. □

Proposition 9.1.27 *Let x_1, x_2, x_3 be three collinear Eckardt points. Then the involutions τ_i corresponding to these points generate a subgroup of automorphisms isomorphic to $\mathfrak{S}_3$. If two Eckardt points x_1, x_2 lie on a line $\ell \subset S$, then the involutions commute, and the product fixes the line and the other line which contains the tangency points of three tritangent planes through ℓ.*

Proof Suppose three Eckardt points lie on a line ℓ. Obviously, each τ_i leaves the line $\ell = \overline{x_1x_2}$ invariant. Thus the subgroup G generated by the three involutions leaves the line invariant and permutes three Eckardt points. This defines a homomorphism $G \to \mathfrak{S}_3$ which is obviously surjective. Let g be a nontrivial element from the kernel. Then it leaves three points fixed, and hence leaves all points on the line fixed. Without loss of generality, we may assume that $g = \tau_1\tau_2$ or $g = \tau_1\tau_2\tau_3$. Since τ_1 and τ_2, and $\tau_1\tau_2, \tau_3$ act differently on ℓ, we get $g = 1$.

Now suppose that two Eckardt points lie on a line ℓ contained in the surface. Obviously, τ_i fixes both points x_1 and x_2. Since a finite automorphism group of $\mathbb{P}^1$ fixing two points is cyclic, the product $\tau = \tau_1\tau_2$ is of order 2; it fixes ℓ pointwise, and also fixes the line ℓ' equal to intersection of the planes of fixed points of τ_1, τ_2. This line intersects each tritangent plane through ℓ at some point. Hence each such plane is invariant with respect to τ, and the tangency points of the remaining three tritangent planes lie on ℓ'. □

Let us project S from a point $x \in S$ that is not an Eckardt point. Suppose x does not lie on any line in S. Then the blow-up S' of S at x is a del Pezzo surface of degree 2. The projection map lifts to a finite double cover of $\mathbb{P}^2$ branched along a nonsingular quartic curve B. The 27 lines, together with the exceptional curve E of the blow-up, map to the 28 bitangents of B. The image of a sixer of lines and the curve E is an Aronhold set of seven bitangents. This relationship between 27 lines on a cubic surface and 28 bitangents of a plane quartic was first discovered by C. Geiser [237] in 1860.

If x lies on one line, S' is a weak del Pezzo surface of degree 2 with one (-2)-curve R. The projection map lifts to a degree map $S' \to \mathbb{P}^2$ which factors through the blowing down map $S \to X$ of R and a finite map of degree 2 $X \to \mathbb{P}^2$ branched over a 1-nodal quartic curve. If x lies on two lines, then we have a degree 2 map $S' \to X \to \mathbb{P}^2$, where X has two A_1-singularities, and the branch curve of $X \to \mathbb{P}^2$ is a 2-nodal quartic.

9.2 Singularities

9.2.1 Non-normal cubic surfaces

Let X be an irreducible cubic surface in $\mathbb{P}^3$. Assume that X is not normal and is not a cone over a singular cubic curve. Then its singular locus contains a 1-dimensional part C of some degree d. Let m be the multiplicity of a general point of C. By Bertini's Theorem, a general plane section H of X is an irreducible plane cubic that contains d singular points of multiplicity m. Since an irreducible plane cubic curve has only one singular point of multiplicity 2, we obtain that the singular locus of X is a line.

Let us choose coordinates in such a way that C is given by the equations $t_0 = t_1 = 0$. Then the equation of X must look like

$$l_0 t_0^2 + l_1 t_0 t_1 + l_2 t_1^2 = 0,$$

where $l_i, i = 0, 1, 2$, are linear forms in t_0, t_1, t_2. This shows that the left-hand side contains t_2 and t_3 only in degree 1. Thus we can rewrite the equation in the form

$$t_2 f + t_3 g + h = 0, \tag{9.16}$$

where f, g, h are binary forms in t_0, t_1, the first two of degree 2, and the third one of degree 3.

Suppose f, g are proportional. Then, the equation can be rewritten in the form $(at_1 + bt_2)f + h = 0$, which shows that X is a cone. A pair of non-proportional binary quadratic forms f, g can be reduced to the form $t_0^2 +$

$t_1^2, at_0^2+bt_1^2$, or $t_0t_1, at_0^2+t_0t_1$ (corresponding to the Segre symbol (2)). After making a linear change of variables t_2, t_3, we arrive at two possible equations

$$t_2t_0^2 + t_3t_1^2 + (at_0 + bt_1)t_0^2 + (ct_0 + dt_1)t_1 = 0,$$
$$t_2t_0t_1 + t_3t_0^2 + (at_0 + bt_1)t_0^2 + (ct_0 + dt_1)t_1 = 0.$$

Replacing t_2 with $t_2' = t_2 + at_0 + bt_1$ and t_3 with $t_3' = t_3 + ct_0 + dt_1$, we obtain two canonical forms of non-normal cubic surfaces that are not cones.

The plane sections through the singular line of the surface define a structure of a scroll on the surface.

Theorem 9.2.1 *Let X be an irreducible non-normal cubic surface. Then, either X is a cone over an irreducible singular plane cubic, or it is projectively equivalent to one of the following cubic surfaces singular along a line:*

(i) $t_0^2t_2 + t_1^2t_3 = 0$;

(ii) $t_2t_0t_1 + t_3t_0^2 + t_1^3 = 0$.

The two surfaces are not projectively isomorphic.

The last assertion follows from considering the normalization $\bar{X}$ of the surface X. In both cases it is a nonsingular surface, however in (i), the preimage of the singular line is irreducible, but in the second case it is reducible.

We have already seen two cubic scrolls in $\mathbb{P}^3$ in Subsection 2.1.1. They are obtained as projections of the cubic scroll $S_{1,4}$ in $\mathbb{P}^4$ isomorphic to the rational minimal ruled surface $\mathbb{F}_1$ (a del Pezzo surface of degree 8). There are two possible centers of the projection: on the exceptional (-1)-curve or outside of this curve. Case (i) corresponds to the first possibility, and case (ii) to the second one.

9.2.2 Lines and singularities

From now on we assume that S is a normal cubic surface that is not a cone. Thus its singularities are rational double points, and S is a del Pezzo surface of degree 3.

Let X be a minimal resolution of singularities of S. All possible Dynkin

curves on X can be easily found from the list of root bases in $\mathbf{E}_6$.

$$
\begin{aligned}
&(r=6)\ \ E_6,\ A_6,\ D_4+A_2, \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=6,\\
&(r=5)\ \ D_5,\ D_4+A_1,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=5,\\
&(r=4)\ \ D_4,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=4,\\
&(r=3)\ \ A_3,\ A_2+A_1,\ 3A_1,\\
&(r=2)\ \ A_2,\ A_1+A_1,\\
&(r=1)\ \ A_1.
\end{aligned}
$$

The following Lemma is easily verified, and we omit its proof.

Lemma 9.2.2 *Let $x_0=(1,0,0,0)$ be a singular point of $S=V(f_3)$. Write*

$$f_3 = t_0 g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3),$$

where g_2, g_3 are homogeneous polynomials of degrees 2 and 3, respectively. Let $x=[a_0,a_1,a_2,a_3]\in S$. If the line $\overline{x_0x}$ is contained in S, then the point $q=[a_1,a_2,a_3]$ is a common point of the conic $V(g_2)$ and the cubic $V(g_3)$. If, moreover, x is a singular point of S, then the conic and the cubic intersect at q with multiplicity >1.

Corollary 9.2.3 *$V(f_3)$ has at most four singular points. Moreover, if $V(f_3)$ has four singular points, then each point is of type A_1.*

Proof Let x_0 be a singular point which we may assume to be the point $[1,0,0,0]$ and apply Lemma 9.2.2. Suppose we have more than 4 singular points. The conic and the cubic will intersect at least in four singular points with multiplicity >1. Since they do not share an irreducible component (otherwise f_3 is reducible), this contradicts Bézout's Theorem. Suppose we have four singular points and x_0 is not of type A_1. Since x_0 is not an ordinary double point, the conic $V(g_2)$ is reducible. Then the cubic $V(g_3)$ intersects it at three points with multiplicity >1 at each point. It is easy to see that this also contradicts Bézout's Theorem. □

Lemma 9.2.4 *The cases, $A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k=6$, except the cases $3A_2, A_5+A_1$ do not occur.*

Proof Assume $M=A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k=6$. Then the discriminant d_M of the lattice M is equal to $(i_1+1)\cdots(i_k+1)$. By Lemma 8.2.1, $3|d_M$,

one of the numbers, say $i_1 + 1$, is equal either to 3 or 6. If $i_1 + 1 = 6$, then $M = A_5 + A_1$. If $i_1 + 1 = 3$, then $(i_2 + 1) \dots (i_k + 1)$ must be a square, and $i_2 + \dots + i_k = 4$. It is easy to see that the only possibilities are $i_2 = i_3 = 2$ and $i_2 = i_3 = i_4 = i_5 = 1$. The last possibility is excluded by applying Corollary 9.2.3. □

Lemma 9.2.5 *The cases $D_4 + A_1$ and $D_4 + A_2$ do not occur.*

Proof Let x_0 be a singular point of S of type D_4. Again, we assume that $x_0 = [1, 0, 0, 0]$ and apply Lemma 9.2.2. As we have already noted, the singularity of type D_4 is analytically (or formally) isomorphic to the singularity $z^2 + xy(x + y) = 0$. This shows that the conic $V(g_2)$ is a double line ℓ. The plane $z = 0$ cuts out a germ of a curve with 3 different branches. Thus there exists a plane section of $S = V(f_3)$ passing through x_0 which is a plane cubic with 3 different branches at x_0. Obviously, it must be the union of 3 lines with a common point at x_0. Now the cubic $V(g_3)$ intersects the line ℓ at 3 points corresponding to the lines through x_0. Thus S cannot have more singular points. □

Let us show that all remaining cases are realized. We will exhibit the corresponding del Pezzo surface as the blow-up of six bubble points $p_1, \dots, p_6$ in $\mathbb{P}^2$.

A_1: 6 proper points in $\mathbb{P}^2$ on an irreducible conic;

A_2: $p_3 \succ_1 p_1$;

$2A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_3$;

A_3: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

$A_2 + A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$;

A_4: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

$3A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$;

$2A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$;

$A_3 + A_1$: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$;

A_5: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

D_4: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$ and p_1, p_3, p_5 are collinear;

$A_2 + 2A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$, and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_4 + A_1$: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2h - p_1 - \dots - p_6| \neq \emptyset$;

D_5: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|h - p_1 - p_2 - p_6| \neq \emptyset$;

$4A_1$: $p_1, \dots, p_6$ are the intersection points of 4 lines in a general linear position;

$2A_2 + A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_3 + 2A_1$: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_5 + A_1$: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2h - p_1 - \cdots - p_6| \neq \emptyset$;

E_6: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$3A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$, $|h - p_1 - p_2 - p_3| \neq \emptyset$, $|h - p_4 - p_5 - p_6| \neq \emptyset$;

Projecting from a singular point and applying Lemma 9.2.2 we see that each singular cubic surface can be given by the following equations.

A_1: $V(t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3))$, where $V(g_2)$ is a nonsingular conic which intersects $V(g_3)$ transversally;

A_2: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$,where $V(t_1 t_2)$ intersects $V(g_3)$ transversally;

$2A_1$: $V(t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3))$, where $V(g_2)$ is a nonsingular conic which is simply tangent to $V(g_3)$ at one point;

A_3: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(t_1 t_2)$ intersects $V(g_3)$ at the point $[0, 0, 1]$ and at other 4 distinct points;

$A_2 + A_1$: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(g_3)$ is tangent to $V(t_2)$ at $[1, 0, 0]$;

A_4: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0, 0, 1]$;

$3A_1$: $V(t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 2 points;

$2A_2$: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ intersects $V(g_3)$ transversally and $V(t_2)$ is an inflection tangent to $V(g_3)$ at $[1, 0, 0]$;

$A_3 + A_1$: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(g_3)$ passes through $[0, 0, 1]$ and $V(t_1)$ is tangent to $V(g_3)$ at the point $[1, 0, 0]$;

A_5: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$ at the point $[0, 0, 1]$;

D_4: $V(t_0 t_1^2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ intersects transversally $V(g_3)$;

$A_2 + 2A_1$: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(g_3)$ is tangent $V(t_1 t_2)$ at two points not equal to $[0, 0, 1]$;

$A_4 + A_1$: $V(t_0 t_1 t_2 + g_3(t_1, t_2, t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0, 0, 1]$ and is tangent to $V(t_2)$ at $[1, 0, 0]$;

D_5: $V(t_0 t_1^2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ is tangent to $V(g_3)$ at $[0, 0, 1]$;

$4A_1$: $V(t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 3 points;

$2A_2 + A_1$: $V(t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3))$, where $V(g_2)$ is tangent to $V(g_3)$ at 2 points $[1, 0, 0]$ with multiplicity 3;

A_3+2A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ passes through $[0,0,1]$ and is tangent to $V(t_1)$ and to $V(t_2)$ at one point not equal to $[0,0,1]$;

A_5+A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$ at the point $[0,0,1]$ and $V(t_2)$ is tangent to $V(g_3)$;

E_6: $V(t_0t_1^2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$.

$3A_2$: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1), V(t_2)$ are inflection tangents of $V(g_3)$ at points different from $[0,0,1]$.

Remark 9.2.6 Applying a linear change of variables, one can simplify the equations. For example, in type XXI (see Table 9.1), we may assume that the inflection points are $[1,0,0]$ and $[0,1,0]$. Then $g_3=t_3^3+t_1t_2L(t_1,t_2,t_3)$. Replacing t_0 with $t_0'=t_0+L(t_1,t_2,t_3)$, we reduce the equation to the form

$$t_0t_1t_2+t_3^3=0. \tag{9.17}$$

Another example is the E_6-singularity (type XX). We may assume that the inflection point is $[0,0,1]$. Then $g_3=t_2^3+t_1g_2(t_1,t_2,t_3)$. The coefficient at t_3^2 is not equal to zero, otherwise the equation is reducible. After a linear change of variables we may assume that $g_2=t_3^2+at_1^2+bt_1t_2+ct_2^2$. Replacing t_0 with $t_0+at_1+bt_2$, we may assume that $a=b=0$. After scaling the unknowns, we get

$$t_0t_1^2+t_1t_2^2+t_1t_3^2+t_2^3=0. \tag{9.18}$$

Table 9.1 gives the classification of possible singularities of a cubic surface, the number of lines and the class of the surface.

Type	Singularity	Lines	Class	Type	Singularity	Lines	Class
I	$\emptyset$	27	12	XII	D_4	6	6
II	A_1	21	10	XIII	A_2+2A_1	8	5
III	A_2	15	9	XIV	A_4+A_1	4	5
IV	$2A_1$	16	8	XV	D_5	3	5
V	A_3	10	8	XVI	$4A_1$	9	4
VI	A_2+A_1	11	7	XVII	$2A_2+A_1$	5	4
VII	A_4	6	7	XVIII	A_3+2A_1	5	4
VIII	$3A_1$	12	6	XIX	A_5+A_1	2	4
IX	$2A_2$	7	6	XX	E_6	1	4
X	A_3+A_1	7	6	XXI	$3A_2$	3	3
XI	A_5	3	6				

Table 9.1 *Singularities of cubic surfaces*

Note that the number of lines can be checked directly by using the equations. The map from $\mathbb{P}^2$ to S is given by the linear system of cubics generated by $V(g_3), V(t_1g_2), V(t_2g_2), V(t_3g_2)$. The lines are images of lines or conics that

intersect a general member of the linear system with multiplicity 1 outside base points. The class of the surface can be computed by applying the Plücker-Teissier formula from Theorem 1.2.7. We use that the Milnor number of an A_n, D_n, E_n singularity is equal to n, and the Milnor number of the singularity of a general plane section through the singular point is equal to 1 if type is A_n and 2 otherwise.

Example 9.2.7 The cubic surface with three singular points of type A_2 given by Equation (9.17) plays an important role in the Geometric Invariant Theory of cubic surfaces. It represents the unique isomorphism class of a strictly semistable point in the action of the group $\mathrm{SL}(4)$ in the space of cubic surfaces. Table 9.1 shows that it is the only normal cubic surface whose dual surface is also a cubic surface. By the Reciprocity Theorem $X \cong (X^\vee)^\vee$, the dual surface cannot be a cone or a scroll. Thus the surface of type XXI is the only self-dual cubic surface.

Another interesting special case is the surface with four A_1-singularities. In notation above, let us choose coordinates such that the three tangency points of the conic $V(g_2)$ and the cubic $V(g_3)$ are $[1,0,0],[0,1,0],[0,0,1]$. After scaling the coordinates, we may assume that $g_2 = t_1t_2 + t_1t_3 + t_2t_3$. An example of a cubic tangent to the conic at the three points is the union of three tangent lines

$$h = (t_1+t_2)(t_1+t_3)(t_2+t_3) = g_2(t_1+t_2+t_3) - t_1t_2t_3.$$

Any other cubic can be given by equation $lg_2 + h = 0$, where l is a linear form. Replacing t_0 with $t_0' = -(t_0 + l + t_1 + t_2 + t_3)$, we reduce the equation to the form

$$t_0(t_1t_2 + t_1t_3 + t_2t_3) + t_1t_2t_3 = t_0t_1t_2t_3\Big(\frac{1}{t_0} + \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\Big) = 0. \quad (9.19)$$

A cubic surface with four nodes is called the *Cayley cubic surface*. As we see, all Cayley cubics are projectively equivalent. They admit $\mathfrak{S}_4$ as its group of automorphisms.

Let us find the dual surface of a Cayley surface. Table 9.1 shows that it must be a quartic surface. The equation of a tangent plane at a general point $[a,b,c,d]$ is

$$\frac{t_0}{a^2} + \frac{t_1}{b^2} + \frac{t_2}{c^2} + \frac{t_3}{d^2} = 0.$$

Thus the dual surface is the image of S under the map

$$[t_0,t_1,t_2,t_3] \mapsto [\xi_0,\xi_1,\xi_2,\xi_3] = [1/t_0^2, 1/t_1^2, 1/t_2^2, 1/t_3^2].$$

Write $t_i = 1/\sqrt{\xi_i}$ and plug in Equation (9.19). We obtain the equation

$$(\sqrt{\xi_0\xi_1\xi_2\xi_3})^{-2}(\sqrt{\xi_0}+\sqrt{\xi_1}+\sqrt{\xi_2}+\sqrt{\xi_3}) = 0.$$

This shows that the equation of the dual quartic surface is obtained from the equation

$$\sqrt{\xi_0}+\sqrt{\xi_1}+\sqrt{\xi_2}+\sqrt{\xi_3} = 0$$

by getting rid of the irrationalities. We get the equation

$$(\sum_{i=0}^{3}\xi_i^2 - 2\sum_{0\le i<j\le 3}\xi_i\xi_j)^2 - 64\xi_0\xi_1\xi_2\xi_3 = 0.$$

The surface has three singular lines $t_i + t_k = t_l + t_m = 0$. They meet at one point $[1,1,1,1]$. The only quartic surface with this property is a Steiner quartic surface from Subsection 2.1.1. Thus the dual of the Cayley cubic surface is a Steiner quartic surface.

9.3 Determinantal equations

9.3.1 Cayley-Salmon equation

Let S' be a minimal resolution of singularities of a del Pezzo cubic surface S. Choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S')$ and consider the image of one of 120 conjugate pairs of triples of tritangent trios from (9.9). Write them as a matrix:

$$\begin{matrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23}. \\ e_{31} & e_{32} & e_{33} \end{matrix} \tag{9.20}$$

Suppose the divisor classes e_{ij} are the classes of (-1)-curves on S'. Then their images in S are lines ℓ_{ij}. The lines defined by the i-th row (the j-th column) lie in a plane Λ_i (Λ'_j), a tritangent plane of S. The union of the planes Λ_i contains all nine lines ℓ_{ij}. The same is true for the planes Λ'_j. The pencil of cubic surfaces spanned by the cubics $\Lambda_1+\Lambda_2+\Lambda_3$ and $\Lambda'_1+\Lambda'_2+\Lambda'_3$ must contain the cubic S. This shows that we can choose the equations $l_i = 0$ of Λ_i and the equations $m_j = 0$ of Λ'_j such that S can be given by equation

$$l_1l_2l_3 + m_1m_2m_3 = \det\begin{pmatrix} l_1 & m_1 & 0 \\ 0 & l_2 & m_2 \\ -m_3 & 0 & l_3 \end{pmatrix} = 0. \tag{9.21}$$

The equation of a cubic surface of the form (9.21), where the nine lines defined by the equations $l_i = m_j = 0$ are all different, is called *Cayley-Salmon equation*. Note that the lines ℓ_{ii} are skew (otherwise we have four lines in one plane). We say that two Cayley-Salmon equations are equivalent if they define the same unordered sets of three planes $V(l_i)$ and $V(m_j)$.

Suppose a cubic surface can be given by a Cayley-Salmon equation. Each plane $V(l_i)$ contains three different lines $\ell_{ij} = V(l_i) \cap V(m_j), j = 1, 2, 3$ and hence is a tritangent plane. After reindexing, we may assume that the lines ℓ_{ii} are skew lines. Let e_{ij} be the divisor classes of the preimages of the lines in S'. Then they form the image of a conjugate pair of tritangent trios under some geometric marking of S'.

Theorem 9.3.1 *Let S be a normal cubic surface. The number of the equivalence classes of Cayley-Salmon equations for S is equal to 120 (type I), 10 (type II), 1 (type III, IV, VIII), and zero otherwise.*

Proof We know that the number of conjugate pairs of triads of tritangent trios of exceptional vectors is equal to 120. Thus the number of conjugate triads of triples of tritangent planes on a nonsingular cubic surface is equal to 120.

Suppose S has one node. We take a blow-up model of S' as the blow-up of six proper points on an irreducible conic. We have 10 matrices of type II in (9.9) which give us 10 pairs of conjugate triples of tritangent planes.

Suppose S has three nodes. We take the blow-up model corresponding to a bubble cycle $x_1 + \cdots + x_6$ with $x_4 \succ x_1, x_5 \succ x_2, x_6 \succ x_3$. The set of lines ℓ_{ij} are represented by the divisor classes of

$$e_0 - e_1 - e_4, e_0 - e_2 - e_5, e_0 - e_3 - e_6, e_k, 2e_0 - e_1 - \cdots - e_6 + e_k, k = 4, 5, 6.$$

It is easy to see that this is the only possibility.

We leave it to the reader to check the assertion in the remaining cases. □

Suppose a normal cubic surface S contains three skew lines ℓ_1, ℓ_2, ℓ_3. Consider the pencil of planes $\mathcal{P}_i$ through the line ℓ_i. For any general point $x \in \mathbb{P}^3$ one can choose a unique plane $\Pi_i \in \mathcal{P}_i$ containing the point x. This defines a rational map

$$f : \mathbb{P}^3 \dashrightarrow \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3. \tag{9.22}$$

Suppose the intersection of the planes Π_i contains a line ℓ. Then ℓ intersects ℓ_1, ℓ_2, ℓ_3, and hence either belongs to S or does not intersect S at a point outside the three lines. This shows that the restriction of the map f to S is a birational map onto its image X.

The composition of map (9.22) with the Segre map defines a rational map

$$f : \mathbb{P}^3 \dashrightarrow \mathcal{S} \subset \mathbb{P}^7,$$

where $\mathcal{S}$ is isomorphic to the Segre variety $\mathsf{s}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Since the pencils $\mathcal{P}_i$ generate the complete linear system $|V|$ of cubic surfaces containing the lines ℓ_i, the map f is given by $|V|$. Since our cubic S is a member of $|V|$, it is equal to the preimage of a hyperplane section H of X. The Segre variety $\mathcal{S}$ is of degree 6, so H is isomorphic to a surface S_6 of degree 6 in $\mathbb{P}^6$ and the restriction of f to S is a birational map onto S_6. The hyperplane section H is defined by a divisor of tridegree $(1, 1, 1)$ on the Segre variety. This gives the following.

Theorem 9.3.2 (F. August) *Any cubic surface containing three skew lines ℓ_1, ℓ_2, ℓ_3 can be generated by three pencils $\mathcal{P}_i$ of planes with base locus ℓ_i in the following sense. There exists a correspondence R of degree $(1, 1, 1)$ on $\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ such that*

$$S = \{x \in \mathbb{P}^3 : x \in \Pi_1 \cap \Pi_2 \cap \Pi_3 \text{ for some } (\Pi_1, \Pi_2, \Pi_3) \in R\}.$$

Note that a del Pezzo surface S_6 of degree 6 containing in the Segre variety $\mathsf{s}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ has three different pencils of conics. So, it is either nonsingular, or has one node. In the first case, it is a toric surface which can be given by the equation

$$u_0v_0w_0 + u_1v_1w_1 = 0,$$

where $(u_0, u_1), (v_0, v_1), (w_0, w_1)$ are projective coordinates in each factor of $(\mathbb{P}^1)^3$. The equation can be considered as a linear equations in the space $\mathbb{P}^7$ with coordinates $u_iv_jw_k$. If S_6 is singular, it is not a toric surface. The corresponding weak del Pezzo surface is the blow-up of three collinear points. It contains only three lines.

Suppose S_6 is a nonsingular surface. Then we can identify the coordinates (u_0, u_1) with the coordinates in the pencil $\mathcal{P}_1$ of planes through ℓ_1 and, similarly, for the other two pairs of coordinates. Thus the cubic surface is equal to the set of solutions of the system of linear equations

$$\begin{aligned} u_0l_1(t_0, t_1, t_2, t_3) &= u_1m_1(t_0, t_1, t_2, t_3), \\ v_0l_2(t_0, t_1, t_2, t_3) &= v_1m_2(t_0, t_1, t_2, t_3), \\ w_0l_3(t_0, t_1, t_2, t_3) &= w_1m_3(t_0, t_1, t_2, t_3), \end{aligned}$$

where l_i, m_i are linear forms and $u_0v_0w_0 + u_1v_1w_1 = 0$. This immediately gives Cayley-Salmon equation of S. Conversely, the choice of Cayley-Salmon equation gives August's projective generation of S.

Remark 9.3.3 The rational map $f : \mathbb{P}^3 \dashrightarrow \mathcal{S} \subset \mathbb{P}^7$ is a birational map. To see this, we take a general line ℓ_4 in $\mathbb{P}^3$. The image of this line is a rational curve of degree 3 in X. The composition of f and the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^3$ from the subspace spanned by $f(\ell_4)$ is a rational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. It is given by the linear system of cubics passing through the lines $\ell_1, \ell_2, \ell_3, \ell_4$. Since four skew lines in $\mathbb{P}^3$ have two *transversals* (i.e. lines intersecting the four lines), the base locus of the linear system also contains the two transversal lines. The union of the six lines is a reducible projectively normal curve of arithmetic genus 3. The transformation T is a bilinear Cremona transformation. The P-locus of T consists of the union of four quadrics Q_k, each containing the lines $\ell_i, i \neq k$ (since it must be of degree 8, there is nothing else). In particular, the map f blows down the quadric Q_4 to a curve parameterizing the ruling of Q_4 that does not contain the lines ℓ_1, ℓ_2, ℓ_3. If S_6 is a nonsingular surface, we must blow down exactly three lines on S, hence the lines ℓ_1, ℓ_2, ℓ_3 have three transversals contained in S. Also, if S_6 is nonsingular, the singular points of S must lie on the lines ℓ_1, ℓ_2, ℓ_3. This can be checked in all cases where Cayley-Salmon equation applies.

Corollary 9.3.4 *Let S be a nonsingular cubic surface. Then S is projectively equivalent to a surface*

$$V(t_0t_1t_2 + t_3(t_0 + t_1 + t_2 + t_3)l(t_0, \dots, t_3)).$$

A general S can be written in this form in exactly 120 ways (up to projective equivalence).

Proof We will prove later that a nonsingular cubic surface has at most 18 Eckardt points. Thus we can choose the linear forms m_1, m_2, m_2 such that the lines $\ell_{1j}, \ell_{2j}, \ell_{3j}, j = 1, 2, 3$ do not intersect. This implies that the linear forms l_1, l_2, l_3, m_j are linearly independent. Similarly, we may assume that the linear forms $m_1, m_2, m_3, l_j, j = 1, 2, 3$ are linearly independent. Choose coordinates such that $l_1 = t_0, l_2 = t_1, l_3 = t_2, m_1 = t_3$. The equation of S can be written in the form $t_0t_1t_2 + t_3m_2m_3 = 0$. Let $m_2 = \sum a_it_i$. It follows from the previous assumption, that the coefficients a_i are all nonzero. After scaling the coordinates, we may assume that $m_2 = t_0 + t_1 + t_2 + t_3$ and we take $l = m_3$. □

9.3.2 Hilbert-Burch Theorem

The Cayley-Salmon equation has a determinantal form and hence gives a determinantal representation of a cubic surface. Unfortunately, it applies to only a few of the 21 different types of cubics. By other methods we will see that a

determinantal representation exists for any normal cubic surface of type different from XX. We will use the following result from commutative algebra (see [208]).

Theorem 9.3.5 (Hilbert-Burch) *Let I be an ideal in a polynomial ring R such that* $\text{depth}(I) = \text{codim} I = 2$ *(thus R/I is a Cohen-Macaulay ring). Then there exists a projective resolution*

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0.$$

The i-th entry of the vector $(a_1, \ldots, a_n)$ defining ϕ_1 is equal to $(-1)^i c_i$, where c_i is the complementary minor obtained from the matrix A defining ϕ_2 by deleting its i-th row.

We apply this Theorem to the case when $R = \mathbb{C}[t_0, t_1, t_2]$ and I is the homogeneous ideal of a closed 0-dimensional subscheme Z of $\mathbb{P}^2 = \text{Proj}(R)$ generated by four linearly independent homogeneous polynomials of degree 3. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then $(I_Z)_m = H^0(\mathbb{P}^2, \mathcal{I}_Z(m))$. By assumption

$$H^0(\mathbb{P}^2, \mathcal{I}_Z(2)) = 0. \tag{9.23}$$

Applying the Hilbert-Burch Theorem, we find a resolution of the graded ring R/I

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} R \longrightarrow R/I \to 0,$$

where ϕ_2 is given by a 3×4-matrix A whose entries are linear forms in t_0, t_1, t_2. Passing to the projective spectrum, we get an exact sequence of sheaves

$$0 \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \xrightarrow{\phi_2} V \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\phi_1} \mathcal{I}_Z \longrightarrow 0,$$

where U, V are vector spaces of dimension 3 and 4. Twisting by $\mathcal{O}_{\mathbb{P}^2}(3)$, we get the exact sequence

$$0 \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\tilde{\phi}_2} V \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\tilde{\phi}_1} \mathcal{I}_Z(3) \longrightarrow 0. \tag{9.24}$$

Taking global sections, we obtain

$$V = H^0(\mathbb{P}^2, \mathcal{I}_Z(3)).$$

Twisting fact sequence (9.24) by $\mathcal{O}_{\mathbb{P}^2}(-2)$, and using the canonical trace isomorphism $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, we obtain that

$$U = H^1(\mathbb{P}^2, \mathcal{I}_Z(1)).$$

The exact sequence

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_Z \to 0$$

shows that

$$U \cong \text{Coker}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_Z)) \cong \text{Coker}\big(\mathbb{C}^3 \to \mathbb{C}^{h^0(\mathcal{O}_Z)}\big).$$

Since $\dim U = 3$, we obtain that $h^0(\mathcal{O}_Z) = 6$. Thus Z is a 0-cycle of length 6.

Now we see that the homomorphism $\tilde{\phi}_2$ of vector bundles is defined by a linear map

$$\phi : E \to \text{Hom}(U, V) = U^\vee \otimes V, \tag{9.25}$$

where $\mathbb{P}^2 = |E|$. We can identify the linear map ϕ with the tensor $\mathfrak{t} \in E^\vee \otimes U^\vee \otimes V$. Let us now view this tensor as a linear map

$$\psi : V^\vee \to \text{Hom}(E, U^\vee) = E^\vee \otimes U^\vee. \tag{9.26}$$

The linear map (9.25) defines a rational map, the right kernel map,

$$f : |E| \to |V^\vee| = |\mathcal{I}_Z(3)|^\vee, \quad [v] \mapsto |\phi(v)(U)^\perp|.$$

It is given by the linear system $|\mathcal{I}_Z(3)|$. In coordinates, it is given by maximal minors of the matrix A defining ϕ_2. Thus S is contained in the locus of $[\alpha]$ such that α belongs to the preimage of the determinantal locus in $\text{Hom}(E, U^\vee)$. It is a cubic hypersurface in the space $\text{Hom}(E, U^\vee)$. Thus the image of f is contained in a determinantal cubic surface S. Since the intersection scheme of two general members C_1, C_2 of the linear system $|\mathcal{I}_Z(3)|$ is equal to the 0-cycle Z of degree 6, the image of f is a cubic surface. This gives a determinantal representation of S.

Theorem 9.3.6 *Assume S is a normal cubic surface. Then S admits k equivalence classes of linear determinantal representations, where k depends on type of S, and is given in the following Table.*

I	II	III	IV	V	VI	VII	VIII	IX	X	XI
72	50	24	34	60	64	52	66	60	58	42
XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	
48	62	50	32	64	58	56	40	0	54	

Table 9.2 *Number of determinantal representations*

Proof Let S' be a minimal resolution of singularities of S. It follows from the previous construction that a blowing-down structure defined by a bubble cycle of six points not containing in a conic, gives a determinantal representation of S. Conversely, suppose $S \subset \mathbb{P}(V)$, and we have a linear map (9.26) for some 3-dimensional vector spaces E and U defining a determinantal representation of S. Then ψ defines a map of vector bundles $U \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \to V \otimes \mathcal{O}_{\mathbb{P}^2}(-3)$, and the cokernel of this map is the ideal sheaf of a 0-cycle of length 6. Its blow-up is isomorphic to S. Since S is normal, the ideal sheaf is integrally closed, and hence corresponds to a bubble cycle η whose blow-up is isomorphic to S'.

So, the number of equivalence classes of linear determinantal representations is equal to the number of nef linear systems $|e_0|$ on S' which define a birational morphism $S' \to \mathbb{P}^2$ isomorphic to the blow-up of bubble cycle η of six points not lying on a conic. It follows from the proof of Lemma 9.1.1 that there is a bijection between the set of vectors $v \in I^{1,6}$ with $v^2 = 1, v \cdot \mathbf{k}_6 = -3$ and the set of roots in $\mathbf{E}_6$. The corresponding root α can be written in the form $\alpha = 2e - v_1 - \cdots - v_6$, where $(v_1, \ldots, v_6)$ is a unique sixer of exceptional vectors. If we choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S')$ defined by this sixer, then $\phi(\alpha)$ is an effective root if and only if the bubble cycle corresponding to this marking lies on a conic. Thus the number of determinantal representations is equal to the number of non-effective roots in $K_{S'}^{\perp}$ modulo the action of the subgroup W_0 of $W(E_6)$ generated by the reflections in nodal roots. In other words, this is the number of roots in $\mathbf{E}_6$ (equal to 72) minus the number of roots in the root sublattice defining the types of singularities on S. Now we use the known number of roots in root lattices and get the result. Note the exceptional case of a surface with a E_6-singularity. Here all roots are effective, so S does not admit a determinantal representation. □

Consider the left and right kernel maps for the linear map (9.26)

$$\mathfrak{l} : S \dashrightarrow |E|, \quad \mathfrak{r} : S \dashrightarrow |U|.$$

The composition of these maps with the resolution of singularities $S' \to S$ is the blowing-down map $\mathfrak{l}' : S' \to |E|$ and $\mathfrak{r}' : S' \to |U|$. When S is nonsingular, these are two maps defined by a double-six. The corresponding Cremona transformation $|E| \dashrightarrow |U|$ is given by the homaloidal linear system $|\mathcal{I}_Z^2(5)| = |5e_0 - 2(e_1 - \cdots - e_6)|$. To identify the space U with $H^0(|E|, \mathcal{I}_Z^2(5))^\vee$, we consider the linear map

$$S^2(\psi) : S^2(V^\vee) \to S^2(E^\vee \otimes U^\vee) \to \bigwedge^2 E^\vee \otimes \bigwedge^2 U^\vee \cong E \otimes V,$$

where the last isomorphism depends on a choice of volume forms on $E^\vee$ and

$U^\vee$. Dualizing, we get a linear pairing

$$E^\vee \otimes U^\vee \to S^2(V).$$

If we identify $H^0(S, \mathcal{O}_S(-2K_S))$ with S^2V, and $E^\vee$ with $H^0(S, \mathcal{O}_S(e_0))$, then $U^\vee$ can be identified with $H^0(S, \mathcal{O}_S(-2K_S - e_0))$. Note that we have also identified U with the cokernel of the map $r : E \to H^0(\mathbb{P}^2, \mathcal{O}_Z)$. Let us choose a basis in $E \cong \mathbb{C}^3$ and an order of points in Z, hence a basis in $H^0(\mathbb{P}^2, \mathcal{O}_Z) \cong \mathbb{C}^6$. The map $\mathbb{C}^6 \to U = \mathrm{Coker}(r)$ gives six vectors in U. The corresponding six points in $|U|$ is the bubble cycle defining the blowing-down structure $\mathfrak{r} : S \to |U|$. This is a special case of the construction of *associated* sets of points (see [176], [209], [569]).

Remark 9.3.7 We can also deduce Theorem 9.3.6 from the theory of determinantal equations from Chapter 4. Applying this theory we obtain that S admits a determinantal equation with entries linear forms if it contains a projectively normal curve C such that

$$H^0(S, \mathcal{O}_S(C)(-1)) = H^2(S, \mathcal{O}_S(C)(-2)) = 0. \tag{9.27}$$

Moreover, the set of non-equivalent determinantal representations is equal to the set of divisor classes of such curves. Let $\pi : S' \to S$ be a minimal resolution of singularities and $C' = \pi^*(C)$. Since $\pi^*\mathcal{O}_S(-1) = \mathcal{O}_{S'}(K_{S'})$, the conditions (9.27) are equivalent to

$$H^0(S', \mathcal{O}_{S'}(C' + K_{S'})) = H^2(S', \mathcal{O}_{S'}(C' + 2K_{S'})) = 0. \tag{9.28}$$

Since C' is nef, $H^1(S', \mathcal{O}_{S'}(C' + K_{S'})) = 0$. Also

$$H^2(S', \mathcal{O}_{S'}(C' + K_X S')) = H^0(S', \mathcal{O}_{S'}(-C')) = 0.$$

By Riemann-Roch,

$$0 = \chi(\mathcal{O}_{S'}(C' + K_{S'})) = \tfrac{1}{2}((C' + K_{S'})^2 - (C' + K_{S'}) \cdot K_{S'}) + 1$$

$$= \tfrac{1}{2}(C'^2 + C' \cdot K_{S'}) + 1.$$

Thus C' is a smooth rational curve, hence C is a smooth rational curve. It is known that a rational normal curve in $\mathbb{P}^n$ must be of degree n. Thus $-K_{S'} \cdot C' = 3$, hence $C'^2 = 1$. The linear system $|C'|$ defines a birational map $\pi : S' \dashrightarrow \mathbb{P}^2$. Let $e_0 = [C'], e_1, \ldots, e_6$ be the corresponding geometric basis of $\mathrm{Pic}(S')$. The condition

$$0 = H^2(X, \mathcal{O}_{S'}(C' + 2K_{S'})) = H^0(S', \mathcal{O}_{S'}(-C' - K_{S'})) = 0$$

is equivalent to

$$|2e_0 - e_1 - \cdots - e_6| = \emptyset. \tag{9.29}$$

9.3.3 Cubic symmetroids

A *cubic symmetroid* is a hypersurface in $\mathbb{P}^n$ admitting a representation as a symmetric (3×3)-determinant whose entries are linear forms in $n+1$ variables. Here we will be interested in cubic symmetroid surfaces. An example of a cubic symmetroid is the Cayley 4-nodal cubic surface

$$t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3 = \det \begin{pmatrix} t_0+t_3 & t_3 & t_3 \\ t_3 & t_1+t_3 & t_3 \\ t_3 & t_3 & t_2+t_3 \end{pmatrix},$$

which we have already encounter before. By choosing the singular points to be the reference points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, it is easy to see that cubic surfaces with 4 singularities of type A_1 are projectively isomorphic. Since the determinantal cubic hypersurface in $\mathbb{P}^5$ is singular along a surface, a nonsingular cubic surface does not admit a symmetric determinantal representation.

Lemma 9.3.8 *Let $L \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ be a pencil of conics. Then it is projectively isomorphic to one of the following pencils:*

(i) $\lambda(t_0t_1 - t_0t_2) + \mu(t_1t_2 - t_0t_2) = 0$;
(ii) $\lambda(t_0t_1 + t_0t_2) + \mu t_1t_2 = 0$;
(iii) $\lambda(t_0t_1 + t_2^2) + \mu t_0t_2 = 0$;
(iv) $\lambda t_2^2 + \mu t_0t_1 = 0$;
(v) $\lambda t_0^2 + \mu(t_0t_2 + t_1^2) = 0$;
(vi) $\lambda t_0^2 + \mu t_1^2 = 0$;
(vii) $\lambda t_0t_1 + \mu t_0t_2 = 0$;
(viii) $\lambda t_0t_1 + \mu t_0^2 = 0$.

Proof The first five cases correspond to the Segre symbols $[1,1,1]$, $[(2)1]$, $[(3)]$, $[(11)1]$, $[(12)]$, respectively. For the future use, we chose different bases. The last three cases correspond to pencils of singular conics. □

Theorem 9.3.9 *A cubic symmetroid is a del Pezzo surface if and only if it is projectively isomorphic to one of the following determinantal surfaces:*

(i) $\mathcal{C}_3 = V(t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3)$ *with four RDP of type* A_1;
(ii) $\mathcal{C}_3' = V(t_0t_1t_2 + t_1t_3^2 - t_2t_3^2)$ *with two RDP of type* A_1 *and one RDP of type* A_3;
(iii) $\mathcal{C}_3'' = V(t_0t_1t_2 - t_3^2(t_0+t_2) - t_1t_2^2)$ *with one RDP of type* A_1 *and one RDP of type* A_5.

Proof Let $A = (l_{ij})$ be a symmetric 3×3 matrix with linear entries l_{ij} defining the equation of S. It can be written in the form $A(t) = t_0A_0 + t_1A_1 + t_2A_2 + t_3A_3$, where $A_i, i = 1, 2, 3, 4$, are symmetric 3×3 matrices. Let W be a linear system of conics spanned by the conics

$$C_i = [t_0, t_1, t_2] \cdot A \cdot \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = 0.$$

Each web of conics is apolar to a unique pencil of conics. Using the previous Lemma, we find the following possibilities. We list convenient bases in corresponding dual 4-dimensional spaces of quadratic forms.

(i) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2(\xi_0\xi_1 + \xi_1\xi_2 + \xi_0\xi_2)$;
(ii) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2(\xi_0\xi_1 - \xi_0\xi_2)$;
(iii) $\xi_0^2,\ \xi_1^2,\ 2\xi_0\xi_1 - \xi_2^2,\ 2\xi_1\xi_2$;
(iv) $\xi_0^2,\ \xi_1^2,\ 2\xi_0\xi_2,\ 2(\xi_1\xi_2 - \xi_0\xi_1)$;
(v) $2\xi_0\xi_2 - \xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_1,\ 2\xi_1\xi_2$;
(vi) $\xi_2^2,\ 2\xi_0\xi_1,\ 2\xi_1\xi_2,\ 2\xi_0\xi_2$;
(vii) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_1$;
(viii) $\xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_2,\ 2\xi_1\xi_2$.

The corresponding determinantal varieties are the following.

(i)

$$\det \begin{pmatrix} t_0 & t_3 & t_3 \\ t_3 & t_1 & t_3 \\ t_3 & t_3 & t_2 \end{pmatrix} = t_0t_1t_2 + t_3^2(-t_0 - t_2 - t_1 + 2t_3) = 0.$$

It has four singular points $[1, 0, 0, 0],\ (0, 1, 0, 0],\ [0, 0, 1, 0]$, and $[1, 1, 1, 1]$. The surface is a Cayley 4-nodal cubic.

(ii)

$$\det \begin{pmatrix} t_0 & t_3 & -t_3 \\ t_3 & t_1 & 0 \\ -t_3 & 0 & t_2 \end{pmatrix} = t_0t_1t_2 - t_1t_3^2 - t_2t_3^2 = 0.$$

It has two ordinary nodes $[0, 1, 0, 0],\ [0, 0, 1, 0]$ and a RDP $[1, 0, 0, 0]$ of type A_3.

(iii)

$$\det \begin{pmatrix} t_0 & t_2 & 0 \\ t_2 & t_1 & t_3 \\ 0 & t_3 & -t_2 \end{pmatrix} = -t_0t_1t_2 - t_0t_3^2 + t_2^3 = 0.$$

The surface has an ordinary node at $[1,0,0,0]$ and a RDP of type A_5 at $[0,1,0]$.

(iv)

$$\det\begin{pmatrix} t_0 & -t_3 & t_2 \\ -t_3 & t_1 & t_3 \\ t_2 & t_3 & 0 \end{pmatrix} = t_3^2(-t_0 - 2t_2) - t_1 t_2^2 = 0.$$

It has a double line $t_3 = t_2 = 0$.

(v)

$$\det\begin{pmatrix} 0 & t_2 & t_0 \\ t_2 & -t_0 & t_3 \\ t_0 & t_3 & t_1 \end{pmatrix} = -t_1 t_2^2 + 2t_0 t_2 t_3 + t_0^3 = 0.$$

The surface has a double line $t_0 = t_2 = 0$.

(vi)

$$\det\begin{pmatrix} 0 & t_1 & t_3 \\ t_1 & 0 & t_2 \\ t_3 & t_2 & t_0 \end{pmatrix} = -t_0 t_1^2 + 2t_1 t_2 t_3 = 0.$$

The surface is the union of a plane and a nonsingular quadric.

(vii)

$$\det\begin{pmatrix} t_0 & 0 & 0 \\ 0 & t_1 & t_3 \\ 0 & t_3 & t_2 \end{pmatrix} = t_0(t_1 t_2 - t_3^2) = 0.$$

The surface is the union of a plane and a quadratic cone.

(viii)

$$\det\begin{pmatrix} 0 & 0 & t_2 \\ 0 & t_0 & t_3 \\ t_2 & t_3 & t_1 \end{pmatrix} = t_0 t_2^2 = 0.$$

The surface is reducible.

□

Remark 9.3.10 Let S be a cone over a plane cubic curve C. We saw in Example 4.2.18 that any irreducible plane cubic curve admits a symmetric determinantal representation. This gives a symmetric determinantal representation of the cone over the cubic; however, it is not defined by a web of conics. In fact, we see from the list in the above that no irreducible cone is given by a web of conics. I have not seen an a priori proof of this.

If S is irreducible non-normal surface, then S admits a symmetric determinantal representation. This corresponds to cases (iv) and (v) from the proof of

the previous Theorem. Case (iv) (resp. (v)) gives a surface isomorphic to the surface from case (i) (resp. (ii)) of Theorem 9.3.9. We also see that a reducible cubic surface that is not a cone admits a symmetric determinantal representation only if it is the union of an irreducible nonsingular (singular) quadric and its tangent (non-tangent) plane. The plane is a tangent if the quadric is nonsingular; it intersects the quadric transversally.

Remark 9.3.11 The three symmetroid del Pezzo cubic surfaces S can be characterized among all del Pezzo cubics by the property that they admit a double cover $\pi : \bar{S} \to S$ ramified only over the singular points. They can be obtained by a projection of a quadric surface from Example 8.6.6.

9.4 Representations as sums of cubes

9.4.1 Sylvester's pentahedron

Counting constants, we see that it is possible that a general homogeneous cubic form in four variables can be written as a sum of five cubes of linear forms in finitely many ways. Since there are no cubic surfaces singular at five general points, the theory of apolarity tells us that the count of constants gives a correct answer. The following result of J. Sylvester gives more.

Theorem 9.4.1 *A general homogeneous cubic form f in four variables can be written as a sum:*

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3 + l_5^3, \tag{9.30}$$

where l_i are linear forms in four variables, no two are proportional. The forms are defined uniquely, up to scaling by a cubic root of unity.

Proof The variety of cubic forms $f \in S^3(E^\vee)$ represented as a sum of five cubes lies in the image of the dominant map of 20-dimensional spaces $(E^\vee)^5 \to S^3(E^\vee)$. The subvariety of $(E^\vee)^5$ that consists of 5-tuples of linear forms containing four linearly dependent forms is a hypersurface. Thus, we may assume that in a representation of f as a sum of five cubes of linear forms, any set of four linear forms are linearly independent.

Suppose

$$f = \sum_{i=1}^{5} l_i^3 = \sum_{i=1}^{5} m_i^3.$$

Let x_i, y_i be the points in the dual space $(\mathbb{P}^3)^\vee$ corresponding to the hyperplanes $V(l_i), V(m_i)$. The first five and the last five are distinct points. Consider the linear system of quadrics in $(\mathbb{P}^3)^\vee$ which pass through the points x_5,

$y_1, \dots, y_5$. Its dimension is larger than or equal 3. Choose a web $|W|$ contained in this linear system. Applying the corresponding differential operator to f we find a linear relations between the linear forms l_1, l_2, l_3, l_4. Since we assumed that they are linearly independent, we obtain that all quadrics in the web contain $x_1, \dots, x_4$. Thus all quadrics in the web pass through x_i, y_j.

Suppose the union of the sets $X = \{x_1, \dots, x_5\}$ and $Y = \{y_1, \dots, y_5\}$ contains nine distinct points. Since three quadrics intersect at ≤ 8 points unless they contain a common curve, the web $|W|$ has a curve B in its base locus. Because an irreducible nondegenerate curve of degree 3 is not contained in the base locus of a web of quadrics, $\deg B \leq 2$. Suppose B contains a line ℓ_0. Since neither X nor Y is contained in a line, we can find a point x_i outside ℓ_0. Consider a plane Π spanned by ℓ_0 and x_i. The restriction of quadrics to Π is a pencil of conics with fixed line ℓ_0 and the base point x_i. This implies that $|W|$ contains a pencil of quadrics of the form $\Pi \cup \Pi'$, where Π' belongs to a pencil of planes containing a line ℓ passing through x_i. Since $X \cup Y$ is contained in the base locus of any pencil in $|W|$, we see that $X \cup Y \subset \Pi \cup \ell$. Now we change the point x_i to some other point y_j not in Π. We find that $X \cup Y$ is contained in $\Pi' \cup \ell'$. Hence the set is contained in $(\Pi \cup \ell) \cap (\Pi' \cup \ell')$. It is the union of ℓ_0 and a set Z consisting of either the line $\overline{x_i y_j}$ or a set of ≤ 3 points. This implies that one of the sets X and Y has four points on ℓ_0. Then X or Y spans a plane, a contradiction.

Suppose B is a conic. Then we restrict $|W|$ to the plane Π it spans, and obtain that $|W|$ contains a net of quadrics of the form $\Pi \cup \Pi'$, where Π' is a net of planes. This implies that $X \cup Y$ is contained in $\Pi \cup Z$, where Z is either empty or consists of one point. Again this leads to contradiction with the assumption on linear independence of the points.

We may now assume that $m_5 = \lambda_5 \ell_5, m_4 = \lambda_4 l_4$, for some nonzero constants λ_4, λ_5, and get

$$\sum_{i=1}^{3} l_i^3 + (1 - \lambda_4^3) l_4^3 + (1 - \lambda_5^3) l_5^3 = \sum_{i=1}^{3} m_i^3.$$

Take the linear differential operator of the second order corresponding to the double plane containing the points y_1, y_2, y_3. It gives a linear relation between $l_1, \dots, l_5$ which must be trivial. Since the points $y_1, y_2, y_3, y_4 = x_4$ and $y_1, y_2, y_3, y_5 = x_5$ are not coplanar, we obtain that $\lambda_4^3 = \lambda_5^3 = 1$. Taking the differential operator of the first order corresponding to the plane through y_1, y_2, y_3, we obtain a linear relation between the quadratic forms l_1^2, l_2^2, l_3^2. Since no two of l_1, l_2, l_3 are proportional, this is impossible. Thus all the coef-

ficients in the linear relation are equal to zero, hence $x_1, x_2, x_3, y_1, y_2, y_3$ are coplanar.

The linear system of quadrics through $y_1, \dots, y_5$ is 4-dimensional. By an argument from above, each quadric in the linear system contains x_1, x_2, x_3 in its base locus. Since $x_1, x_2, x_3, y_1, y_2, y_3$ lie in a plane Π, and no three points x_i's or y_j's are collinear, the restriction of the linear system to the plane is a fixed conic containing the six points. This shows that the dimension of the linear system is less than or equal to 3. This contradiction shows that the sets $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}$ have two points in common. Thus, we can write

$$l_1^3 + (1 - \lambda_2^3)l_2^3 + (1 - \lambda_3^3)l_3^3 = m_1^3.$$

The common point of the planes $V(l_1), V(l_2), V(l_3)$ lies on $V(m_1)$. After projecting from this point, we obtain that the equation of a triple line can be written as a sum of cubes of three linearly independent linear forms. This is obviously impossible. So, we get $\lambda_2^3 = \lambda_3^3 = 1$, hence $m_1 = \lambda_1 l_1$, where $\lambda_1^3 = 1$. So, all cubes λ_i^3 are equal to 1.

□

Corollary 9.4.2 *A general cubic surface is projectively isomorphic to a surface in $\mathbb{P}^4$ given by equations*

$$\sum_{i=0}^{4} a_i z_i^3 = \sum_{i=0}^{4} z_i = 0. \tag{9.31}$$

The coefficients $(a_0, \dots, a_4)$ are determined uniquely up to permutation and a common scaling.

Proof Let $S = V(f)$ be a cubic surface given by Equation (9.31). Let $b_0 l_1 + \cdots + b_4 l_5 = 0$ be a unique, up to proportionality, linear relation. Consider the embedding of $\mathbb{P}^3$ into $\mathbb{P}^4$ given by the formula

$$[y_0, \dots, y_4] = [l_1(t_0, \dots, t_3), \dots, l_5(t_0, \dots, t_3)].$$

Then the image of S is equal to the intersection of the cubic hypersurface $V(\sum y_i^3)$ with the hyperplane $V(\sum b_i y_i)$. Now make the change of coordinates $z_i = b_i y_i$, if $b_i \neq 0$ and $z_i = y_i$ otherwise. In the new coordinates, we get Equation (9.31), where $a_i = b_i^3$. The Sylvester presentation is unique, up to permutation of the linear functions l_i, multiplication l_i by third roots of 1, and a common scaling. It is clear that the coefficients $(a_0, \dots, a_4)$ are determined uniquely up to permutation and common scaling. □

We refer to equations (9.31) as *Sylvester equations* of a cubic surface.

Recall from Subsection 6.3.5, that a cubic surface $V(f)$ is called Sylvester

nondegenerate if it admits Equation (9.30), where any four linear forms are linearly independent.

It is clear that in this case the coefficients $a_1, \dots, a_5$ are all nonzero.

If four of the linear forms in (9.30) are linearly dependent, after a linear change of variables, we may assume that $l_1 = t_0, l_2 = t_1, l_3 = t_2, l_4 = t_3, l_5 = at_0 + bt_1 + ct_2$. The equation becomes

$$f = t_3^3 + g(t_0, t_1, t_2), \tag{9.32}$$

where g_3 is a ternary cubic form. We called such surfaces $V(f)$ cyclic.

Remark 9.4.3 Suppose a cubic surface $V(f)$ admits Sylvester equations. Then any net of polar quadrics admits a common polar pentahedron. The condition that a net of quadrics admits a common polar pentahedron is given by the vanishing of the Toeplitz invariant Λ from (1.64). Using this fact, Toeplitz gave another proof of the existence of Sylvester pentahedron for a general cubic surface [564].

9.4.2 The Hessian surface

Suppose S is given by the Sylvester equations (9.31). Let us find the equation of its Hessian surface. Recall that this is the locus of points whose polar quadric is singular. For our surface S in the plane $H = V(\sum z_i) \subset \mathbb{P}^4$ this means that this is the locus of points $z = [\alpha_0, \dots, \alpha_4] \in H$ with $\sum \alpha_i = 0$ such that the polar quadric is tangent to H at some point. The equation of the polar quadric is $\sum \alpha_i a_i z_i^2 = 0$.

It is tangent to H if the point $[1, \dots, 1]$ lies in the dual quadric $\sum \frac{1}{\alpha_i a_i} u_i^2 = 0$. Here we omit the term with $a_i = 0$. Thus, the equation of the Hessian surface is

$$\sum_{i=0}^{4} \frac{1}{z_i a_i} = 0, \quad \sum z_i = 0,$$

where we have to reduce to the common denominator to get an equation of a quartic hypersurface. If all $a_i \neq 0$, we get the equation

$$z_0 \cdots z_5 \Big(\sum_{i=0}^{4} \frac{A_i}{z_i}\Big) = \sum_{i=0}^{4} z_i = 0, \tag{9.33}$$

where $A_i = (a_0 \cdots a_5)/a_i$. If some coefficients a_i are equal to zero, say $a_0 = \dots = a_k = 0$, the Hessian surface becomes the union of planes $V(z_i) \cap V(\sum z_i), i = 0, \dots, k$, and a surface of degree $3 - k$.

Assume that $S = V(f)$ is Sylvester nondegenerate, so the Hessian surface $\mathrm{He}(S)$ is irreducible. The 10 lines

$$\ell_{ij} = V(z_i) \cap V(z_j) \cap V(\sum z_i)$$

are contained in $\mathrm{He}(S)$. The 10 points

$$p_{ijk} = V(z_i) \cap V(z_j) \cap V(z_j) \cap V(\sum z_i)$$

are singular points of $\mathrm{He}(S)$.

The union of the planes $V(z_i) \cap V(\sum z_i)$ is called the *Sylvester pentahedron*, the lines ℓ_{ij} are its *edges*, the points p_{ijk} are its *vertices*.

Remark 9.4.4 Recall that the Hessian of any cubic hypersurface admits a birational automorphism σ which assigns to the polar quadric of corank 1 its singular point. Let X be a minimal nonsingular model of $\mathrm{He}(S)$. It is a K3 surface. The birational automorphism σ extends to a biregular automorphism of X. It exchanges the proper inverse transforms of the edges with the exceptional curves of the resolution. One can show that for a general S, the automorphism of X has no fixed points, and hence the quotient is an Enriques surface.

We know that a cubic surface admitting a degenerate Sylvester equation must be a cyclic surface. Its Hessian is the union of a plane and the cone over a cubic curve. A cubic form may not admit a polar pentahedral, so its equation may not be written as a sum of powers of linear forms. For example, consider a cubic surface given by equation

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 + 3t_3^2(at_0 + bt_1 + ct_2) = 0.$$

For a general choice of the coefficients, the surface is nonsingular and noncyclic. Its Hessian has the equation

$$t_0t_1t_2t_3 + t_0t_1t_2(at_0 + bt_1 + ct_2) - t_3^2(a^2t_1t_2 + b^2t_0t_2 + c^2t_0t_1) = 0.$$

It is an irreducible surface with an ordinary node at $[0, 0, 0, 1]$ and singular points $[0, 0, 1, 0], [0, 1, 0, 0], [1, 0, 0, 0]$ of type A_3. So we see that the surface cannot be Sylvester nondegenerate. The surface does not admit a polar pentahedral, it admits a generalized polar pentahedral in which two of the planes coincide. We refer to [472] and [151] for more examples of cubic surfaces with degenerate Hessian.

Proposition 9.4.5 *A cubic surface given by a nondegenerate Sylvester Equation* (9.31) *is nonsingular if and only if, for all choices of signs,*

$$\sum_{i=0}^{4} \pm \frac{1}{\sqrt{a_i}} \neq 0. \tag{9.34}$$

Proof The surface is singular at a point $(z_0, \dots, z_4)$ if and only if

$$\operatorname{rank}\begin{pmatrix} a_0z_0^2 & a_1z_1^2 & a_2z_2^2 & a_3z_3^2 & a_4z_4^2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1.$$

This gives $a_iz_i^2 = c, i = 0, \dots, 3$, for some $c \neq 0$. Thus $z_i = \pm c/\sqrt{a_i}$ for some choice of signs, and the equation $\sum z_i = 0$ gives (9.34). Conversely, if (9.34) holds for some choice of signs, then $[\pm\frac{1}{\sqrt{a_0}}, \dots, \pm\frac{1}{\sqrt{a_4}}]$ satisfies $\sum z_i = 0$ and $\sum a_iz_i^3 = 0$. It also satisfies the equations $a_it_i^2 = a_jt_j^2$. Thus it is a singular point. □

9.4.3 Cremona's hexahedral equations

The Sylvester Theorem has the deficiency that it cannot be applied to any nonsingular cubic surface. The Cremona's hexahedral equations that we consider here work for any nonsingular cubic surface.

Theorem 9.4.6 (L. Cremona) *Assume that a cubic surface S is not a cone and admits a Cayley-Salmon equation (e.g. S is a nonsingular surface). Then S is isomorphic to a cubic surface in $\mathbb{P}^5$ given by the equations*

$$\sum_{i=0}^{5} t_i^3 = \sum_{i=0}^{5} t_i = \sum_{i=0}^{5} a_it_i = 0. \tag{9.35}$$

Proof Let $S = V(l_1l_2l_3 + m_1m_2m_3)$ be a Cayley-Salmon equation of S. Let us try to find some constants such that the linear forms, after scaling, add up to zero. Write

$$l_i' = \lambda_i l_i, \quad m_i' = \mu_i m_i, \ i = 1, 2, 3.$$

Since S is not a cone, four of the linear forms are linearly independent. After reordering the linear forms, we may assume that the linear forms l_1, l_2, l_3, m_1 are linearly independent. Let

$$m_2 = al_1 + bl_2 + cl_3 + dm_1, \ m_3 = a'l_1 + b'l_2 + c'l_3 + d'l_4.$$

The constants λ_i, μ_i must satisfy the following system of equations

$$\begin{aligned} \lambda_1 + a\mu_2 + a'\mu_3 &= 0, \\ \lambda_2 + b\mu_2 + b'\mu_3 &= 0, \\ \lambda_3 + c\mu_2 + c'\mu_3 &= 0, \\ \mu_1 + d\mu_2 + d'\mu_3 &= 0, \\ \lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3 &= 0. \end{aligned}$$

The first four linear equations allow us to express linearly all unknowns in terms of μ_2, μ_3. Plugging in the last equation, we get a cubic equation in μ_2/μ_3. Solving it, we get a solution. Now set

$$z_1 = l_2' + l_3' - l_1', \quad z_2 = l_3' + l_1' - l_2', \quad z_3 = l_1' + l_2' - l_3',$$

$$z_4 = \mu_2' + \mu_3' - \mu_1', \quad z_5 = \mu_3' + \mu_1' - \mu_2', \quad z_6 = \mu_1' + \mu_2' - \mu_3'.$$

One checks that these six linear forms satisfy the equations from the assertion of the Theorem. □

Equations (9.35) of a del Pezzo cubic surface are called *Cremona's hexahedral equations*.

Corollary 9.4.7 (T. Reye) *A general homogeneous cubic form f in four variables can be written as a sum of six cubes in ∞^4 different ways. In other words,*

$$\dim \mathrm{VSP}(f, 6)^o = 4.$$

Proof This follows from the proof of the previous theorem. Consider the map

$$(\mathbb{C}^4)^6 \to \mathbb{C}^{20}, \quad (l_1, \dots, l_6) \mapsto l_1^3 + \cdots + l_6^3.$$

It is enough to show that it is dominant. We show that the image contains the open subset of nonsingular cubic surfaces. In fact, we can use a Cayley-Salmon equation $l_1 l_2 l_3 + m_1 m_2 m_3$ for $S = V(f)$ and apply the proof of the Theorem to obtain that, up to a constant factor,

$$f = z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3.$$

□

Suppose a nonsingular S is given by equations (9.35). They allow us to locate 15 lines on S such that the remaining lines form a double-six. The equations of these lines in $\mathbb{P}^5$ are

$$z_i + z_j = 0,\ z_k + z_l = 0,\ z_m + z_n = 0,\ \sum_{i=1}^{6} a_i z_i = 0,$$

where $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$. Let us denote the line given by the above equations by $l_{ij,kl,mn}$.

Let us identify a pair a, b of distinct elements in $\{1, 2, 3, 4, 5, 6\}$ with a transposition (ab) in $\mathfrak{S}_6$. We have the product $(ij)(kl)(mn)$ of three commuting transpositions corresponding to each line $l_{ij,kl,mn}$. The group $\mathfrak{S}_6$ admits a unique (up to a composition with a conjugation) outer automorphism

which sends each transposition to the product of three commuting transpositions. In this way we can match lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab} of the $\mathbf{E}_6$-lattice. To do it explicitly, one groups together five products of three commuting transpositions in such a way that they do not contain a common transposition. Such a set is called a *total* and the triples (ij, kl, mn) are called *synthemes*. Here is the set of six totals

$$\begin{aligned}
T_1 &= (12)(36)(45),\ (13)(24)(56),\ (14)(26)(35),\ (15)(23)(46),\ (16)(25)(34), \\
T_2 &= (12)(36)(45),\ (13)(25)(46),\ (14)(23)(56),\ (15)(26)(34),\ (16)(24)(35), \\
T_3 &= (12)(35)(46),\ (13)(24)(56),\ (14)(25)(36),\ (15)(26)(34),\ (16)(23)(45), \\
T_4 &= (12)(34)(56),\ (13)(25)(46),\ (14)(26)(35),\ (15)(24)(36),\ (16)(23)(45), \\
T_5 &= (12)(34)(56),\ (13)(26)(45),\ (14)(25)(36),\ (15)(23)(46),\ (15)(24)(35), \\
T_6 &= (12)(35)(46),\ (13)(26)(45),\ (14)(23)(56),\ (15)(24)(36),\ (16)(25)(34).
\end{aligned} \tag{9.36}$$

Two different totals T_a, T_b contain one common product $(ij)(kl)(mn)$. The correspondence $(a, b) \mapsto (ij)(kl)(mn)$ defines the outer automorphism

$$\alpha : \mathfrak{S}_6 \to \mathfrak{S}_6. \tag{9.37}$$

For example, $\alpha((12)) = (12)(36)(45)$ and $\alpha((23)) = (15)(26)(34)$.

After we matched the lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab}, we check that this matching defines an isomorphism of the incidence subgraph of the lines with the subgraph of the incidence graph of 27 lines on a cubic surface whose vertices correspond to exceptional vectors c_{ab} .

Theorem 9.4.8 *Cremona's hexahedral equations of a nonsingular cubic surface S defines an ordered double-six of lines. Conversely, a choice of an ordered double-six defines uniquely Cremona hexahedral equations of S.*

Proof We have seen already the first assertion of the theorem. If two surfaces given by hexahedral equations define the same double-six, then they have in common 15 lines. Obviously, this is impossible. Thus the number of different hexahedral equations of S is less than or equal to 36. Now consider the identity

$$\begin{aligned}
&(z_1+\cdots+z_6)\big((z_1+z_2+z_3)^2+(z_4+z_5+z_6)^2-(z_1+z_2+z_3)(z_4+z_5+z_6)\big) \\
&= (z_1+z_2+z_3)^3+(z_4+z_5+z_6)^3 = z_1^3+\cdots+z_6^3 \\
&+3(z_2+z_3)(z_1+z_3)(z_1+z_2)+3(z_4+z_5)(z_5+z_6)(z_4+z_6).
\end{aligned}$$

It shows that Cremona hexahedral equations define a Cayley-Salmon equation

$$(z_2+z_3)(z_1+z_3)(z_1+z_2)+(z_4+z_5)(z_5+z_6)(z_4+z_6)=0,$$

where we have to eliminate one unknown with help of the equation $\sum a_i z_i =$

0. Applying permutations of $z_1, \dots, z_6$, we get 10 Cayley-Salmon equations of S. Each set of nine lines formed by the corresponding conjugate pair of triads of tritangent planes are among the 15 lines determined by the hexahedral equation. It follows from the classification of the conjugate pairs that we have 10 such pairs of lines c_{ij}'s (type II). Thus a choice of Cremona hexahedral equations defines exactly 10 Cayley-Salmon equations of S. Conversely, it follows from the proof of Theorem 9.4.6 that each Cayley-Salmon equation gives three Cremona hexahedral equations (unless the cubic equation has a multiple root). Since we have 120 Cayley-Salmon equations for S we get $36 = 360/10$ hexahedral equations for S. They match with 36 double-sixes. □

9.4.4 The Segre cubic primal

Let $p_1, \dots, p_m$ be a set of points in $\mathbb{P}^n$, where $m > n+1$. For any ordered subset $(p_{i_1}, \dots, p_{i_{n+1}})$ of $n+1$ points we denote by $(i_1 \dots i_{n+1})$ the determinant of the matrix whose rows are projective coordinates of the points $(p_{i_1}, \dots, p_{i_{n+1}})$ in this order. We consider $(i_1 \dots i_{n+1})$ as a section of the invertible sheaf $\otimes_{j=1}^{n+1} p_{i_j}^* \mathcal{O}_{\mathbb{P}^n}(1)$ on $(\mathbb{P}^n)^m$. It is called a *bracket-function*. A monomial in bracket-functions such that each index $i \in \{1, \dots, m\}$ occurs exactly d times defines a section of the invertible sheaf

$$\mathcal{L}_d = \bigotimes_{i=1}^{n} p_i^* \mathcal{O}_{\mathbb{P}^n}(d).$$

According to the Fundamental Theorem of Invariant Theory (see [182]) the subspace $(R_n^m)(d)$ of $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)$ generated by such monomials is equal to the space of invariants $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)^{\mathrm{SL}(n+1}$, where the group $\mathrm{SL}(n+1)$ acts linearly on the space of sections via its diagonal action on $(\mathbb{P}^n)^m$. The graded ring

$$R_n^m = \bigoplus_{d=0}^{\infty} (R_n^m)(d) \tag{9.38}$$

is a finitely generated algebra. Its projective spectrum is isomorphic to the GIT-quotient

$$P_n^m := (\mathbb{P}^n)^m // \mathrm{SL}(n+1)$$

of $(\mathbb{P}^n)^m$ by $\mathrm{SL}(n+1)$. The complement U^{ss} of the set of common zeros of generators of the algebra R_n^m admits a regular map to P_n^m. The set U^{ss} does not depend on the choice of generators. Its points are called *semi-stable*. Let U^{s} be the largest open subset such that the fibres of the restriction map $U^{\mathrm{s}} \to P_n^m$ are orbits. Its points are called *stable*.

It follows from the Hilbert-Mumford numerical stability criterion that a points set $(p_1, \dots, p_m)$ in $\mathbb{P}^1$ is semi-stable (resp. stable) if and only if at most $\frac{1}{2}m$ (resp. $< \frac{1}{2}m$) points coincide. We have already seen the definition of the bracket-functions in the case $m = 4$. They define the cross ratio of four points

$$[p_1, p_2, p_3, p_4] = \frac{(12)(34)}{(13)(24)}.$$

The cross ratio defines the rational map $(\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$. It is defined on the open set U^s of points where no more than two coincide and it is an orbit space over the complement of three points $0, 1, \infty$.

In the case of points in $\mathbb{P}^2$ the condition of stability (semi-stability) is that at most $\frac{1}{3}m$ (resp. $< \frac{1}{3}m$) coincide and at most $\frac{2}{3}m$ (resp. $< \frac{2}{3}m$) points are on a line.

Proposition 9.4.9 *Let $\mathcal{P} = (p_1, \cdots, p_6)$ be an ordered set of distinct points in $\mathbb{P}^1$. The following conditions are equivalent.*

(i) There exists an involution of $\mathbb{P}^1$ such that the pairs $(p_1, p_2), (p_3, p_4), (p_5, p_6)$ are orbits of the involution.

(ii) The binary forms $g_i, i = 1, 2, 3$, with zeros $(p_1, p_2), (p_3, p_4), (p_5, p_6)$ are linearly dependent.

(iii) Let x_i be the image of p_i under a Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$. Then the lines $\overline{x_1x_2}, \overline{x_3x_4}, \overline{x_5x_6}$ are concurrent.

(iv) The bracket-function $(14)(36)(25) - (16)(23)(54)$ vanishes at $\mathcal{P}$.

Proof (i) $\Leftrightarrow$ (ii) Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 2 map defined by the involution. Let f be given by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$, where g_1, g_2 are binary forms of degree 2. By choosing coordinates in the target space, we may assume that $f(p_1) = f(p_2) = 0, f(p_3) = f(p_4) = 1, f(p_5) = f(p_6) = \infty$, i.e. $g_1(p_1) = g_1(p_2) = 0, g_2(p_3) = g_2(p_4) = 0, (g_1 - g_2)(p_5) = (g_1 - g_2)(p_6) = 0$. Obviously, the binary forms $g_1, g_2, g_3 = g_1 - g_2$ are linearly dependent. Conversely, suppose g_1, g_2, g_3 are linearly dependent. By scaling, we may assume that $g_3 = g_1 - g_2$. We define the involution by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$.

(ii) $\Leftrightarrow$ (iii) Without loss of generality, we may assume that $p_i = [1, a_i]$ and $g_1 = t_1^2 - (a_1 + a_2)t_0t_1 + a_1a_2t_0^2, g_2 = t_1^2 - (a_3 + a_4)t_0t_1 + a_3a_4t_0^2, g_3 = t_1^2 - (a_5 + a_6)t_0t_1 + a_5a_6t_0^2$. The condition that the binary forms are linearly dependent is

$$D = \det \begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} = 0. \tag{9.39}$$

The image of p_i under the Veronese map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$ is the point $x_i = [1, a_i, a_i^2]$. The line $\overline{x_ix_j}$ has equation

$$\det \begin{pmatrix} t_0 & t_1 & t_2 \\ 1 & a_i & a_i^2 \\ 1 & a_j & a_j^2 \end{pmatrix} = (a_j - a_i)(a_ia_jt_0 - (a_i + a_j)t_1 + t_2) = 0.$$

Obviously, the three lines are concurrent if and only if (9.39) is satisfied.

(iii) $\Leftrightarrow$ (iv) We have

$$\begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} \cdot \begin{pmatrix} a_1^2 & a_3^2 & a_5^2 \\ -a_1 & -a_3 & -a_5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (a_3 - a_1)(a_3 - a_2) & (a_5 - a_1)(a_5 - a_2) \\ (a_1 - a_3)(a_1 - a_4) & 0 & (a_5 - a_3)(a_5 - a_4) \\ (a_1 - a_5)(a_1 - a_6) & (a_3 - a_5)(a_3 - a_6) & 0 \end{pmatrix}.$$

Taking the determinant, we obtain

$$D(a_1 - a_3)(a_1 - a_5)(a_3 - a_5) =$$

$$= \det \begin{pmatrix} 0 & (a_3 - a_1)(a_3 - a_2) & (a_5 - a_1)(a_5 - a_2) \\ (a_1 - a_3)(a_1 - a_4) & 0 & (a_5 - a_3)(a_5 - a_4) \\ (a_1 - a_5)(a_1 - a_6) & (a_3 - a_5)(a_3 - a_6) & 0 \end{pmatrix} =$$

$$= (a_3-a_5)(a_5-a_1)(a_1-a_3)[(a_1-a_4)(a_3-a_6)(a_5-a_2)+(a_6-a_1)(a_2-a_3)(a_4-a_5)].$$

Since the points are distinct, canceling by the product $(a_3-a_5)(a_5-a_1)(a_1-a_3)$, we obtain

$$(a_1 - a_4)(a_3 - a_6)(a_5 - a_2) + (a_6 - a_1)(a_2 - a_3)(a_4 - a_5) =$$

$$= (14)(36)(25) - (16)(23)(54) = 0.$$

□

We let

$$[ij, kl, mn] := (il)(kn)(jm) - (jk)(lm)(ni). \tag{9.40}$$

For example, $[12, 34, 56] = (14)(36)(25) - (16)(23)(54)$. Note that determinant (9.39) does not change if we permute $(a_i, a_{i+1}), i = 1, 3, 5$. It also does not change if we apply an even permutation of the pairs, and changes the sign if we apply an odd permutation.

Let us identify the set $(1, 2, 3, 4, 5, 6)$ with points $(\infty, 0, 1, 2, 3, 4, 5)$ of the

projective line $\mathbb{P}^1(\mathbb{F}_5)$. The group $\mathrm{PSL}(2,\mathbb{F}_5) \cong \mathfrak{A}_5$ acts on $\mathbb{P}^1(\mathbb{F}_5)$ via Moebius transformations $z \mapsto \frac{az+b}{cz+d}$. Let $u_0 = [\infty 0, 14, 23]$ and let $u_i, i = 1, \dots, 4$, be obtained from u_0 via the action of the transformation $z \mapsto z + i$. Let

$$U_1 := u_0 + u_1 + u_2 + u_3 + u_4$$

$$= \big([\infty 0, 14, 23] + [\infty 1, 20, 34] + [\infty 2, 31, 40] + [\infty 3, 42, 01] + [\infty 4, 03, 12]\big).$$

Obviously, U_1 is invariant under the subgroup of order 5 generated by the transformation $z \mapsto z+1$. It is also invariant under the transformation $\tau : z \mapsto -1/z$. It is well known that $\mathfrak{A}_5$ is generated by these two transformations. The orbit of U_∞ under the group $\mathfrak{A}_6$ acting by permutations of $\infty, 0, \dots, 4$ consists of six functions $U_1, U_2, U_3, U_4, U_5, U_6$. We will rewrite them now returning to our old notation of indices by the set $(1, 2, 3, 4, 5, 6)$.

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{pmatrix} = \begin{pmatrix} 0 & [12,36,45] & [13,24,56] & [14,35,26] & [15,46,23] & [16,25,34] \\ & 0 & [15,26,34] & [13,46,25] & [16,35,24] & [14,23,56] \\ & & 0 & [16,23,45] & [14,25,36] & [12,35,46] \\ & & & 0 & [12,34,56] & [15,36,24] \\ & & & & 0 & [13,45,26] \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{9.41}$$

where the matrix is skew-symmetric. We immediately observe that

$$U_1 + U_2 + U_3 + U_4 + U_5 + U_6 = 0. \tag{9.42}$$

Next observe that the triples of pairs $[ij, kl, mn]$ in each row of the matrix constitute a total from (9.36). One easily computes the action of $\mathfrak{S}_6$ on U_i's. For example,

$$(12) : (U_1, U_2, U_3, U_4, U_5, U_6) \mapsto (-U_2, -U_1, -U_6, -U_5, -U_4, -U_3).$$

Its trace is equal to 1.

Recall that there are four isomorphism classes of irreducible 5-dimensional linear representations of the permutation group $\mathfrak{S}_6$. They differ by the trace of a transposition (ij).

If the trace is equal to 3, the representation is isomorphic to the standard representation V_{st} in the space

$$V = \{(z_1, \dots, z_6) \in \mathbb{C}^6 : z_1 + \cdots + z_6 = 0\}.$$

It coincides with the action of the Weyl group $W(A_6)$ on the root lattice A_6. It corresponds to the partition $(5, 1)$ of 6.

If the trace is equal to -3, the representation is isomorphic to the tensor product of the standard representation and the 1-dimensional sign representation. It corresponds to the dual partition $(2, 1, 1, 1, 1)$.

If the trace is equal to 1, the representation is isomorphic to the composition of the outer automorphism $\alpha : \mathfrak{S}_6 \to \mathfrak{S}_6$ and the standard representation. It corresponds to the partition $(3,3)$.

If the trace is equal to -1, the representation is isomorphic to the tensor product of the previous representation and the sign representation. It corresponds to the partition $(2,2,2)$.

So, our representation on the linear space $V = (R_1^6)(1)$ associated with the partition $(3,3)$.

One checks that the involution $(12)(34)(56)$ acts as

$$(U_1, U_2, U_3, U_4, U_5, U_6) \mapsto (-U_1, -U_2, -U_3, -U_5, -U_4, -U_6). \quad (9.43)$$

Its trace is equal to -3. A well-known formula from the theory of linear representations

$$\dim V^G = \frac{1}{\#G} \sum_{g \in G} \text{Trace}(g)$$

shows that the dimension of the invariant subspace for the element $(12)(34)(56)$ is equal to 1. It follows from (9.43) that the function $U_4 - U_5$ is invariant. On the other hand, we also know that the function $[12, 34, 56]$ is invariant too. This gives $U_4 - U_5 = c[12, 36, 54]$ for some scalar c. Evaluating these functions on a point set $(p_1, \ldots, p_6)$ with $p_1 = p_2, p_3 = p_6, p_4 = p_5$ we find that $c = 6$. Now applying permutations we obtain:

$$\begin{aligned}
&U_1 - U_2 = 6[12,36,45],\ U_1 - U_3 = 6[13,24,56],\ U_1 - U_4 = 6[14,35,26], \quad (9.44)\\
&U_1 - U_5 = 6[15,46,23],\ U_1 - U_6 = 6[16,25,34],\ U_2 - U_3 = 6[15,26,34],\\
&U_2 - U_4 = 6[13,46,25],\ U_2 - U_5 = 6[16,35,24],\ U_2 - U_6 = 6[14,23,56],\\
&U_3 - U_4 = 6[16,45,23],\ U_3 - U_5 = 6[14,25,36],\ U_3 - U_6 = 6[12,46,35],\\
&U_4 - U_5 = 6[12,43,56],\ U_4 - U_6 = 6[15,36,24],\ U_5 - U_6 = 6[13,45,26].
\end{aligned}$$

Similarly, we find that $U_1 + U_2$ is the only anti-invariant function under σ and hence coincides with $c(12)(36)(45)$. After evaluating the functions at a point set $(p_1, \ldots, p_6)$ with $p_1 = p_3, p_2 = p_4, p_5 = p_6$ we find that $c = 4$. In this way we get the relations:

$$\begin{aligned}
&U_1 + U_2 = 4(12)(36)(45),\ U_1 + U_3 = 4(13)(42)(56),\ U_1 + U_4 = 4(41)(53)(26), \quad (9.45)\\
&U_1 + U_5 = 4(15)(46)(32),\ U_1 + U_6 = 4(16)(25)(34),\ U_2 + U_3 = 4(15)(26)(43),\\
&U_2 + U_4 = 4(13)(46)(25),\ U_2 + U_5 = 4(16)(35)(42),\ U_2 + U_6 = 4(14)(23)(56),\\
&U_3 + U_4 = 4(16)(54)(32),\ U_3 + U_5 = 4(14)(25)(63),\ U_3 + U_6 = 4(12)(46)(53),\\
&U_4 + U_5 = 4(12)(34)(56),\ U_4 + U_6 = 4(15)(36)(24),\ U_5 + U_6 = 4(13)(45)(62).
\end{aligned}$$

By using (9.42), we obtain

$$\begin{aligned}
U_1 &= (12)(36)(45) + (13)(42)(56) + (14)(35)(26) + (15)(46)(32) + (16)(25)(34), \\
U_2 &= (12)(36)(45) + (13)(46)(25) + (14)(56)(23) + (15)(26)(43) + (16)(24)(53), \\
U_3 &= (12)(53)(46) + (13)(42)(56) + (14)(52)(36) + (15)(26)(43) + (16)(23)(45), \\
U_4 &= (12)(34)(56) + (13)(46)(25) + (14)(35)(26) + (15)(24)(36) + (16)(23)(45), \\
U_5 &= (12)(34)(56) + (13)(54)(26) + (14)(52)(36) + (15)(46)(32) + (16)(24)(53), \\
U_6 &= (12)(53)(46) + (13)(54)(26) + (14)(56)(23) + (15)(36)(24) + (16)(25)(34).
\end{aligned} \tag{9.46}$$

We see that our functions are in bijective correspondence with six totals from above. The functions $U_1, \dots, U_6$ are known as the *Joubert functions*.

It is easy to see that the functions U_i do not vanish simultaneously on semi-stable point sets. Thus they define a morphism

$$J : P_1^6 \to \mathbb{P}^5.$$

Theorem 9.4.10 *The morphism J defined by the Joubert functions is an isomorphism onto the subvariety S_3 of $\mathbb{P}^5$ given by the equations*

$$\sum_{i=0}^{5} z_i = \sum_{i=0}^{5} z_i^3 = 0. \tag{9.47}$$

Proof It is known that the graded ring R_1^6 is generated by the following bracket-functions (*standard tableaux*):

$$(12)(34)(56),\ (12)(35)(46),\ (13)(24)(56),\ (13)(25)(46),\ (14)(25)(36)$$

(see [176]). The subspace of $R_1^6(1)$ generated by the Joubert functions is invariant with respect to $\mathfrak{S}_6$. Since $R_1^6(1)$ is an irreducible representation, this implies that the relation $\sum U_i = 0$ spans the linear relations between the Joubert functions. Consider the sum $\Sigma = \sum U_i^3$. Obviously, it is invariant with respect to $\mathfrak{A}_6$. One immediately checks that an odd permutation in $\mathfrak{S}_6$ transforms each sum Σ to $-\Sigma$. This implies that $\Sigma = 0$ whenever two points p_i and p_j coincide. Hence Σ must be divisible by the product of 15 functions (ij). This product is of degree 5 in coordinates of each point but Σ is of degree 3. This implies that $\Sigma = 0$. Since the functions U_i generate the graded ring R_1^6, by definition of the space P_1^6, we obtain an isomorphism from P_1^6 to a closed subvariety of S_3. Since the latter is irreducible and of dimension equal to the dimension of P_1^6, we obtain the assertion of the theorem. □

The cubic threefold S_3 is called the *Segre cubic primal*. We will often consider it as a hypersurface in $\mathbb{P}^4$.

It follows immediately by differentiating that the cubic hypersurface S_3 has 10 double points. They are the points $p = [1, 1, 1, -1, -1, -1]$ and others obtained by permuting the coordinates. A point p is given by the equations

$z_i + z_j = 0, 1 \leq i \leq 3, 4 \leq j \leq 6$. By using (9.42) this implies that p is the image of a point set with $p_1 = p_4 = p_6$ or $p_2 = p_3 = p_5$. Thus the singular points of the Segre cubic primal are the images of semi-stable, but not stable, point sets.

Also S_3 has 15 planes with equations $z_i + z_j = z_k + z_l = z_l + z_m = 0$. Let us see that they are the images of point sets with two points that coincide. Without loss of generality, we may assume that $z_1 + z_2 = z_3 + z_4 = z_5 + z_6 = 0$. Again from (9.42), we obtain that $(12)(36)(45), (16)(23)(45)$ and $(13)(26)(45)$ vanish. This happens if and only if $p_4 = p_5$.

We know that the locus of point sets $(q_1, \ldots, q_6)$ such that the pairs (q_i, q_j), (q_k, q_l), and (q_m, q_n) are orbits of an involution are defined by the equation $[ij, kl, mn] = 0$. By (9.44), we obtain that they are mapped to a hyperplane section of S_3 defined by the equation $z_a - z_b = 0$, where $\alpha((ab)) = (ij)(kl)(mn)$.

It follows from Cremona's hexahedral equations that a nonsingular cubic surface is isomorphic to a hyperplane section of the Segre cubic. In a Theorem below we will make it more precise. But first we need some lemmas.

Lemma 9.4.11 *Let $x_1, \ldots, x_6$ be six points in $\mathbb{P}^2$. Let $\{1, \ldots, 6\} = \{i, j\} \cup \{k, l\} \cup \{m, n\}$. The condition that the lines $\overline{x_i x_j}, \overline{x_k x_l}, \overline{x_m x_n}$ are concurrent is*

$$(ij, kl, mn) := (kli)(mnj) - (mni)(klj) = 0. \tag{9.48}$$

Proof The expression $(kli)(mnx) - (mni)(klx)$ can be considered as a linear function defining a line in $\mathbb{P}^2$. Plugging in $x = x_i$ we see that it passes through the point x_i. Also if x is the intersection point of the lines $\overline{x_k x_l}$ and $\overline{x_m x_n}$, then, writing the coordinates of x as a linear combination of the coordinates of $x_k x_l$, and of $x_m x_n$, we see that the line passes through the point x. Now equation (9.48) expresses the condition that the point x_j lies on the line passing through x_i and the intersection point of the lines $\overline{x_k x_l}$ and $\overline{x_m x_n}$. This proves the assertion. □

The functions (ij, kl, mn) change the sign after permuting two numbers in one pair. They change sign after permuting two pairs of numbers.

It is known (see [176]) that the space $R_2^6(1)$ is generated by bracket-functions $(ijk)(lmn)$. Its dimension is equal to 5 and it has a basis corresponding to standard tableaux

$$(123)(456),\ (124)(356),\ (125)(346),\ (134)(256),\ (135)(246).$$

The group $\mathfrak{S}_6$ acts linearly on this space via permuting the numbers $1, \ldots, 6$.

Let

$$\begin{pmatrix}\bar{U}_1\\ \bar{U}_2\\ \bar{U}_3\\ \bar{U}_4\\ \bar{U}_5\\ \bar{U}_6\end{pmatrix} = \begin{pmatrix} 0 & (12,36,45) & (13,24,56) & (14,26,35) & (15,46,23) & (16,34,25)\\ & 0 & (15,26,34) & (13,25,46) & (16,35,24) & (14,56,23)\\ & & 0 & (16,45,23) & (14,36,25) & (12,46,35)\\ & & & 0 & (12,56,34) & (15,36,24)\\ & & & & 0 & (13,45,26)\\ & & & & & 0 \end{pmatrix} \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}.$$

Equations (9.44) extend to the functions $\bar{U}_i$.

Note that the transposition (12) acts on the functions $\bar{U}$ as

$$(\bar{U}_1, \bar{U}_2, \bar{U}_3, \bar{U}_4, \bar{U}_5, \bar{U}_6) \mapsto (\bar{U}_2, \bar{U}_1, \bar{U}_6, \bar{U}_5, \bar{U}_4, \bar{U}_3).$$

The trace is equal to -1. This shows that the representation $(R_2^6)(1)$ is different from the representation $(R_1^6)(1)$; it is associated to the partition $(2,2,2)$. One checks that the substitution $(12)(34)(56)$ acts by

$$(\bar{U}_1, \bar{U}_2, \bar{U}_3, \bar{U}_4, \bar{U}_5, \bar{U}_6) \mapsto (\bar{U}_1, \bar{U}_2, \bar{U}_3, \bar{U}_5, \bar{U}_4, \bar{U}_6).$$

The trace is equal to 3. This implies that the sign representation enters the representation of the cyclic group $\langle (12)(34)(56)\rangle$ on $(R_2^6)(1)$ with multiplicity 1. Thus the space of anti-invariant elements is one-dimensional. It is spanned by $\bar{U}_4 - \bar{U}_5$. Since the function $(12)(34)(56)$ is anti-invariant, we obtain that $\bar{U}_4 - \bar{U}_5 = c(12)(34)(56)$. Again, as above, we check that $c = 6$. In this way, the equations (9.44) extend to the functions $\bar{U}_i$ with $[ij, kl.mn]$ replaced with $(ij)(kl)(mn)$.

Lemma 9.4.12 *We have the relation*

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = -6(146)(253) \tag{9.49}$$

and similar relations obtained from this one by permuting the set $(1, \ldots, 6)$.

Proof Adding up, we get

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = \big((14,26,35) + (14,56,23) + (14,25,36)\big) + \big((16,34,25)$$

$$+(16,35,24) + (16,45,23)\big) + \big((15,46,23) + (13,25,46) + (12,46,35)\big).$$

Next we obtain

$$(14,26,35) + (14,56,23) + (14,25,36) = (142)(536) - (146)(532) + (146)(523)$$

$$-(143)(526) + (142)(563) - (143)(562) = -2(146)(253),$$

$$(16,34,25) + (16,35,24) + (16,45,23) = (163)(524) - (164)(523) + (165)(243)$$

$$-(163)(245) + (164)(325) - (165)(324) = -2(146)(253),$$

$$(15,46,23) + (13,25,46) + (12,46,35) = (465)(312) - (462)(315) + (463)(152)$$

$$-(461)(153)+(465)(123)-(463)(125) = 2\big((465)(312)-(462)(315)+(463)(152)\big).$$

Now we use the Plücker relation (10.3)

$$(ijk)(lmn) - (ijl)(kmn) + (ijm)(kln) - (ijn)(klm) = 0. \qquad (9.50)$$

It gives

$$(465)(312) - (462)(315) + (463)(152) = -(146)(253).$$

Collecting all of this together, we get the assertion. □

Let $(p_1, \dots, p_6)$ be a fixed ordered set of six points in $\mathbb{P}^2$. Consider the following homogeneous cubic polynomials in coordinates $x = (t_0, t_1, t_2)$ of a point in $\mathbb{P}^2$.

$$\begin{aligned}
F_1 &= (12x)(36x)(45x) + (13x)(42x)(56x) + (14x)(26x)(35x) + (15x)(46x)(32x) + (16x)(34x)(25x),\\
F_2 &= (12x)(36x)(45x) + (13x)(25x)(46x) + (14x)(56x)(23x) + (15x)(26x)(43x) + (16x)(24x)(53x),\\
F_3 &= (12x)(35x)(46x) + (13x)(42x)(56x) + (14x)(52x)(36x) + (15x)(26x)(43x) + (16x)(45x)(23x),\\
F_4 &= (12x)(34x)(56x) + (13x)(46x)(25x) + (14x)(35x)(26x) + (15x)(36x)(24x) + (16x)(23x)(45x),\\
F_5 &= (12x)(34x)(56x) + (13x)(54x)(26x) + (14x)(52x)(36x) + (15x)(46x)(32x) + (16x)(24x)(53x),\\
F_6 &= (12x)(53x)(46x) + (13x)(54x)(26x) + (14x)(56x)(23x) + (15x)(36x)(24x) + (16x)(25x)(34x).
\end{aligned}$$

Theorem 9.4.13 *The rational map*

$$\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5,\ x \mapsto [(F_1(x), \dots, F_6(x)]$$

has the image given by the equations

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 = 0, \qquad (9.51)$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0,$$

$$a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 + a_5 z_5 + a_6 z_6 = 0,$$

where $(a_1, \dots, a_6)$ *are the values of* $(\bar{U}_1, \dots, \bar{U}_6)$ *at the point set* $(p_1, \dots, p_6)$. *They satisfy* $a_1 + \dots + a_6 = 0$.

Proof Take $x = (1, 0, 0)$, then each determinant (ijx) is equal to the determinant (ij) for the projection of $p_1, \dots, p_6$ to $\mathbb{P}^1$. Since all the bracket-functions are invariant with respect to $\mathrm{SL}(3)$ we see that any (ijx) is the bracket-function for the projection of the points to $\mathbb{P}^1$ with center at x. This shows that the relations for the functions U_i imply similar relations for the polynomials F_i. This is an example of Clebsch's transfer principle, which we discussed in Subsection

3.4.2. Let us find the additional relation of the form $\sum_{i=0}^{5} a_i z_i = 0$. Consider the cubic curve

$$C = a_1 F_1(x) + \cdots + a_6 F_6(x) = 0,$$

where $a_1, \ldots, a_6$ are as in the assertion of the theorem. We have already noted that (ij, kl, mn) are transformed by $\mathfrak{S}_6$ in the same way as $(ij)(kl)(mn)$ up to the sign representation. Thus the expression $\sum_i a_i F_i(x)$ is transformed to itself under an even permutation and is transformed to $-\sum_i a_i F_i(x)$ under an odd permutation. Thus the equation of the cubic curve is invariant with respect to the order of the points $p_1, \ldots, p_6$. Obviously, C vanishes at the points p_i. Suppose we prove that C vanishes at the intersection point of the lines $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, then by symmetry it vanishes at the intersection points of all possible pairs of lines, and hence contains five points on each line. Since C is of degree 3 this implies that C vanishes on 15 lines, hence C is identically zero and we are done.

So, let us prove that the polynomial C vanishes at $p = \overline{p_1 p_2} \cap \overline{p_3 p_4}$. Recall from analytic geometry (or multi-linear algebra) that p can be represented by the vector $(v_1 \times v_2) \times (v_3 \times v_4) = (v_1 \wedge v_2 \wedge v_3)v_4 - (v_1 \wedge v_2 \wedge v_4)v_3 = (123)v_4 - (124)v_3$. Thus the value of (ijx) at p is equal to

$$(ijp) = (123)(ij4) - (124)(ij3) = (12)(ij)(34). \tag{9.52}$$

Applying Clebsch's transfer principle to (9.45), we obtain

$$F_1(x) + F_2(x) = 4(12x)(36x)(45x),\ F_4(x) + F_5(x) = 4(12x)(34x)(56x),$$

$$F_1(x) + F_6(x) = 4(16x)(25x)(34x),\ F_3(x) + F_6(x) = 4(12x)(53x)(46x),$$

$$F_2(x) + F_3(x) = (15x)(26x)(43x).$$

This implies that $F_1 + F_2,\ F_4 + F_5,\ F_1 + F_6,\ F_3 + F_6,\ F_2 + F_3$ all vanish at p. Thus the value of C at p is equal to

$$(a_4 - a_5)F_4(p) + (a_2 + a_6 - a_1 - a_3)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_6 + a_5 - a_1 - a_3 - a_4)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_5 + a_6)(F_1(p) + F_3(p)).$$

Here we used that $a_1 + \cdots + a_6 = 0$ and $F_1(p) + F_3(p) + 2F_6(p) = 0$. By Lemma 9.4.12,

$$a_4 - a_5 = (a_4 + a_1 + a_2) - (a_5 + a_1 + a_2) = 6(125)(436) - 6(126)(435) = 6(12, 43, 56).$$

$$a_2 + a_5 + a_6 = 6(346)(125).$$

By using reftrit2 and (9.52), we get

$$F_4(p) + F_6(p) = (51p)(42p)(36p) = (42p)(12,34,15)(12,36,34),$$

$$F_1(p) + F_3(p) = (13p)(42p)(56p) = (42p)(12,56,34)(12,13,34).$$

Collecting this together, we obtain that the value of $\frac{1}{6}C$ at p is equal to

$$(12,43,56)(42p)[(12,34,15)(12,36,34) + (125)(436)(12,13,34)).$$

It remains for us to check that

$$(12,34,15)(12,36,34) + (125)(436)(12,13,34)$$

$$= (125)(314)(123)(364) + (125)(463)(123)(134) = 0.$$

□

Recall that the Segre cubic contains 15 planes defined by equations

$$\Pi_{ij,kl,mn} : z_i + z_j = z_k + z_l = z_m + z_n = 0,$$

where $\{i,j\} \cup \{k,l\} \cup \{m,n\} = [1,6]$. The intersection of this plane with the hyperplane $H : \sum a_i z_i = 0$ is the union of three lines on the cubic surface. In this way we see 15 lines. Each hyperplane $H_{ij} : z_i + z_j = 0$ cuts out the Segre cubic S_3 along the union of three planes $\Pi_{ij,kl,mn}$, where the union of $\{k,l\}$ and $\{m,n\}$ is equal to $[1,6] \setminus \{i,j\}$. The hyperplane H intersects $H_{ij} \cap \mathsf{S}_3$ along the union of three lines. Thus we see 15 tritangent planes and 15 lines forming a configuration (15_3). This is a subconfiguration of the configuration $(27_5, 45_3)$ of 27 lines and 45 tritangent planes on a nonsingular cubic surface.

The Segre cubic is characterized by the property that it has 10 nodes.

Theorem 9.4.14 *Let S be a normal cubic hypersurface in $\mathbb{P}^4$ with 10 ordinary double nodes. Then S is isomorphic to the Segre cubic primal.*

Proof Choose projective coordinates such that one of the singular points is the point $[1,0,0,0,0]$. The equation of S can be written in the form

$$t_0 A(t_1,\dots,t_4) + B(t_1,\dots,t_4) = 0.$$

By taking the partials, we obtain that the degree 6 curve $C = V(A,B)$ in $\mathbb{P}^3$ has nine singular points. Since $[1,0,0,0,0]$ is an ordinary double point, the quadratic form A is nondegenerate. Thus the curve C is a curve of bidegree $(3,3)$ on a nonsingular quadric $V(A)$. It is a curve of arithmetic genus with

nine singular points. It is easy to see that this is possible only if C is the union of six lines, two triples of lines from each of the two rulings. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts transitively on the set of ordered triple of points, we can fix the curve C. Two cubics $V(B)$ and $V(B')$ cut out the same curve C on $V(A)$ if and only if $B' - B = AL$, where L is a linear form. Replacing t_0 by $t_0 + L$, we can fix B. □

It follows from the proof that no cubic hypersurface in $\mathbb{P}^4$ has more than ten ordinary double points. Thus the Segre cubic primal can be characterized, up to projective equivalence, by the property that it has maximal number of ordinary double points.

Proposition 9.4.15 *Let $p_1, \dots, p_5$ be points in $\mathbb{P}^3$ in general linear position. The linear system of quadrics through these points defines a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ whose image is isomorphic to the Segre cubic primal.*

Proof It is clear that the dimension of the linear system is equal to 4. To compute the degree of the image, we have to compute the number of intersection points of three general quadrics from the linear system and subtract the number of base points. Three general quadrics intersect at eight points, subtracting five, we get three. So, the image of the rational map is a cubic hypersurface S in $\mathbb{P}^4$. For each line $\ell_{ij} = \overline{p_i p_j}$, the general member of the linear system intersects ℓ_{ij} only at the points p_i, p_j. This implies that the image of the line in $\mathbb{P}^4$ is a point. It is easy to see that no other line in $\mathbb{P}^4$, except the ten lines ℓ_{ij}, is blown down to a point. This implies that the image of ℓ_{ij} is an isolated singular point of S. Let $Y \to \mathbb{P}^3$ be the blow-up of the points $p_1, \dots, p_5$. The composition $f : Y \to \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ defines a regular birational map from Y to S. It is a small resolution of S in the sense that the preimages of the singular points are not divisors but curves. Let $Y' \to Y$ be the blow-up of the proper transforms of the lines ℓ_{ij} in Y. The normal bundle of a line ℓ n $\mathbb{P}^3$ is isomorphic to $\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(1)$. It follows from some elementary facts of the intersection theory (see [231], Appendix B.6) that the normal bundle of the proper transform $\bar{\ell}_{ij}$ of ℓ_{ij} is isomorphic to $\mathcal{O}_{\bar{\ell}_{ij}}(-1) \oplus \mathcal{O}_{\bar{\ell}_{ij}}(-1)$. This implies that the preimage of $\bar{\ell}_{ij}$ in Y' is isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the composition $Y' \to Y \to S$ is a resolution of singularities with the exceptional divisor over each singular point of S isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It is well known that it implies that each singular point of S is an ordinary double point of S. Applying Theorem 9.4.14, we obtain that S is isomorphic to the Segre cubic primal. □

Remark 9.4.16 According to [219], the Segre cubic primal admits 1024 small resolutions in the category of complex manifolds. By the action of $\mathfrak{S}_6$ they

are divided into 13 isomorphism classes. Six of the classes give projective resolutions.

The coefficients $(a_1, \dots, a_6)$ in Theorem 9.4.13 can be viewed as elements of the 5-dimensional linear space $V = (R_2^6)(1)$. Since the functions $\bar{U}_i$ add up to zero, $a_1 + \cdots + a_6 = 0$. They map the moduli space P_2^6 of ordered sets of six points in $\mathbb{P}^2$ to the hyperplane $V(\sum t_i)$ in $\mathbb{P}^5$. We know that the action of $\mathfrak{S}_6$ on P_2^6 defines an irreducible representation of type $(2, 2, 2)$ on V and the functions $\bar{U}_i$ are transformed according to the same representation. It is known that the algebra R_2^6 is generated by the space $(R_2^6)(1)$ and one element Υ from $(R_2^6)(2)$. We have

$$\Upsilon = (123)(145)(246)(356) - (124)(135)(236)(456). \tag{9.53}$$

If we replace 6 with x and consider this as an equation of a conic in $\mathbb{P}^2$, we observe that the expression vanishes when $x = p_1, p_2, p_3, p_4, p_5$. Thus the conic passes through the points $p_1, p_2, p_3, p_4, p_5, x$. So, the function Υ vanishes on the set of points $(p_1, \dots, p_6)$ lying on a conic. This is a hypersurface X in P_2^6. One shows that Υ^2 is a polynomial of degree 4 in generators of $(R_2^6)(1)$. This implies that the image of X is a quartic hypersurface in $\mathbb{P}(V)$. Since the map $X \to \mathbb{P}(V)$ is $\mathfrak{S}_6$-equivariant, the image of X can be given by a $\mathfrak{S}_6$-invariant polynomial in t_i. Since the representation V is self-dual, and is obtained from the standard representation of $\mathfrak{S}_6$ on V by composing with the outer automorphism, the invariant functions are symmetric polynomials. So, the equation of the image of X is equal to

$$s_2^2 + \lambda s_4 = 0,$$

where $s_k = \sum_{i=0}^5 t_i^k$. The coefficient λ can be found from the fact that the hypersurface X is singular at the locus of strictly semi-stable points represented by points sets $p_i = p_j$ and the remaining four points are collinear. The locus consists of 15 lines. A simple computation shows that the only symmetric quartic with this property is the quartic $V(s_2^2 - 4s_4)$ (see [**?**], Theorem 4.1).

The quartic threefold CR_4 in $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^5 t_i = 0, \quad (\sum_{i=0}^5 t_i^2)^2 - 4\sum_{i=0}^5 t_i^4 = 0 \tag{9.54}$$

will be called the *Castelnuovo-Richmond quartic*.

Corollary 9.4.17 *The variety P_2^6 is isomorphic to the double cover of $\mathbb{P}^4$ ramified over the Castelnuovo-Richmond quartic. It can be given by the equa-*

tions

$$t_5^2 + (\sum_{i=0}^{5} t_i^2)^2 - 4\sum_{i=0}^{5} t_i^4 = 0, \quad \sum_{i=0}^{5} t_i = 0. \tag{9.55}$$

in $\mathbb{P}(1,1,1,1,1,2)$.

The involution $(t_0,\dots,t_6) \mapsto (t_0,\dots,t_5,-t_6)$ is the *association involution*. Applying it to the projective equivalence class of a general point set $(p_1,\dots,p_6)$ we obtain the projective equivalence class of a set $(q_1,\dots,q_6)$ such that the blow-ups of the two sets are isomorphic cubic surfaces, and the two geometric markings are defined by a double-six. We refer for all of this to [176].

Consider the dual variety $(\mathsf{S}_3)^\vee$ of the Segre cubic primal. Since S_3 has 10 ordinary nodes, the Plücker-Teissier Formula shows that $(\mathsf{S}_3)^\vee$ is a quartic hypersurface. The duals of the hyperplanes H_{ij} define 15 points in the dual $\mathbb{P}^4$. The duals of the planes $\Pi_{ij,kl,mn}$ are 15 lines. They are singular lines of CR_4. The 15 lines and 15 points form a configuration (15_3) in the dual space.

Proposition 9.4.18 *The dual variety of the Segre cubic primal is isomorphic to the Castelnuovo-Richmond quartic hypersurface:*

$$\mathsf{CR}_4 \cong (\mathsf{S}_3)^\vee.$$

Proof We may assume that S_3 is given by the equation $\sum_{i=0}^4 t_i^3 - (\sum_{i=0}^4 t_i)^3 = 0$ in $\mathbb{P}^4$, and the group $\mathfrak{S}_6$ acts by letting its subgroup $\mathfrak{S}_5$ permute $t_0,\dots,t_4$ and sending the transposition (56) to the transformation $t_i \mapsto t_i, i \le 4, t_4 \mapsto -L$, where $L = t_0 + \cdots + t_4$. The polar map is given by polynomials $F_i = t_i^2 - L^2, i = 0,\dots,4$. After a linear change of the coordinates y_i in the target space

$$y_i' = y_i - \frac{1}{3}(y_0 + y_1 + y_2 + y_3), \quad i = 0,\dots,4,$$

we obtain that the linear representation of $\mathfrak{S}_6$ on the target space is isomorphic to the representation on the t_i's. Thus the dual hypersurface is isomorphic to a quartic threefold in $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^{5} y_i = 0, \quad s_2^2 + \lambda s_4 = 0,$$

where $s_k = \sum_{i=0}^5 y_i^k$. Under the polar map, the 15 planes in $\mathfrak{S}_3$ are mapped to 15 singular lines on the dual variety. A straightforward computation shows that this implies that the parameter λ is equal to -4 (see [236], Theorem 4.1). □

9.4.5 Moduli spaces of cubic surfaces

The methods of the Geometric Invariant Theory (GIT) allows one to construct the moduli space of nonsingular cubic surfaces $\mathcal{M}_{\text{cub}}$ as an open subset of the GIT-quotient

$$\mathbb{P}(S^3((\mathbb{C}^4)^\vee)//\mathrm{SL}(4) = \mathrm{Proj}\bigoplus_{d=0}^{\infty} S^d(S^3((\mathbb{C}^4)^\vee)^\vee)^{\mathrm{SL}(4)}. \tag{9.56}$$

The analysis of stability shows that, except one point, the points of this variety represent the orbits of cubic surfaces with ordinary double points. The exceptional point corresponds to the isomorphism class of a unique surface with three A_2-singularities. So, the GIT-quotient can be taken as a natural compactification $\bar{\mathcal{M}}_{\text{cub}}$ of the moduli space $\mathcal{M}_{\text{cub}}$. The computations from the classical invariant theory due to G. Salmon [490], [494] and A. Clebsch [103] (see a modern exposition in [308]) show that the graded ring of invariants is generated by elements I_d of degrees $d = 8, 16, 24, 32, 40$ and 100 (a modern proof of completeness can be found in [38]). The first four basic invariants are invariants with respect to the group G of invertible matrices with the determinant equal to ± 1. This explains why their degrees are divisible by 8 (see [182]). The last invariant is what the classics called a skew invariant, it is not an invariant of G but an invariant of $\mathrm{SL}(4)$. There is one basic relation expressing I_{100}^2 as a polynomial in the remaining invariants. The graded subalgebra generated by elements of degree divisible by 8 is freely generated by the first five invariants. Since the projective spectrum of this subalgebra is isomorphic to the projective spectrum of the whole algebra, we obtain an isomorphism

$$\bar{\mathcal{M}}_{\text{cub}} \cong \mathbb{P}(8, 16, 24, 32, 40) \cong \mathbb{P}(1, 2, 3, 4, 5). \tag{9.57}$$

This, of course, implies that the moduli space of cubic surfaces is a rational variety.

The discriminant Δ of a homogeneous cubic form in four variables is expressed in terms of the basic invariants by the formula

$$\Delta = (I_8^2 - 64I_{16})^2 - 2^{14}(I_{32} + 2^{-3}I_8I_{24}) \tag{9.58}$$

(the exponent -3 is missing in Salmon's formula and also the coefficient at I_{32} was wrong, it has been corrected in [151]).

We may restrict the invariants to the open Zariski subset of Sylvester non-degenerate cubic surfaces, It allows one to identify the first four basic invariants with symmetric functions of the coefficients of the Sylvester equations.

Salmon's computations give

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_1\sigma_5^3, \quad I_{24} = \sigma_4\sigma_5^4, \quad I_{32} = \sigma_2\sigma_5^6, \quad I_{40} = \sigma_5^8, \tag{9.59}$$

where σ_i are elementary symmetric polynomials. Evaluating Δ from above, we obtain a symmetric polynomial of degree 8 obtained from (9.34) by eliminating the irrationality.

The invariant I_{40} restricts to $(a_0a_1a_2a_3a_4)^8$. It does not vanish on the set of Sylvester nondegenerate cubic surfaces. Its locus of zeros is the closure of the locus of Sylvester-degenerate nonsingular cubic surfaces.

The skew-invariant I_{100} is given by the equation

$$I_{100} = (a_0a_1a_2a_3a_4)^{19} \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0^{-1} & a_1^{-1} & a_2^{-1} & a_3^{-1} & a_4^{-1} \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It vanishes on the closure of the locus of nonsingular surfaces with an Eckardt point. Observe that it vanishes if $a_i = a_j$ and that agrees with Example 9.1.25.

Following [151] we can interpret (9.59) as a rational map

$$\mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5 \cong \mathbb{P}(1,2,3,4,5) \dashrightarrow \bar{\mathcal{M}}_{\text{cub}} \cong \mathbb{P}(1,2,3,4,5).$$

We have

$$\sigma_1 = \frac{I_{16}}{\sigma_5^3}, \quad \sigma_2 = \frac{I_{32}}{\sigma_5^6}, \quad \sigma_3 = \frac{I_{24}^2 - I_8I_{40}}{\sigma_5^9}, \quad \sigma_4 = \frac{I_{24}I_{40}}{\sigma_5^{12}}, \quad \sigma_5 = \frac{I_{40}^2}{\sigma_5^{15}}.$$

This gives the inverse rational map

$$\bar{\mathcal{M}}_{\text{cub}} \dashrightarrow \mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5.$$

The map is not defined at the set of points where all the invariants I_{8d} vanish except I_8. It is shown in [151], Theorem 6.1 that the set of such points is the closure of the orbit of a Fermat cubic surface.

Remark 9.4.19 One should compare the moduli space P_2^6 of ordered sextuples of points in the plane and the moduli space $\bar{\mathcal{M}}_{\text{cub}}$ of cubic surfaces. The blow-up of a set of six points in general position is isomorphic to a cubic surface. It comes equipped with a geometric basis. The Weyl group $W(\mathbf{E}_6)$ acts transitively on geometric bases, and the birational quotient of P_2^6 by the action of $W(\mathbf{E}_6)$ is isomorphic to $\bar{\mathcal{M}}_{\text{cub}}$. The forgetful map

$$P_2^6 \dashrightarrow \bar{\mathcal{M}}_{\text{cub}} \tag{9.60}$$

is of degree equal to $\#W(\mathbf{E}_6)$. The action of the subgroup $\mathfrak{S}_6$ of the Weyl

group is easy to describe. It is a regular action on P_2^6 via permuting the points. In the model of P_2^6 given by Equation (9.55), the action is achieved by permuting the coordinates $t_0, \dots, t_5$ according to the representation of type $(2,2,2)$. The quotient is isomorphic to the double cover

$$(\mathbb{P}_2^6)/\mathfrak{S}_6 \to \mathbb{P}^4/\mathfrak{S}_6 \cong \mathbb{P}(2,3,4,5,6).$$

It is ramified over the image of the hypersurface $V(\Upsilon) \subset P_2^6$ parameterizing points sets on a conic. The branch locus is the image of the Castelnuovo-Richmond quartic CR_4 in the quotient. It is isomorphic to $\mathbb{P}(2,3,5,6)$. In the cubic surface interpretation the ramification locus is birationally isomorphic to cubic surfaces with a node. This shows that the moduli space of singular cubic surfaces is birationally isomorphic to $\mathbb{P}(2,3,5,6)$, and hence is a rational variety.

The quotient $(\mathbb{P}_2^6)/\mathfrak{S}_6$ can be viewed as a birational model of the moduli space of cubic surfaces together with a choice of a double-six. The previous isomorphism shows that this moduli space is rational. It is not known whether the moduli space of cubic surfaces together with a sixer of lines is a rational variety.

Note that the functions $\bar{U}_i$, taken as generators of the space $(R_2^6)(1)$, allow one to identify some special loci in P_2^6 with ones in $\bar{\mathcal{M}}_{\text{cub}}$. For example, we know from (9.44) that $\bar{U}_1 - \bar{U}_2 = 0$ represents the locus of points sets $(p_1, \dots, p_6)$ such that the lines $\overline{p_1p_2}, \overline{p_3p_6}, \overline{p_4p_5}$ are collinear. This corresponds to a cubic surface with an Eckardt point. Changing the order of points, this gives 15 hypersurfaces in P_2^6 permuted by $\mathfrak{S}_6$. Another example is a hypersurface $V(\bar{U}_1 + \bar{U}_2 + \bar{U}_3)$. According to Lemma 9.4.12, it corresponds to the locus of points set $(p_1, \dots, p_6)$, where the points p_1, p_4, p_6 or p_2, p_3, p_5 are collinear. They are permuted by $\mathfrak{S}_6$ and give 20 hypersurfaces in P_2^6. The image of these hypersurfaces under the map (9.60) is contained in the locus of singular surfaces.

A cubic surface in $\mathbb{P}^3$ can be given as a hyperplane section of a cubic threefold in $\mathbb{P}^4 = |W|$. In this way the theory of projective invariants of cubic surfaces becomes equivalent to the theory of projective invariants of $\mathrm{PGL}(5)$ in the space $S^3(W^\vee) \times W^\vee$. The Cremona hexahedral equations of a cubic surface represents a subvariety of this representation isomorphic to $\mathbb{C}^6$. Clebsch's transfer principle (for a modern explanation see [308]) allows one to express projective invariants of $\mathrm{GL}(4)$ as polynomial functions on $\mathbb{C}^6$. The degree of an invariant polynomial of degree m equal to their weights $3m/4$. In particular, the basic polynomials $I_8, \dots, I_{100}$ become polynomials $J_6, J_{12}, J_{18}, J_{24}, J_{30}, J_{75}$ in $(a_1, \dots, a_6)$ of degrees indicated in the subscript.

The first five polynomials are symmetric polynomials in $a_1, \dots, a_6$, the last one is a skew-symmetric polynomial. For example,

$$J_6 = 24(4\sigma_2^3 - 3\sigma_3^2 - 16\sigma_2\sigma_4 + 12\sigma_6)$$

(see [117] Part III, p. 336, and [538]).

The skew-invariant J_{75} defining the locus of cubic surfaces with an Eckardt points is reducible. It contains as a factor of degree 15 the discriminant $\prod_{i<j}(a_i - a_j)$ of the polynomial $(X - a_1)\cdots(X - a_6)$. The remaining factor of degree 60 is equal to the product of 30 polynomials of the form

$$T_{1256;3} = (126)(356)(134)(253) - (136)(256)(123)(354), \tag{9.61}$$

where we use Lemma 9.4.12 to express the product of two brackets as a function $a_i+a_j+a_k$. The vanishing of $T_{1256;3}$ expresses the condition that the conic through the points p_1, p_2, p_3, p_5, p_6 is touched at p_3 by the line $\overline{p_3p_4}$ (equivalently, the tritangent plane defined by the lines $e_3, 2e_0 - \sum e_i, e_0 - e_3 - e_4$ has an Eckardt point). Together with 15 polynomials $\bar{U}_i - \bar{U}_j$, this accounts for 45 hypersurfaces defining the locus of cubic surfaces with an Eckardt point. Note that the formulas for $\bar{U}_i - \bar{U}_j$ and $T_{1256;3}$ allow one to compute the number of Eckardt points on a surface given by Cremona's hexahedral equations. For example, if we have one pair of equal coefficients a_i, we have an Eckardt point on the surface. However, it is not a necessary condition, because an Eckardt point may arise from the vanishing of a function of type $T_{1256;3}$. For example, a cyclic cubic surface has nine Eckardt points, and they cannot be found only from the equalities of the coefficients a_i.

We can also find the expression of the discriminant invariant Δ (9.58) in terms of the coefficients $a_0, \dots, a_5$.

We know that the quartic symmetric polynomials $\sigma_2^2 - 4\sigma_4$ in $a_1, \dots, a_6$ equal to the squares of the function Υ from (9.53) representing points sets on a conic. Thus we see that the discriminant invariant in $(a_0, \dots, a_5)$, being of of degree 24, must be a scalar multiple of the product of powers of $(\sigma_2 - 4\sigma_4)$ and powers of $(a_i + a_j + a_k)$, $1 \le i < j < k \le 5$ representing points sets with three collinear points. The only way to make a symmetric polynomial of degree 24 in this way is to take all factors in the first power. We also use that σ_1 vanishes on $(a_1, \dots, a_6)$. The computer computation gives the following expression in terms of the elementary symmetric polynomials:

$$\Delta = (\sigma_2^2 - 4\sigma_4)(\sigma_3^4\sigma_4^2 - 2\sigma_2\sigma_3^3\sigma_4\sigma_6 + \sigma_2^2\sigma_3^2\sigma_5^2 + 2\sigma_3^2\sigma_4\sigma_5^2 - 2\sigma_2\sigma_3\sigma_5^4 + 2\sigma_2\sigma_3\sigma_4\sigma_6 -$$

$$8\sigma_3^2\sigma_4^2\sigma_6 - 2\sigma_2^3\sigma_3\sigma_5\sigma_6 + 8\sigma_2\sigma_3\sigma_4\sigma_5\sigma_5\sigma_6 + 2\sigma_2^2\sigma_5^2\sigma_6 + \sigma_2^4\sigma_6^2 - 8\sigma_2^2\sigma_4\sigma_6^2 + 16\sigma_4^2\sigma_6^2).$$

Remark 9.4.20 The story goes on. The group $W(\mathbf{E}_6)$ acts birationally on the space P_2^6 by changing the markings and Coble describes in [117], Part III, rational invariants of this action. He also defines a linear system of degree 10 of elements of degree 3 in R_2^6 which gives a $W(\mathbf{E}_6)$-equivariant embedding of a certain blow-up of P_2^6 in $\mathbb{P}^9$ corresponding to some irreducible 10-dimensional linear representation of the Weyl group. For a modern treatment of this construction we refer to [124] and [234]. Other $W(\mathbf{E}_6)$-equivariant birational models of R_2^6 were given in [412] and [274].

We also refer to a recent construction of the GIT-moduli space of cubic surfaces as a quotient of a complex 4-dimensional ball by a reflection group [6],[185]. The embedding of the moduli spaces in $\mathbb{P}^9$ by means of automorphic forms on the 4-dimensional complex ball is discussed in [7] and [225].

9.5 Automorphisms of cubic surfaces

9.5.1 Cyclic groups of automorphisms

Let W be the Weyl group of a simple root system of type A, D, E. The conjugacy classes of elements of finite order can be classified (see table 9.3). We will use the classification of conjugacy classes of elements in W. This can be found in [125], [65] or [377]. We also include the information about the characteristic polynomial of the action of an element $w \in W$ on the root space, in particular, its trace. The fourth column gives the order of the centralizer subgroup of the conjugacy class.

We know that the automorphism group $\mathrm{Aut}(S)$ of a nonsingular cubic surface acts faithfully on $\mathrm{Pic}(S)$. We will identify elements $g \in \mathrm{Aut}(S)$ with the corresponding elements $g^* \in W(S)$. We fix a geometric basis to identify element of $W(S)$ with the Weyl group $W(\mathbb{E}_6)$.

To apply the previous information about elements of the Weyl group to automorphisms of a cubic surface, we use the following two Lemmas. The first one is the well-known *Lefschetz's fixed-point formula*, which can be found in most of topology text-books. From now on, S denotes a nonsingular cubic surface.

Lemma 9.5.1 *Let σ be an automorphism of a nonsingular cubic surface S and* $\mathrm{Fix}(\sigma)$ *be its fixed locus. Let* $\mathrm{Tr}(\sigma)$ *be the trace of σ in its action on $K_S^{\perp}$. Then the topological Euler-Poincaré characteristic of* $\mathrm{Fix}(\sigma)$ *is given by the formula*

$$\chi(\mathrm{Fix}(\sigma)) = 3 + \mathrm{Tr}(\sigma).$$

Since any automorphism of S is a restriction of a projective automorphism

	Atlas	Carter	Manin	Ord	$\#C$	Tr	Char
x	1A	$\emptyset$	c_1	1	51840	6	$(t-1)^6$
x	2A	$4A_1$	c_3	2	1152	-2	$p_1^4(t-1)^2$
x	2B	$2A_1$	c_2	2	192	2	$p_1^2(t-1)^4$
	2C	A_1	c_{16}	2	1440	4	$p_1(t-1)^5$
	2D	$3A_1$	c_{17}	2	96	0	$p_1^3(t-1)^3$
x	3A	$3A_2$	c_{11}	3	648	-3	p_2^3
x	3C	A_2	c_6	3	216	3	$p_2(t-1)^4$
x	3D	$2A_2$	c_9	3	108	0	$p_2^2(t-1)^2$
x	4A	$D_4(a_1)$	c_4	4	96	2	$(t^2+1)^2(t-1)^2$
x	4B	A_1+A_3	c_5	4	16	0	$p_1p_3(t-1)^2$
	4C	$2A_1+A_3$	c_{19}	4	96	-2	$p_1^2p_3(t-1)^2$
	4D	A_3	c_{18}	4	32	2	$p_3(t-1)^3$
x	5A	A_4	c_{15}	5	10	1	$p_4(t-1)^2$
x	6A	$E_6(a_2)$	c_{12}	6	72	1	$p_2(t^2-t+1)^2$
x	6C	D_4	c_{21}	6	36	1	$p_1(t^3+1)(t-1)^2$
x	6E	A_1+A_5	c_{10}	6	36	-2	p_1p_5
x	6F	$2A_1+A_2$	c_8	6	24	-1	$p_1^2p_2(t-1)^2$
	6G	A_1+A_2	c_7	6	36	1	$p_1p_2(t-1)^3$
	6H	A_1+2A_2	c_{22}	6	36	-2	$p_1p_2^2(t-1)$
	6I	A_5	c_{23}	6	12	0	$p_5(t-1)$
x	8A	D_5	c_{20}	8	8	0	$p_1(t^4+1)(t-1)$
x	9A	$E_6(a_1)$	c_{14}	9	9	0	(t^6+t^3+1)
	10A	A_1+A_4	c_{25}	10	10	-1	$p_1p_4(t-1)$
x	12A	E_6	c_{13}	12	12	-1	$p_2(t^4-t^2+1)$
	12C	$D_5(a_1)$	c_{24}	12	12	1	$(t^3+1)(t^2+1)(t-1)$

Table 9.3 *Conjugacy classes in* $W(\mathbf{E}_6)$

of $\mathbb{P}^3$ (because $|-K_S| = |\mathcal{O}_S(1)|$ is invariant) we will identify automorphisms of S with projective automorphisms of $\mathbb{P}^3$. It is clear that the fixed locus $\mathrm{Fix}(\sigma)$ of σ in S is equal to the intersection of the fixed locus $F(\sigma)$ of σ in $\mathbb{P}^3$ with S.

Let $n > 1$ be the order of σ in $\mathrm{PGL}(4)$. We can represent σ by a matrix $A \in \mathrm{GL}(4)$ such that $A^n = cI_4$. Multiplying A by $1/\alpha$, where $\alpha^n = 1/\lambda$, we may assume that A is of order n. Let $\lambda_1, \ldots, \lambda_4$ be eigenvalues of A. They determine the type of the fixed locus $F(\sigma)$ of σ. Let k be the number of distinct eigenvalues. We have

- $k = 4$: $F(\sigma)$ consists of four isolated points;
- $k = 3$: $F(\sigma)$ consists of a line and two isolated points;
- $k = 2$: $F(\sigma)$ consists of two lines or a plane and an isolated point.

Let us see what the possibilities are for $\mathrm{Fix}(\sigma) = S \cap F(\sigma)$. If $k = 4$, $\mathrm{Fix}(\sigma)$ may consist of $N \leq 4$ isolated fixed points. If $k = 3$, $\mathrm{Fix}(\sigma)$ may consist of

a line and $N \leq 2$ isolated points, or $N \leq 5$ isolated fixed points. If $k = 2$, $\mathrm{Fix}(\sigma)$ may consist of a plane cubic curve, necessarily nonsingular and $N \leq 1$ isolated points, or two lines, or a line and $N \leq 3$ points, or $N \leq 6$ points.

Some of these possibilities can be excluded. If $\mathrm{Fix}(\sigma)$ consists of two lines, then a general line intersecting the two lines contains a third intersection point with S. It must be a fixed point contradicting the assumption. Also, a line ℓ in $F(\sigma)$ either is contained in S or intersects S at three points. In fact, if ℓ is tangent to S at a point x, then the action of σ in the tangent space $T_x(S)$ has one eigenvector with eigenvalue equal to 1. So, one can choose local coordinates u, v at x such that the action is locally given by $(u, v) \mapsto (u, \epsilon v)$. The curve locally given by $v = 0$ belongs to $\mathrm{Fix}(\sigma)$, contradicting the assumption that x is an isolated fixed point.

We will start from the case when σ is of order 2. In this case, $F(\sigma)$ is a plane plus a point, or two lines. In the first case, if $\mathrm{Fix}(\sigma)$ consists of a curve only, we have $\chi(\mathrm{Fix}(\sigma)) = 0, \mathrm{Tr}(\sigma) = -3$, and Table 9.3 shows that there is no element of order 2 with trace -3. Thus $\mathrm{Fix}(\sigma)$ consists of a plane and a point, and σ belongs to the conjugacy class $2A$. We choose the equation of the plane to be $V(t_3)$ and the isolated fixed point to be $[1, 0, 0, 0]$. We may assume that the action is defined by the diagonal matrix $\mathrm{diag}[1, 1, 1, -1]$. The equation of S becomes

$$at_3^3 + t_3^2 l(t_0, t_1, t_2) + t_3 q(t_0, t_1, t_2) + g_3(t_0, t_1, t_2) = 0,$$

where $V(g_3)$ is a nonsingular plane curve in $V(x_3)$. Since the surface is σ-invariant, we must have $a = q = 0$. Reducing the equation $V(g_3)$ to a Hesse form, we get the equation

$$t_3^2(at_0 + bt_1 + ct_2) + t_0^3 + t_1^3 + t_2^3 + ct_0t_1t_2 = 0. \tag{9.62}$$

Suppose that $F(\sigma)$ is the union of two lines. If $\mathrm{Fix}(\sigma)$ consists of six points, then $\chi(\mathrm{Fix}(\sigma)) = 6, \mathrm{Tr}(\sigma) = 3$, and Table 9.3 shows that there are no elements of order 2 with trace 3. So, $\mathrm{Fix}(\sigma)$ must consist of a line and 3 points, and σ is of type $2B$. We can choose coordinates such that $\ell_1 = V(t_0) \cap V(t_1)$ and $\ell_2 = V(t_1) \cap V(t_2)$. The element σ is represented by a diagonal matrix $\mathrm{diag}[1, 1, -1, -1]$. The equation of S becomes

$$t_0q_1(t_2, t_3) + t_1q_2(t_2, t_3) + \sum_{i+j=2} t_0^i t_1^j a_{ij}(t_2, t_3) + g_3(t_0, t_1) = 0. \tag{9.63}$$

Since S is σ-invariant, the linear forms $a_{ij}(t_2, t_3)$ must be equal to zero. Since S is nonsingular, g_3 has no multiple zeros, so it can be reduced to the form $t_0^3 + t_1^3$. For the same reason, t_2^2, t_3^2 enter in q_1 or q_2. If q_1 and q_2 have the same roots, one checks that the surface is singular. So, after a linear change of

variables t_2, t_3, we reduce q_1, q_2 to the form $q_1 = t_2(at_3 + ct_2), q_2 = t_3(bt_2 + dt_3)$, where $c, d \neq 0$ (otherwise, t_2 or t_3 enters the equation in degree ≤ 1, and the surface is singular). After scaling t_2, t_3, we may assume $c = d = 1$. This gives the equation

$$t_0t_2(t_2 + at_3) + t_1t_3(t_3 + bt_2) + t_0^3 + t_1^3 = 0. \tag{9.64}$$

Now, the information about the characteristic polynomial tells us the conjugacy class of a power of σ. Suppose σ is of order $2k$. It follows from above that σ^k must be of type $2A$ or $2B$. Going through the list of the conjugacy classes, we find that the classes

$$4C,\ 4D,\ 6G,\ 6H,\ 6I,\ 10A,\ 12C$$

do not occur.

Let us tabulate all possible cases, the corresponding values of $Tr(\sigma)$, and possible conjugacy classes with this trace.

Order	$F(g)$	Fix(σ)	χ	Tr(σ)	Conjugacy class
$n \geq 2$	plane+1 pt.	curve+point	1	-2	2A,6E
		curve	0	-3	3A
$n \geq 2$	2 lines	line+3 pts.	5	2	2B,4A
		6 pts.	6	3	3C
$n \geq 3$	line+2pts.	line+2pts.	4	1	5A,6A,6C
		line+1 pt.	3	0	3D,4B,8A,9A
		line	2	-1	6F, 12A
		5 pts.	5	2	2B, 4A
		4 pts.	4	1	5A,6A, 6C
		3 pts.	3	0	3D,8A,9A
$n \geq 4$	4 pts.	4 pts.	4	1	5A,6A,6C,6G
		3pts.	3	0	4B,6I,8A,9A
		2pts.	2	-1	6F,12A
		1 pt.	1	-2	6E

Table 9.4 *Fixed points*

Now, we are in business. In the following, ϵ_n denotes a primitive n-th root of unity.

- σ is of type 3A.

In this case $F(\sigma)$ is the union of a plane and a point, and Fix(σ) consists of a single curve. We can choose coordinates such that σ is represented by the diagonal matrix diag$[1, 1, 1, \epsilon_3]$. It is easy to see that the equation must be

$$t_3^3 + g_3(t_0, t_1, t_2) = 0.$$

The surface is a cyclic surface. Since $V(g_3)$ must be nonsingular, we can reduce it to a Hesse form to obtain the equation

$$t_3^3 + t_0^3 + t_1^3 + t_2^3 + at_0t_1t_2 = 0. \tag{9.65}$$

- σ is of type 3C.

In this case $F(\sigma)$ is the union of two lines, and Fix(σ) consists of six points. We can choose coordinates such that σ is represented by the diagonal matrix diag$[1, 1, \epsilon_3, \epsilon_3]$. Write the equation of S in the form

$$\sum_{k=0}^{3}\sum_{i=0}^{k} t_0^i t_1^{k-i} a_{ik}(t_2, t_3) = 0,$$

where a_{ik} is a binary form of degree $3 - k$. Applying σ, we see that $a_{i1} = a_{i2} = 0$. The equations become

$$g_3(t_0, t_1) + h_3(t_2, t_3) = 0.$$

Since S is nonsingular, the binary forms g_3, h_3 have no multiple zeros. Thus, after a linear change of coordinates, we reduce equation to the Fermat form.

- σ is of type 3D.

In this case $F(\sigma)$ is the union of a line and two points, Fix(σ) consists of either three points or a line plus one point. Consider the first case. We can choose coordinates such that σ is represented by the diagonal matrix diag$[1, 1, \epsilon_3, \epsilon_3^2]$, where the line is $t_2 = t_3 = 0$. Since the isolated fixed points $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$ are not in S, we can write the equation of S in the form

$$t_2^3 + t_3^3 + t_2t_3a_1(t_0, t_1) + g_3(t_0, t_1) + \sum t_2^i t_3^j a_{ij}(t_0, t_1) = 0,$$

where $(i, j) \neq (3, 0), (0, 3), (1, 1), (0, 0)$. Since all monomials $t_0^i t_1^j$ with such (i, j) are not σ-invariant, we obtain that $a_{ij} = 0$. Thus S can be given by the equation

$$t_2^3 + t_3^3 + t_2t_3a_1(t_0, t_1) + g_3(t_0, t_1) = 0.$$

To make it different from case 3C, we have to assume that $a_1 \neq 0$. By a further change of coordinates, we can reduce it to the equation

$$t_2^3 + t_3^3 + t_2t_3(t_0 + at_1) + t_0^3 + t_1^3 = 0. \tag{9.66}$$

Consider the second case. We keep the coordinates from the previous case. The equation becomes

$$f = \sum_{0<i+j\leq 3} t_2^i t_3^j a_{3-i-j}(t_0, t_1) = 0.$$

As we saw in the previous case, each monomial entering in f must be divisible by t_2^3, t_3^3, t_2t_3. This shows that the variables t_0, t_1 enter with degree ≤ 1. This implies that the surface is singular. So, this case does not occur.

- σ is of type 4A.

Note that σ^2 belongs to the conjugacy class $2A$. Hence we can choose coordinates such that σ acts via the diagonal matrix $[1, 1, -1, i]$ and S is given by the equation

$$t_3^2 l(t_0, t_1, t_2) + g_3(t_0, t_1, t_2) = 0.$$

This equation is σ-invariant only if $l(t_0, t_1, t_2) = at_2$ and $g_3(t_0, t_2, -t_2) = g_3(t_0, t_1, t_2)$. This gives the equation

$$t_3^2 t_2 + t_2^2 g(t_0, t_1) + h(t_0, t_1) = 0.$$

We can further reduce it to the form

$$t_3^2 t_2 + t_2^2 t_1 + t_0^3 + at_0 t_1^2 + bt_1^3 = 0. \tag{9.67}$$

Note that $\mathrm{Fix}(\sigma)$ consists of five isolated points.

- σ is of type 4B.

In this case $F(\sigma)$ consists of four isolated points and $\mathrm{Fix}(\sigma)$ consists of three points. Since σ^2 is of type 2B, we may write equation in the form (9.64).

$$t_0 q_1(t_2, t_3) + t_1 q_2(t_2, t_3) + g_3(t_0, t_1) = 0.$$

We may assume that σ acts via the matrix $\mathrm{diag}[1, -1, i, -i]$. Since S is non-singular, one of the monomials $t_0^3, t_0^2 t_1$ and one of the monomials t_1^3, t_1^2 must enter in g_3. It is clear that both t_0^3 and t_1^3 cannot enter in g_3. After switching the variables, we may assume that t_0^3 enters. In order for S to remain invariant with respect to σ, we must have $q_1 = ct_2t_3, q_2 = at_2^2 + bt_3^2$ and $g_3(t_0, t_1) = dt_0^3 + et_0 t_1^2$. If $c = 0$, we get a singular surface. So, we may assume that $c = 1$, and, after scaling the coordinates, we get the equation[1]

$$\lambda t_0 t_2 t_3 + t_1(t_2^2 + t_3^2) + t_0^3 + t_0 t_1^2 = 0. \tag{9.68}$$

If we choose the new coordinates to transform t_2t_3 to $t_2^2 + t_3^2$, after rescaling, we get the equation

$$t_0^3 + t_0(t_1^2 + t_2^2 + t_3^2) + \lambda t_1 t_2 t_3 = 0. \tag{9.69}$$

- σ is of type 5A.

[1] This corrects the mistake in [188], where the equation defines a singular surface.

In this case $F(\sigma)$ is either the union of a line and two points, or the union of four points. Let us see that the first case does not occur. If $\mathrm{Fix}(\sigma)$ contains a line ℓ, we can reduce the equation to the form (9.63), where $\ell = V(t_2) \cap V(t_3)$. The quadratic forms q_1, q_2 must be invariant with respect to an automorphism of order 5 of the line $V(t_0) \cap V(t_1)$. This forces $q_1 = q_2 = 0$, and shows that this case does not occur.

So, we may assume that $F(\sigma)$ consists of four points. In this case we may assume that σ acts via the matrix $\mathrm{diag}[1, \epsilon_5, \epsilon_5^2, \epsilon_5^3]$. Since $[1,0,0,0], \dots, [0,0,0,1]$ are on the surface, the equation does not contain $t_0^3, \dots, t_3^3$. Since every monomial contains one of the powers t_i^2 (otherwise S is singular), we can write the equation in the form

$$t_0^2 a_0(t_1,t_2,t_3) + t_1^2 a_1(t_0,t_2,t_3) + t_2^2 a_2(t_0,t_1,t_3) + t_3^2 a_3(t_0,t_1,t_2) = 0. \tag{9.70}$$

The only way to make it invariant is to assume (after permuting the variables, if necessary) that $a_0 = c_0 t_2, a_1 = c_1 t_0, a_2 = c_2 t_3, a_3 = c_3 t_1$. After scaling the coordinates, we get the equation

$$t_0^2 t_2 + t_1^2 t_0 + t_2^2 t_3 + t_3^2 t_1 = 0. \tag{9.71}$$

- σ is of type 6A.

Similarly to the previous case, we prove that $\mathrm{Fix}(\sigma)$ cannot contain a line. So Table 9.4 shows that it consists of four points. Note that σ^3 is of type 2A and σ^2 is of type 3A. So, we can choose coordinates to assume that σ acts via $\mathrm{diag}[1, 1, -1, \epsilon_3]$, and the equation is in the form (9.65)

$$t_3^3 + g_3(t_0,t_1,t_2) = 0.$$

The only way to make it invariant is to assume that $g_3 = t_2^2(at_0 + bt_1) + h_3(t_0,t_1)$. Reducing h_3 to the sum of cubics, and scaling t_2, we get the equation

$$t_3^3 + t_2^2(t_0 + bt_1) + t_0^3 + t_1^3 = 0. \tag{9.72}$$

- σ is of type 6C.

In this case σ^3 is of type 2A. We can choose the equation of the form

$$t_3^2(at_0 + bt_1 + ct_2) + g_3(t_0,t_1,t_2) = 0, \tag{9.73}$$

and the action to one defined by the matrix $\mathrm{diag}[1, 1, \epsilon_6^i, \epsilon_6]$, where $i = \pm 2$. In order to make the equation invariant, we must assume that $i = -2, a = b = 0$ and $g_3 = dt_2^3 + h_3(t_0,t_1)$. After additional change of coordinates, we reduce the equation to the form

$$t_3^2 t_2 + t_2^3 + t_0^3 + t_1^3 = 0. \tag{9.74}$$

- σ is of type 6E.

In this case, σ^3 is of type 2A. We can choose the equation of the form

$$t_3^2(at_0 + bt_1 + ct_2) + g_3(t_0, t_1, t_2) = 0.$$

and the action is defined by $\mathrm{diag}[1, \epsilon_3^i, \epsilon_3, \epsilon_6]$, where $i = 1$ or 2. Table 9.4 shows that $F(\sigma)$ consists of four points and $\mathrm{Fix}(\sigma)$ is a single point. This shows that all eigenvalues are different, so $i = 2$. Suppose $a \neq 0$. Then all monomials entering in the equation have eigenvalue ϵ_3. This implies that $t_2^3, t_2^2t_0, t_2^2t_1, t_2^2t_3$ do not enter. This makes S singular at the point $[0, 0, 1, 0]$. So $a = 0$, and after switching t_1, t_2, if necessary, we may assume that $b \neq 0$, $c = 0$. This makes all monomials invariant. After scaling the coordinates, we get the equation

$$t_3^2t_1 + t_0^3 + t_1^3 + t_2^3 + at_0t_1t_2 = 0. \tag{9.75}$$

- σ is of type 6F.

As in case 6A, we show that $\mathrm{Fix}(\sigma)$ does not contain a line. Consulting Table 9.4, we find that $F(g)$ has four isolated fixed points. Thus σ can be represented by the diagonal matrix with all distinct eigenvalues. We find that σ^2 is of type 3C. Thus we can reduce the equation to the form

$$g_3(t_0, t_1) + h_3(t_2, t_3) = 0.$$

The element σ acts via the matrix $\mathrm{diag}[1, \epsilon_6^3, \epsilon_6^4, \epsilon_6]$. To make the equation invariant, we must (after permuting (t_0, t_1) and (t_2, t_3)) take $g_3 = at_0^3 + bt_0t_1^2, h_3 = ct_2^3 + dt_2t_3^2$. After scaling the coordinates, we arrive at the equation

$$t_0^3 + t_0t_1^2 + t_2^3 + t_2t_3^2 = 0. \tag{9.76}$$

- σ is of type 8A.

The square of σ is an element of order 4 from the conjugacy class 4A. So we can choose coordinates such that the equation of S is reduced to the form

$$t_3^2t_2 + t_2^2g(t_0, t_1) + h_3(t_0, t_1) = 0$$

and σ acts via $\mathrm{diag}[1, \epsilon_8^4, \epsilon_8^i, \epsilon_8]$, where $i = 2$ or $i = 6$. If $i = 2$, the equation is invariant only if it is of the form

$$t_3^2t_2 + at_2^2t_0 + t_1(bt_0^2 + ct_1^2) = 0.$$

If $i = 6$, we must have

$$t_3^2t_2 + at_2^2t_1 + t_0(bt_0^2 + ct_1^2) = 0.$$

Switching t_0, t_1, we may choose the second one. Here $a, b, c \neq 0$ because otherwise S is singular. After scaling the unknowns, we get the equation

$$t_3^2 t_2 + t_2^2 t_1 + t_0^3 + t_0 t_1^2 = 0. \tag{9.77}$$

- σ is of type 9A.

The cube of σ is an element of type 3A. So, the surface S is a cyclic surface with Equation (9.65) with σ acting via $\mathrm{diag}[1, \epsilon_3^i, \epsilon_3^j, \epsilon_9]$. All monomials entering the equation must be eigenvectors with eigenvalue ϵ_3. Thus, no cubes of the variables t_0, t_1, t_2 enter the equation. Since S is nonsingular, t_0^2, t_1^2, t_2^2 must divide some of the monomials. This gives, up to a permutation of variables, the equation

$$t_3^3 + t_0^2 t_1 + t_1^2 t_2 + t_2^2 t_0 = 0. \tag{9.78}$$

- σ is of type 12A.

The square of σ is an element of order 6 of type 6A. So, the equation can be reduced to the form

$$t_3^3 + t_2^2(at_0 + bt_1) + h_3(t_0, t_1) = 0.$$

The automorphism acts via the matrix $\mathrm{diag}[1, -1, \epsilon_4^i, \epsilon_6]$, where $i = 1$ or 3. This forces us to take $b = 0$ and h_3 to be a linear combination of monomials $t_1^3, t_0^2 t_1$. After scaling the coordinates, we arrive at the equation

$$t_3^3 + t_1^3 + t_2^2 t_0 + t_0^2 t_1 = 0. \tag{9.79}$$

We sum up our findings in Table 9.5.

9.5.2 Maximal subgroups of $W(\mathbf{E}_6)$

We will need some known information about the structure of the Weyl group of type E_6.

Theorem 9.5.2 *Let H be a maximal subgroup of $W(\mathbf{E}_6)$. Then one of the following cases occurs:*

(i) $H \cong 2^4 : \mathfrak{S}_5$ *of order* $2^4 \cdot 5!$ *and index 27;*
(ii) $H \cong \mathfrak{S}_6 \times 2$ *of order* $2 \cdot 6!$ *and index 36;*
(iii) $H \cong 3_+^{1+2} : 2\mathfrak{S}_4$ *of order* 1296 *and index 40;*
(iv) $H \cong \mathfrak{S}_3 \wr \mathfrak{S}_3 \cong 3^3 : (\mathfrak{S}_4 \times 2)$ *of order* 1296 *and index 40;*
(v) $H \cong \big(2.(\mathfrak{A}_4 \times \mathfrak{A}_4).2\big).2$ *of order* 1152 *and index 45.*

σ	Action	Equation	Fix(σ)
2A	$[1,1,1,-1]$	$t_3^2 l(t_0,t_1,t_2)+t_0^3+t_1^3+t_2^3+ct_0t_1t_2$	cubic+1 pt.
2B	$[1,1,-1,-1]$	$t_0t_2(t_2+at_3)+t_1t_3(t_3+bt_2)+t_0^3+t_1^3$	line+1 pt.
3A	$[1,1,1,\epsilon_3]$	$t_3^3+t_0^3+t_1^3+t_2^3+at_0t_1t_2$	cubic
3C	$[1,1,\epsilon_3,\epsilon_3]$	$t_0^3+t_1^3+t_2^3+t_3^3$	6 pts.
3D	$[1,1,\epsilon_3,\epsilon_3^2]$	$t_2^3+t_3^3+t_2t_3(t_0+at_1)+t_0^3+t_1^3$	3 pts.
4A	$[1,1,-1,i]$	$t_3^2t_2+t_2^2t_1+t_0^3+at_0t_1^2+bt_1^3$	5 pts.
4B	$[1,-1,i,-i]$	$t_0t_2t_3+t_1(at_2^2+bt_3^2)+t_0^3+bt_0t_1^2$	3 pts.
5A	$[1,\epsilon_5,\epsilon_5^2,\epsilon_5^3]$	$t_0^2t_2+t_1^2t_0+t_2^2t_3+t_3^2t_0$	4 pts.
6A	$[1,1,-1,\epsilon_3]$	$t_3^3+t_2^2(t_0+bt_1)+t_0^3+t_1^3$	4 pts.
6C	$[1,1,\epsilon_3^2,\epsilon_6]$	$t_3^2t_2+t_2^3+t_0^3+t_1^3$	4 pts.
6E	$[1,\epsilon_3^2,\epsilon_3,\epsilon_6]$	$t_3^2t_1+t_0^3+t_1^3+t_2^3+at_0t_1t_2$	1 pt.
6F	$[1,-1,\epsilon_3^2,\epsilon_6]$	$t_0^3+t_0t_1^2+t_2^3+t_2t_3^2$	2 pts.
8A	$[1,-1,-i,\epsilon_8]$	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2$	3 pts.
9A	$[1,\epsilon_3,\epsilon_3^2,\epsilon_9]$	$t_3^3+t_0^2t_1+t_1^2t_2+t_2^2t_0$	3 pts.
12A	$[1,-1,i,\epsilon_6]$	$t_3^3+t_1^3+t_2^2t_0+t_0^2t_1$	2 pts.

Table 9.5 *Cyclic groups of automorphisms*

Here we use the notations from the Atlas [125], where $\mathbb{Z}/n\mathbb{Z}=n$, semi-direct products: $H\ltimes G=H:G$, 3_+^{1+2} denotes the group of order 3^3 of exponent p, and $A.B$ denote a group with normal subgroup isomorphic to A and quotient isomorphic to B.

Let us identify the group $W(\mathbf{E}_6)$ with the Weyl group of the lattice $K_S^{\perp}$ defined by a nonsingular cubic surface S. We recognize a maximal subgroup from (i) as the stabilizer subgroup of a line on S.

A maximal subgroup H of type (ii) is the stabilizer subgroup of a double-six. Its subgroup isomorphic to $\mathfrak{S}_6$ permutes lines in one of the sixes.

I do not know a geometric interpretation of a maximal subgroup of type (iii).

By Theorem 9.1.6, a maximal subgroup of type (iv) is isomorphic to the stabilizer subgroup of a Steiner complex of triads of double-sixes. It is also coincides with a stabilizer subgroup of the root sublattice of type $A_2+A_2+A_2$. There is another interpretation of this subgroup in terms of a compactification of the moduli space of cubic surfaces (see [412]).

A maximal subgroup of type (v) is the stabilizer subgroup of a tritangent plane.

Proposition 9.5.3 *$W(\mathbf{E}_6)$ contains a unique normal subgroup $W(\mathbf{E}_6)'$. It is a simple group and its index is equal to 2.*

Proof Choose a root basis $(\alpha_1,\dots,\alpha_6)$ in the root lattice $\mathbf{E}_6$. Let $s_0,\dots,s_5$ be the corresponding simple reflections. Each element $w\in W(\mathbf{E}_6)$ can be

written as a product of the simple reflections. Let $\ell(w)$ be the minimal length of the word needed to write w as such a product. For example, $\ell(1) = 0, \ell(s_i) = 1$. One shows that the function $\ell : W(\mathbf{E}_6) \to \mathbb{Z}/2\mathbb{Z},\ w \mapsto \ell(w) \mod 2$ is a homomorphism of groups. Its kernel $W(\mathbf{E}_6)'$ is a subgroup of index 2. The restriction of the function ℓ to the subgroup $H \cong \mathfrak{S}_6$ generated by the reflections $s_1, \ldots, s_5$ is the sign function. Suppose K is a normal subgroup of $W(\mathbf{E}_6)'$. Then $K \cap H$ is either trivial or equal to the alternating subgroup $\mathfrak{A}_6$ of index 2. It remains for us to use the fact that $H \times (r)$ is a maximal subgroup of $W(\mathbf{E}_6)$ and s is a reflection which does not belong to $W(\mathbf{E}_6)'$. □

Remark 9.5.4 Recall that we have an isomorphism (9.8) of groups

$$W(\mathbf{E}_6) \cong \mathrm{O}(6, \mathbb{F}_2)^-.$$

The subgroup $W(\mathbf{E}_6)'$ is isomorphic to the commutator subgroup of $\mathrm{O}(6, \mathbb{F}_2)^-$.

Let us mention other realizations of the Weyl group $W(\mathbf{E}_6)$.

Proposition 9.5.5

$$W(\mathbf{E}_6)' \cong \mathrm{SU}_4(2),$$

where $\mathrm{SU}_4(2)$ *is the group of linear transformations with determinant 1 of* $\mathbb{F}_4^4$ *preserving a nondegenerate Hermitian product with respect to the Frobenius automorphism of* $\mathbb{F}_4$.

Proof Let $\mathbf{F} : x \mapsto x^2$ be the Frobenius automorphism of $\mathbb{F}_4$. We view the expression

$$\sum_{i=0}^{3} t_i^3 = \sum_{i=0}^{3} t_i \mathbf{F}(t_i)$$

as a nondegenerate Hermitian form in $\mathbb{F}_4^4$. Thus $\mathrm{SU}_4(2)$ is isomorphic to the subgroup of the automorphism group of the cubic surface S defined by the equation

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0$$

over the field $\bar{\mathbb{F}}_2$. The Weyl representation (which is defined for nonsingular cubic surfaces over fields of arbitrary characteristic) of $\mathrm{Aut}(S)$ defines a homomorphism $\mathrm{SU}_4(2) \to W(\mathbf{E}_6)$. The group $\mathrm{SU}_4(2)$ is known to be simple and of order equal to $\frac{1}{2}|W(\mathbf{E}_6)|$. This defines an isomorphism $\mathrm{SU}_4(2) \cong W(\mathbf{E}_6)'$. □

Proposition 9.5.6

$$W(\mathbf{E}_6) \cong \mathrm{SO}(5, \mathbb{F}_3), \quad W(\mathbf{E}_6)' \cong \mathrm{SO}(5, \mathbb{F}_3)^+,$$

where $\mathrm{SO}(5,\mathbb{F}_3)^+$ *is the subgroup of elements of spinor norm 1.*

Proof Let $V = \mathbf{E}_6/3\mathbf{E}_6$. Since the discriminant of the lattice $\mathbf{E}_6$ is equal to 3, the symmetric bilinear form defined by

$$\langle v + 3\mathbf{E}_6, w + 3\mathbf{E}_6\rangle = -(v,w) \mod 3$$

is degenerate. It has a 1-dimensional radical spanned by the vector

$$v_0 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_4 + \boldsymbol{\alpha}_5 \mod 3\mathbf{E}_6.$$

The quadratic form $q(v) = (v,v) \mod 3$ defines a nondegenerate quadratic form on $\bar{V} = V/\mathbb{F}_3 v_0 \cong \mathbb{F}_3^5$. We have a natural injective homomorphism $W(\mathbf{E}_6) \to \mathrm{O}(5,\mathbb{F}_2)$. Comparing the orders, we find that the image is a subgroup of index 2. It must coincide with $\mathrm{SO}(5,\mathbb{F}_3)$. Its unique normal subgroup of index 2 is $\mathrm{SO}(5,\mathbb{F}_3)^+$. □

Remark 9.5.7 Let E be a vector space of odd dimension $2k+1$ over a finite field $\mathbb{F}_q$ equipped with a nondegenerate symmetric bilinear form. An element $v \in E$ is called a *plus vector* (resp. *minus vector*) if (v,v) is a square in $\mathbb{F}_q^*$ (resp. is not a square $\in \mathbb{F}_q^*$). The orthogonal group $\mathrm{O}(E)$ has three orbits in $|E|$: the set of isotropic lines, the set of lines spanned by a plus vector and the set of lines spanned by a minus vector. The isotropic subgroup of a non-isotropic vector v is isomorphic to the orthogonal group of the subspace $v^\perp$. The restriction of the quadratic form to $v^\perp$ is of Witt index k if v is a plus vector and of Witt index $k-1$ if v is a minus vector. Thus the stabilizer group is isomorphic to $\mathrm{O}(2k,\mathbb{F}_q)^\pm$. In our case, when $k=2$ and $q=3$, we obtain that minus vectors correspond to cosets of roots in $V = \mathbf{E}_6/3\mathbf{E}_6$, hence the stabilizer of a minus vector is isomorphic to the stabilizer of a double-six, i.e. a maximal subgroup of $W(\mathbf{E}_6)$ of index 36. The stabilizer subgroup of a plus vector is a group of index 45 and isomorphic to the stabilizer of a tritangent plane. The stabilizer of an isotropic plane is a maximal subgroup of type (iii), and the stabilizer subgroup of an isotropic line is a maximal subgroup of type (iv).

9.5.3 Groups of automorphisms

Now we are ready to classify all possible subgroups of automorphisms of a nonsingular cubic surface.

In Table 9.6 we use the notation $\mathcal{H}_3(3)$ for the *Heisenberg group* of unipotent 3×3-matrices with entries in $\mathbb{F}_3$.

Theorem 9.5.8 *The following is the list of all possible groups of automorphisms of nonsingular cubic surfaces.*

Type	Order	Structure	$f(t_0,t_1,t_2,t_3)$	Eckardt
I	648	$3^3:\mathfrak{S}_4$	$t_0^3+t_1^3+t_2^3+t_3^3$	18
II	120	$\mathfrak{S}_5$	$t_0^2t_1+t_1^2t_2+t_2^2t_3+t_3^2t_0$	10
III	108	$\mathcal{H}_3(3):4$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	9
IV	54	$\mathcal{H}_3(3):2$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	9
V	24	$\mathfrak{S}_4$	$t_0^3+t_0(t_1^2+t_2^2+t_3^2)+at_1t_2t_3$	6
VI	12	$\mathfrak{S}_3\times 2$	$t_2^3+t_3^3+at_2t_3(t_0+t_1)+t_0^3+t_1^3$	4
VII	8	8	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2$	1
VIII	6	$\mathfrak{S}_3$	$t_2^3+t_3^3+at_2t_3(t_0+bt_1)+t_0^3+t_1^3$	3
IX	4	4	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2+at_1^3$	1
X	4	2^2	$t_0^2(t_1+t_2+bt_3)+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	2
XI	2	2	$t_0^2(t_1+bt_2+ct_3)+t_1^3+t_2^3+t_3^3+ct_1t_2t_3$	1

Table 9.6 *Groups of automorphisms of cubic surfaces*

Here, in the third row, α is a root of the equation $8x^6+20x^3-1=0$, and in the next row, $a\neq\alpha$ and also $a\neq a^4$, otherwise the surface is of Type II. Similar restrictions must be made for other parameters. There are also conditions for the surface to be nonsingular.

Proof Let S be a nonsingular cubic surface.

- Suppose $\mathrm{Aut}(S)$ contains an element from the conjugacy class 3C.

Table 9.5 shows that S is isomorphic to the Fermat cubic $V(t_0^3+t_1^3+t_2^3+t_3^3)$. Obviously, its automorphism group contains a subgroup G isomorphic to $3^3:\mathfrak{S}_4$. To see that it coincides with this group, we use that G is a subgroup of index 2 of a maximal subgroup H of type (iv). As we noted before, the group H is the stabilizer subgroup of a root lattice $A_2+A_2+A_2$. It contains an element represented by a reflection in one copy of the lattice, and the identity on other copies. This element has trace equal to 4, so belongs to the conjugacy class 2C. It is not realized by an automorphism of a nonsingular cubic surface. This gives Type I from the Table.

- Suppose $\mathrm{Aut}(S)$ contains an element of order 5.

Table 9.5 shows that S is isomorphic to the surface

$$t_0^2t_1+t_1^2t_2+t_2^2t_3+t_3^2t_0=0. \tag{9.80}$$

Consider the embedding of S in $\mathbb{P}^4$ given by the linear functions

$$\begin{aligned} z_0 &= t_0 + t_1 + t_2 + t_3, \\ z_1 &= \epsilon t_0 + \epsilon^3 t_1 + \epsilon^4 t_2 + \epsilon^2 t_3, \\ z_2 &= \epsilon^2 t_0 + \epsilon t_1 + \epsilon^3 t_2 + \epsilon^4 t_3, \\ z_3 &= \epsilon^3 t_0 + \epsilon^4 t_1 + \epsilon^2 t_2 + \epsilon t_3, \\ z_4 &= \epsilon^4 t_0 + \epsilon^2 t_1 + \epsilon t_2 + \epsilon^3 t_3, \end{aligned} \tag{9.81}$$

where $\epsilon^5 = 1$. One checks that $\sum_{i=0}^4 z_i = 0$ and (9.80) implies that also $\sum_{i=0}^4 z_i^3 = 0$. This shows that S is isomorphic to the following surface in $\mathbb{P}^4$:

$$\sum_{i=0}^4 z_i^3 = \sum_{i=0}^4 z_i = 0. \tag{9.82}$$

These equations exhibit a subgroup G of automorphisms of S isomorphic to $\mathfrak{S}_5$.

Assume that G is a proper subgroup of $\mathrm{Aut}(S)$. Note that the only maximal subgroup of $W(\mathbf{E}_6)$ that contains a subgroup isomorphic to $\mathfrak{S}_5$ is a subgroup H of type (i) or (ii). If H is of type (i), then $\mathrm{Aut}(S)$ contains one of the involutions from the subgroup 2^4. The group H is isomorphic to the Weyl group $W(D_5)$. We encountered it as the Weyl group of a del Pezzo surface of degree 4. It follows from the proof of Proposition 8.6.7 that nontrivial elements of the subgroup 2^4 are conjugate to the composition of reflection $s_{\alpha_1} \circ s_{\alpha_5}$. Its trace is equal to 1. Thus, this element belongs to the conjugacy class 2D that is not realized by an automorphism. If H is of type (ii), then G is contained in $\mathfrak{S}_6$ or contains an element which commutes with G. It is immediately seen that the surface does not admit an involution which commutes with all elements in G. Since $\mathfrak{S}_5$ is a maximal subgroup of $\mathfrak{S}_6$, in the first case, we obtain that $\mathrm{Aut}(S)$ contains a subgroup isomorphic to $\mathfrak{S}_6$. However, a cyclic permutation g of order 6 acts on $\mathbf{E}_6$ by cyclically permuting vectors $\mathbf{e}_1, \dots, \mathbf{e}_6$ and leaving $\mathbf{e}_0$ invariant. Its trace is equal to 1. This shows that g belongs to the conjugacy class 6I and is not realized by an automorphism. This gives us type II from Table 9.5.

- Suppose $\mathrm{Aut}(S)$ contains an element of type 3A.

From Table 9.5, we infer that S is a cyclic surface which is projectively isomorphic to the surface go $S = V(t_0^3 + t_1^3 + t_2^3 + t_3^3 + at_1t_2t_3)$. Obviously, it contains a group of automorphisms G isomorphic to $3.(3^2 : 2)$. The central element of order 3 is realized by the matrix $\mathrm{diag}[1, 1, 1, \epsilon_3]$. The quotient

group is isomorphic to a group of projective automorphisms of the plane cubic $C = S \cap V(t_3)$. In the group law, the group is generated by translations by points of order 3 and the inversion automorphism. For special parameter a we get more automorphisms corresponding to a harmonic or an equianharmonic cubic. Let us see that there is nothing else in $\mathrm{Aut}(S)$. An equianharmonic cubic is projectively isomorphic to the Fermat cubic, so it will give Type I. The remaining two cases will give us surfaces of types III and IV.

The subgroup 3.3^2 is isomorphic to the Heisenberg group $\mathcal{H}_3(3)$ of upper-triangular 3×3 matrices with entries in $\mathbb{F}_3$ with 1 at the diagonal. In the notation of the Atlas, it is group 3_+^{1+2}. We see that it is contained in the only maximal subgroup which is of type (iii). The element generating the center of 3_+^{1+2} is a central element in the maximal subgroup. Thus any extra automorphism commutes with the central element, and hence descends to an automorphism of the cubic curve C. This proves that $G = \mathrm{Aut}(S)$.

- Suppose $\mathrm{Aut}(S)$ contains an element of type 8A.

Consulting Table (9.5), we infer that S is isomorphic to the surface of type VII. The only maximal subgroup of $W(\mathbf{E}_6)$ which contains an element of order 8 is a subgroup H of order 1152. As we know it stabilizes a tritangent plane. In our case the tritangent plane is $t_2 = 0$. It has the Eckardt point $x = [0, 0, 0, 1]$. Thus $G = \mathrm{Aut}(S)$ is a subgroup of the linear tangent space $T_x S$. If any element of G acts identically on the set of lines in the tritangent plane, then it acts identically on the projectivized tangent space, and hence G is a cyclic group. Obviously this implies that G is of order 8. Assume that there is an element τ which permutes cyclically the lines. Let G' be the subgroup generated by σ and τ. Obviously, $\tau^3 = \sigma^k$. Since G does not contain elements of order 24, we may assume that $k = 2$ or 4. Obviously, τ normalizes $\langle\sigma\rangle$ since otherwise we have two distinct cyclic groups of order 8 acting on a line with a common fixed point. It is easy to see that this is impossible. Since $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^2$ this implies that σ and τ commute. Thus $\sigma\tau$ is of order 24, which is impossible. This shows that $\mathrm{Aut}(S) \cong \mathbb{Z}/8\mathbb{Z}$.

By taking powers of elements of order 9 and 12, we obtain surfaces with automorphism groups which we have already classified. So, we may assume that $\mathrm{Aut}(S)$ does not contain elements of order 5, 8, 9, 12. In a similar manner, we may assume that any element of order 3 belongs to the conjugacy class 3D, and an element of order 6 belongs to the conjugacy class of type 6A or 6E.

- Suppose $\mathrm{Aut}(S)$ contains an element of type 3D.

Assume $\mathrm{Aut}(S)$ contains an element σ from conjugacy class $3D$. Then the surface is isomorphic to $V(t_2^3 + t_3^3 + t_2 t_3(t_0 + a t_1) + t_0^3 + t_1^3)$. We assume that

$a \neq 0$. Otherwise, the surface is cyclic, and has an automorphism of type $3A_2$. This has been already taken care of. Let τ be an involution which exchanges the coordinates t_2 and t_3. The subgroup H generated by σ and τ is isomorphic to $\mathfrak{S}_3$. The involution τ is of type 2A, it is a harmonic homology. Thus the three involutions in H define three Eckardt points x_1, x_2, x_3. They are on the line $\ell = V(t_0) \cap V(t_1)$. The group H acts faithfully on the set of the three Eckardt points.

By Proposition 9.1.27, a triple of collinear Eckardt points defines a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$. If the triples are disjoint, then the subgroups do not have a common involution, hence they intersect only at the identity. Otherwise, they have one common involution.

Suppose we have an automorphism $g \notin H$. If $gHg^{-1} = H$, then, replacing g with the product with some involution in H, we may assume that g commutes with σ. This shows that we can simultaneously diagonalize the matrices representing g and σ. It is immediately checked from the equation of the surface that this is possible only if $a = 1$ and g is the transformation which switches t_0 and t_1. So, if $gHg^{-1} \neq H$, we obtain that $\mathrm{Aut}(S)$ is isomorphic to $\mathfrak{S}_3$ or $2 \times \mathfrak{S}_3$. This gives types VI and VIII.

Let us assume that $H' = gHg^{-1} \neq H$. Then H' is the subgroup defined by the three Eckardt points y_i on the line $\ell' = g(\ell)$. Since each of the involutions corresponding to the points x_i commutes with at most one involution corresponding to the points y_i, we obtain that one of the lines $\overline{x_i y_j}$ contains the third Eckardt point and defines a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$ which one common involution with H. Replacing H' with this subgroup, we may assume that the lines ℓ and ℓ' intersect at $x_1 = y_1$ and, hence span a plane Π. Each of the pairs of lines $(\overline{x_i y_2}, \overline{x_i y_3}), i = 2, 3$, contains at most one line contained in S. Applying Proposition 9.1.27, we either get a complete quadrilateral in Π with 6 Eckardt points as its vertices and its three diagonals lying on S, or we get more than 9 Eckardt points on Π. Note that a plane section of S not containing a line on S intersects the 27 lines at 27 points, an Eckardt point is counted with multiplicity 3. This shows that an irreducible plane section of S contains ≤ 9 Eckardt points. If it contains a line with two Eckardt points on it, then the number is at most seven. This eliminates the second possibility. It follows from the structure of $W(E_6)$, that the first possibility gives that the four subgroups isomorphic to $\mathfrak{S}_3$ defined by the sides of the quadrilateraal generate a subgroup G of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_4$. The list of maximal subgroups of $W(E_6)$ shows that either $G = \mathrm{Aut}(S)$, or $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ and hence S is the Clebsch diagonal surface given by equation (9.80).

- Suppose $\mathrm{Aut}(S)$ contains an element of type $2B$.

Then the equation of the surface is (9.64). After a linear change of the variables t_0, t_1 the equation can be reduced to equation as (9.69). So we get the surface of type V.

- Suppose $\mathrm{Aut}(S)$ contains an element of order 4.

If σ belongs to the conjugacy class 4B, then the equation must be as in (9.69). Then $\mathrm{Aut}(S)$ contains an additional cyclic group of type 3D. This leads to a surface of type V with $\mathrm{Aut}(S) \cong \mathfrak{S}_4$. If σ belongs to $4A$ then the equation of the surface is (9.67). This is a cubic surface of type IX with cyclic group of automorphisms of order 4. Here we have to assume that the surface is not isomorphic to the surface of type VII. It follows from the proof of the next Corollary that in all previous cases, except type VII, the automorphism group is generated by involutions of type $4A_1$. Thus our surface cannot be reduced to one of the previous cases.

Finally, it remains for us to consider the following case.

- $\mathrm{Aut}(S)$ contains only involutions of type $2A$, i.e. harmonic homologies.

Suppose we have two such involutions. They define two Eckardt points x_1 and x_2. By Proposition 9.1.27, if the line $\overline{x_1x_2}$ is contained in S, then the involutions commute. If the line does not belong to S, then the two involutions generate $\mathfrak{S}_3$, and hence contains an element of order 3. Suppose we have a third involution defining a third Eckardt point x_3. Then we have a tritangent plane formed by the lines $\overline{x_ix_j}$. Obviously, it must coincide with each tritangent plane corresponding to the Eckardt points x_i. This contradiction shows that we can have at most two commuting involutions. This gives the last two cases of our theorem. The condition that there is only one involution of type $2A$ is that the line $V(t_1 + t_2 + at_3)$ does not pass through an inflection point of the plane curve $V(t_0)$. □

The next Corollary can be checked case by case, and its proof is left as an exercise.

Corollary 9.5.9 *Let* $\mathrm{Aut}(S)^o$ *be the subgroup of* $\mathrm{Aut}(S)$ *generated by involutions of type* $2A$. *Then* $\mathrm{Aut}(S)^o$ *is a normal subgroup of* $\mathrm{Aut}(S)$ *such that the quotient group is either trivial or a cyclic group of order 2 or 4. The order 4 could occur only for the surface of type VII. The order 2 occurs only for surfaces of type IX.*

Finally, we explain the last column of Table 9.6. We already noticed that the Fermat surface has 18 Eckardt points. A harmonic involution of a surface of type II corresponds to a transposition in $\mathfrak{S}_5$. Their number is equal to 10. The

surfaces of types III and IV are cyclic surfaces, we have already explained that they have nine Eckardt points. This can be also confirmed by looking at the structure of the group. A surface of type VI has four Eckardt points. They correspond to four harmonic symmetries. Three of them come from the subgroup $\mathfrak{S}_3$ and the fourth one corresponds to the central involution. Of course, we can see it in the equation. The fourth Eckardt point is $[1, -1, 1, -1]$. Surfaces of type VII and IX have one involution of type 2A. Surfaces of type X have two and surfaces of type XI have only one.

9.5.4 The Clebsch diagonal cubic

We have already defined the Clebsch diagonal surface in Example 9.1.20 as a nonsingular cubic surface given by equations

$$t_0^3 + \cdots + t_3^3 - (t_0 + t_1 + t_2 + t_3)^3 = 0.$$

In the proof of Theorem 9.5.8 we found an explicit isomorphism to the surface in $\mathbb{P}^3$ with equation

$$t_0^2 t_1 + t_1^2 t_2 + t_2^2 t_3 + t_3^2 t_0 = 0.$$

The Sylvester pentahedron of the surface is $V(t_0 t_1 t_2 t_3 (t_0 + t_1 + t_2 + t_3))$. Its ten vertices are the Eckardt points. Each edge is a line going through three Eckardt points.

Each face of the pentahedron intersects the tetrahedron formed by the other four faces along three diagonals, they are lines on S (this explains the name of the surface). In this way we get 15 lines, the $\mathfrak{S}_5$-orbit of the line

$$t_0 = t_1 + t_2 = t_3 + t_4 = 0.$$

The remaining 12 lines form a double-six. Their equations are as follows. Let η be a primitive 5-th root of unity. Let $\sigma = (a_1, \ldots, a_4)$ be a permutation of $\{1, 2, 3, 4\}$. Each line ℓ_σ spanned by the points $[1, \eta^{a_1}, \ldots, \eta^{a_4}]$ and $[1, \eta^{-a_1}, \ldots, \eta^{-a_4}]$ belongs to the surface. This gives $12 = 4!/2$ different lines. Here is one of the ordered double-sixes formed by the twelve lines

$$(\ell_{1234}, \ell_{1243}, \ell_{1324}, \ell_{1342}, \ell_{1432}, \ell_{1423}), (\ell_{2413}, \ell_{2431}, \ell_{3412}, \ell_{3421}, \ell_{2312}, \ell_{2321}). \tag{9.83}$$

The Schur quadric Q corresponding to this double-six is the quadric

$$t_0^2 + \cdots + t_4^2 = 0, \quad t_0 + \cdots + t_4 = 0.$$

For example, the polar line of ℓ_{1234} is the line given by equations

$$\sum_{i=0}^{4} \eta^i t_i = \sum_{i=0}^{4} \eta^{-i} t_i = \sum_{i=0}^{4} t_i = 0$$

and, as is easy to see, it coincides with the line ℓ_{2413}. The Schur quadric intersects ℓ_{ijkl} at two points $[1, \eta^i, \eta^j, \eta^k, \eta^l]$ and $[1, \eta^{-i}, \eta^{-j}, \eta^{-k}, \eta^{-l}]$.

The group $\mathfrak{S}_5$ (as well as its subgroup $\mathfrak{S}_4$) acts transitively on the double-six. The group $\mathfrak{A}_5$ stabilizes a sixer.

The intersection $Q \cap S$ is the *Bring curve* of genus 4 given by the equations

$$t_0^3 + \cdots + t_4^3 = t_0^2 + \cdots + t_4^2 = t_0 + \cdots + t_4 = 0.$$

Its automorphism group is isomorphic to $\mathfrak{S}_5$. The image of this curve under the map $\pi_1 : S \to \mathbb{P}^2$ which blows down the first half $(\ell_1, \dots, \ell_6)$ of the double-six (9.83) is Schur sextic with nodes at the points $p_i = \pi(\ell_i)$.

Consider the stereographic projection from the 2-dimensional sphere $\mathsf{S}^2 : \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$ to the Riemann sphere $(a, b, c) \mapsto z = \frac{a+ib}{1-c}$. A rotation around the axis $\mathbb{R}(a, b, c)$ about the angle 2ϕ corresponds to the Möbius transformation

$$z \mapsto \frac{(\alpha + i\beta)z - (\gamma - \delta i)}{(\gamma + \delta i)z + (\alpha - \beta i)},$$

where $\alpha = \cos\phi, \beta = a \sin\phi, \gamma = b \sin\phi, \delta = c \sin\phi$. The icosahedron group $\mathfrak{A}_5$ acting by rotation symmetries of an icosahedron inscribed in S^2 defines an embedding of $\mathfrak{A}_5$ in the group $\mathrm{PGL}(3)$. One can choose the latter embedding as a subgroup generated by the following transformations S, U, T of orders $5, 2, 3$ represented by the Möbius transformations

$$S : z \mapsto \eta z, \quad U : z \mapsto -z^{-1}, \quad T : z \mapsto \frac{(\eta - \eta^4)z + \eta^2 - \eta^3}{(\eta^2 - \eta^3)z + \eta^4 - \eta}.$$

The orbit of the north pole of the sphere under the corresponding group of rotations is an icosahedron. It is known that the icosahedron group has three exceptional orbits in $\mathbb{P}^1$ with stabilizers of orders $5, 3, 2$. They are the sets of zeros of the homogeneous polynomials

$$\begin{aligned}
\Phi_{12} &= z_0 z_1 (z_1^{10} + 11 z_0^5 z_1^5 - z_0^{10}), \\
\Phi_{20} &= -(z_0^{20} + z_1^{20}) + 228(z_1^{15} z_0^5 - z_1^5 z_0^{15}) - 494 z_1^{10} z_0^{10}, \\
\Phi_{30} &= z_0^{30} + z_1^{30} + 522(z_1^{25} z_0^5 - z_1^5 z_0^{25}) - 10005(z_1^{20} z_0^{10} + z_1^{10} z_0^{20}).
\end{aligned}$$

The isomorphism $\mathrm{SU}(2)/\pm 1 \to \mathrm{SO}(3)$ defines a 3-dimensional complex linear representation of $\mathfrak{A}_5$ which embeds $\mathfrak{A}_5$ in $\mathrm{PGL}(3)$. In an appropriate

coordinate system it leaves the conic $K = V(t_0^2 + t_1t_2)$ invariant. The group $\mathfrak{A}_5$ acts in the plane in such a way that the Veronese map

$$[z_0, z_1] \mapsto [-z_0z_1, z_0^2, -z_1^2] \tag{9.84}$$

is equivariant. The six lines

$$V(t_1), \quad V(t_2), \quad V(t_0 + \eta^i t_1 + \eta^{-i} t_2), \quad i = 0, 1, 2, 3, 4, \tag{9.85}$$

cut out on K the set

$$V(\Phi_{12}) = \{0, \infty, \eta^i(\eta + \eta^{-1}), \eta^i(\eta^2 + \eta^{-2})\}, \quad i = 0, \dots, 4.$$

The poles of the six lines with respect to the conic is the set of six points

$$[1, 0, 0], \ \ [1, 2\eta^i, 2\eta^{-i}], \ i = 0, 1, 2, 3, 4.$$

They are called the *fundamental set of points*.

The image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined by the five cubics $(F_0, F_1, F_2, F_3, F_4)$

$$F_i = \eta^i(4t_0^2t_2 - t_1t_2^2) + \eta^{2i}(-2t_0t_2^2 + t_1^3) + \eta^{3i}(2t_0t_1^2 - t_2^3) + \eta^{4i}(-4t_0^2t_1 + t_1^2t_2),$$

is the Clebsch diagonal cubic given by equations (9.13) (see [340], II, 1, §5).

The equation of the Schur sextic (also called the *Klein sextic* in this case) is

$$B = F_0^2 + \cdots + F_4^2 = 10(4t_0^2t_2 - t_1t_2^2)(-4t_0^2t_1 + t_1^2t_2) + 10(-2t_0t_2^2 + t_1^3)(2t_0t_1^2 - t_2^3)$$

$$= -20(8t_0^4t_1t_2 - 2t_0^2t_1^2t_2^2 + t_1^3t_2^3 - t_0t_1^5 - t_0t_2^5) = 0.$$

The 12 intersection points of the sextic with the conic K are the images of the 12 roots of Φ_{12} under the Veronese map (9.84). The images of 30 roots of Φ_{30} are the intersection points of K with the union of 15 lines joining pairwise the six fundamental points. Let D be the product of the linear forms defining these lines

$$\eta^\nu t_1 - \eta^{-i} t_2, \ (1 + \sqrt{5})t_0 + \eta^i t_1 + \eta^{-i} t_2, \ (1 - \sqrt{5})t_0 + \eta^{-i} t_1 + \eta^i t_2.$$

The images of these lines under the map given by the polynomials F_i's are the 15 diagonals of the Clebsch cubic. The images of 20 roots of Φ_{20} are cut out by an invariant curve of degree 10 given by equation

$$C = G_0^2 + G_1G_2$$

where

$$G_0 = -8t_0^3t_1t_2 + 6t_0t_1^2t_2^2 - t_1^5 - t_2^5,$$
$$G_1 = 16t_0^3t_2^2 - 8t_0^2t_1^3 - 4t_0t_1t_2^3 + 2t_1^4t_2,$$
$$G_2 = 16t_0^2t_1^3 - 8t_0^2t_2^3 - 4t_0t_1^3t_2 + 2t_1t_2^4$$

are quintic polynomials which define the $\mathfrak{A}_5$-equivariant symmetric Cremona transformation of degree 5. The curve $V(C)$ (with the source and the target identified via the duality defined by the conic K) is equal to the image of the conic K. The curve $V(C)$ is a rational curve which has each fundamental point as its singular points of multiplicity 4 with two ordinary cuspidal branches.

The four polynomials of degrees 2, 6, 10 and 15

$$A = t_0^2 + t_1 t_2, \quad B, \quad C, \quad D$$

generate the algebra $\mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5}$ of invariant polynomials. The relation between the fundamental invariants is

$$D^2 = -1728B^5 + C^3 + 720ACB^3 - 80A^2C^2B + 64A^3(5B^2 - AC)^2$$

(see [340], II, 4, §3). The even part of the graded ring $\mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5}$ is freely generated by polynomials A, B, C of degrees $2, 6, 10$, so that

$$\mathbb{P}^2/\mathfrak{A}_5 \cong \text{Proj}\, \mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5} \cong \mathbb{P}(1, 3, 5).$$

Remark 9.5.10 The Clebsch diagonal surface and the Bring curve of genus 4 play a role in the theory of modular forms. Thus, the Bring curve is isomorphic to the modular curve $\overline{\mathcal{H}/\Gamma}$, where $\Gamma = \Gamma_0(2) \cap \Gamma(5)$. It is also realized as the curve of fixed points of the Bertini involution on the del Pezzo surface of degree 1 obtained from the elliptic modular surface $S(5)$ of level 5 by blowing down the zero section [59], [410]. The blow-up of the Clebsch diagonal surface at its 10 Eckardt points is isomorphic to a minimal resolution of the *Hilbert modular surface* $\overline{\mathcal{H} \times \mathcal{H}/\Gamma}$, where Γ is the 2-level principal congruence subgroup of the Hilbert modular group associated to the real field $\mathbb{Q}(\sqrt{5})$ [296]. The curve C of degree 10 is isomorphic to the image of the diagonal in $\mathcal{H} \times \mathcal{H}$ under the involution switching the factors [297].

Remark 9.5.11 The pencil of curves of degree 6 $\lambda A^3 + \mu B = 0$ has remarkable properties, studied by R. Winger [603]. It has 12 base points, each point is an inflection point for all members of the pencil. The curves share common tangents at these points. They are the six lines (9.85). These lines count for 12 inflection tangents and 24 bitangents of each curve. The pencil contains three singular fibres: the curve $V(B)$, the union of the six lines, and a rational curve W with ten nodes forming an orbit of $\mathfrak{A}_5$ with stabilizer subgroup isomorphic to $\mathfrak{S}_3$. The union of the lines corresponds to the parameter $[\lambda, \mu] = [1, -1]$. The rational sextic corresponds to the parameter $[\lambda, \mu] = [32, 27]$. Other remarkable members of the pencil correspond to the parameter $[1 + \alpha, -\alpha]$, where $\alpha = (-9 \pm 3\sqrt{-15})/20$. These are the nonsingular *Valentiner sextics* with automorphism group isomorphic to $\mathfrak{A}_6$.

Let $\mathrm{PSO}(3) \cong \mathrm{SO}(3)$ be the group of projective automorphisms leaving invariant the conic $K = V(t_0^2 + t_1 t_2)$. Via the Veronese map it is isomorphic to $\mathrm{PSL}(2)$. We have described explicitly the embedding $\iota : \mathfrak{A}_5 \hookrightarrow \mathrm{SO}(3)$. There are two non-isomorphic 3-dimensional irreducible representations of $\mathfrak{A}_5$ dual to each other. Note that the transformations S and S^{-1} are not conjugate in $\mathfrak{A}_5$, so that the dual representations are not isomorphic. In our representation the trace of S is equal to $1 + \eta + \eta^{-1} = 1 + 2\cos 2\pi/5 = (1 + \sqrt{5})/2$ and, in the dual representation, the trace of S is equal to $1 + \eta^2 + \eta^{-2} = 2\cos 4\pi/5 = (1 - \sqrt{5})/2$. The polar lines of the fundamental set of six points define the fundamental set in the dual representation. Thus each subgroup of $\mathrm{SO}(3)$ isomorphic to $\mathfrak{A}_5$ defines two sets of fundamental points, one in each of the two dual planes. We call them *icosahedral sets* of six points. The group $\mathrm{SO}(3)$ acts by conjugation on the set of subgroups isomorphic to $\mathfrak{A}_5$, with two conjugacy classes. This shows that the set of dual pairs of fundamental sets is parameterized by the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$.

The six fundamental lines (9.85) form a polar hexagon of the double conic $V(A^2)$ as the following identity shows (see [400], p. 261):

$$30(t_0^2 + t_1 t_2)^2 = 25t_0^4 + \sum (t_0 + \eta^i t_1 + \eta^{-i} t_2)^4.$$

This shows that an icosahedral set in the dual plane is a polar hexagons of A^2. Hence $\mathrm{VSP}(A^2, 6)$ contains a subvariety isomorphic to the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$. As we have explained in Section 1.4.4, this variety embeds into the Grassmannian $G(3, \mathcal{H}^3)$, where $\mathcal{H}^3$ is the 7-dimensional linear space of cubic harmonic polynomial with respect to the quadratic form $q = t_0^2 + t_1 t_2$. Its closure is the subvariety of $G(3, \mathcal{H}^3)_\sigma$ of $G(3, \mathcal{H}^3)$ of subspaces isotropic to the Mukai skew forms σ_{ω, A^2}. It is a smooth irreducible Fano variety of genus 12 (see [400]). A compactification of the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$ isomorphic to $G(3, \mathcal{H}^3)_\sigma$ was constructed earlier by S. Mukai and H. Umemura in [397]. It is isomorphic to the closure of the orbit of $\mathrm{SL}(2)$ acting on the projective space of binary forms of degree 12.

Recall that the dual of the 4-dimensional space of cubic polynomials vanishing at the polar hexagon is a 3-dimensional subspace of $\mathcal{H}^3$ which is isotropic with respect to Mukai's skew forms. It follows from Theorem 6.3.32 that the variety $\mathrm{VSP}(A^2, 6)$ is the closure of $\mathrm{SO}(3)/\mathfrak{A}_5$ and isomorphic to a Fano variety of genus 12.

Observe that the cubic polynomials F_ν are harmonic with respect to the Laplace operator corresponding to the dual quadratic form $q^\vee = -\frac{1}{4}\xi_0^2 + \xi_1\xi_2$. Thus each fundamental set in the plane defines a 4-dimensional subspace of the space ${}'\mathcal{H}^3$ of harmonic cubic polynomials with respect to $q^\vee$. This space is dual to the 3–dimensional subspace in $\mathcal{H}^3$ defined by the dual fundamental set with

respect to the polarity pairing $\mathcal{H}^3 \times' \mathcal{H}^3 \to \mathbb{C}$. Note that the intersection of two 4-dimensional subspaces in the 7-dimensional space $'\mathcal{H}^3$ of cubic polynomials is nonzero. Thus for each two fundamental sets there is a harmonic polynomial vanishing at the both sets. One can show that the set of harmonic cubic curves vanishing at infinitely many fundamental sets is parameterized by a surface in the dual projective space $\mathbb{P}^3$ which is isomorphic to the Clebsch diagonal cubic surface under the map given by the Schur quadric (see [300]).

We refer to [382] for more of the beautiful geometry associated to the Bring curve and the Clebsch diagonal cubic surface.

Exercises

9.1 Let T_{st} be the standard Cremona transformation, considered as a biregular automorphism σ of a nonsingular del Pezzo surface S of degree 6. Show that the orbit space $S/\langle\sigma\rangle$ is isomorphic to a Cayley 4-nodal cubic surface.

9.2 Show that a cubic surface can be obtained as the blow-up of five points on $\mathbb{P}^1 \times \mathbb{P}^1$. Find the conditions on the five points such that the blow-up is isomorphic to a nonsingular cubic surface. Show that each pair of skew lines on a cubic surface is intersected by five skew lines which can be blown down to 5 points on a nonsingular quadric.

9.3 Compute the number of m-tuples of skew lines on a nonsingular surface for $m = 2, 3, 4, 5$.

9.4 Suppose a quadric intersects a cubic surface along the union of three conics. Show that the three planes defined by the conics pass through three lines in a tritangent plane.

9.5 Let Γ and Γ' be two rational normal cubics in $\mathbb{P}^3$ containing a common point p. For a general plane Π through p let $\Pi \cap \Gamma = \{p, p_1, p_2\}, \Pi \cap \Gamma' = \{p, p_1', p_2'\}$ and $f(p) = \overline{p_1 p_1} \cap \overline{p_1' p_2'}$. Consider the set of planes through p as a hyperplane H in the dual space $(\mathbb{P}^3)^\vee$. Show that the image of the rational map $H \dashrightarrow \mathbb{P}^3, \Pi \mapsto f(p)$ is a nonsingular cubic surface and every such cubic surface can be obtained in this way.

9.6 Show that the linear system of quadrics in $\mathbb{P}^3$ spanned by quadrics which contain a degree 3 rational curve on a nonsingular cubic surface S can be spanned by the quadrics defined by the minors of a matrix defining a determinantal representation of S.

9.7 Show that all singular surfaces of type VII, or X, pr XI, or XIII-XXI are isomorphic and there are two non-isomorphic surfaces of type XII.

9.8 Show that a cubic surface with three nodes is isomorphic to a surface $V(w^3 + w(xy + yz + xz) + \lambda xyz)$. Show that the surface admits an Eckardt point if and only if $\lambda = \pm\sqrt{-2}$.

9.9 Let ℓ be a line on a del Pezzo cubic surface and E be its proper inverse transform on the corresponding weak del Pezzo surface X. Let $\mathcal{N}$ be the sublattice of $\text{Pic}(X)$ spanned by irreducible components of exceptional divisors of $\pi : X \to S$. Define the multiplicity of ℓ by

$$m(\ell) = \frac{\#\{\sigma \in \text{O}(\text{Pic}(X)) : \sigma(E) - E \in \mathcal{N}\}}{\#\{\sigma \in \text{O}(\text{Pic}(X)) : \sigma(E) = E\}}.$$

Show that the sum of the multiplicities is always equal to 27.

9.10 Show that the 24 points of intersection of a Schur quadric with the corresponding double-six lie on the Hessian of the surface ([21], vol. 3, p. 211).

9.11 Consider a Cayley-Salmon equation $l_1l_2l_3 - l_1'l_2'l_3' = 0$ of a nonsingular cubic surface.

(i) Show that the six linear polynomials l_i, l_i' satisfy the following linear equations

$$\sum_{j=1}^{3} a_{ij}l_j = \sum_{j=1}^{3} a_{ij}'l_j' = 0,\ i = 1,2,3,$$

where

$$\sum_{i=1}^{3} a_{ij} = 0,\ j = 1,2,3, \quad a_{i1}a_{i2}a_{i3} = a_{i1}'a_{i2}'a_{i3}',\ i = 1,2,3.$$

(ii) Show that for each $i = 1,2,3$ the nine planes

$$a_{ij}l_i - a_{ij}'l_j' = 0,\ i,j = 1,2,3$$

contain 18 lines common to three planes. The 18 lines obtained in this way form three double-sixes associated to the pair of conjugate triads defined by the Cayley-Salmon equation.

(iii) Show that the Schur quadrics defined by the three double-sixes can be defined by the equations

$$\sum_{j=1}^{3} a_{2j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{2j}a_{3j}{l_j'}^2 = 0,$$
$$\sum_{j=1}^{3} a_{1j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{3j}{l_j'}^2 = 0,$$
$$\sum_{j=1}^{3} a_{1j}a_{2j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{2j}{l_j'}^2 = 0$$

([172]).

9.12 Prove the following theorem of Schläfli: given five skew lines in $\mathbb{P}^3$ and a line intersecting them all, there exists a unique cubic surface that contains a double-six including the seven lines ([173]).

9.13 For each type of a cubic surface with nontrivial group of automorphisms, find its Cremona hexahedral equations.

9.14 Show that the pull-back of a bracket-function (ijk) under the Veronese map is equal to $(ij)(jk)(ik)$.

9.15 Let S be a weak del Pezzo surface and R be a Dynkin curve on S. Show that S admits a double cover ramified only over R if and only if the sum of irreducible components in R is divisible by 2 in the Picard group. Using this, classify all del Pezzo surfaces which admit a double cover ramified only over singular points.

9.16 Show that the Segre cubic primal is isomorphic to a tangent hyperplane section of the cubic fourfold with nine lines given by the equation $xyz - uvw = 0$ (*Perazzo primal* [434], [22]).

9.17 Consider the following *Cayley's family of cubic surfaces* in $\mathbb{P}^3$ with parameters l, m, n, k.

$$w[x^2 + y^2 + z^2 + w^2 + (mn + \frac{1}{mn})yz + (ln + \frac{1}{ln})xz + (lm + \frac{1}{lm})xy$$

$$+(l + \frac{1}{l})xw + (m + \frac{1}{m})yw + (n + \frac{1}{n})zw] + kxyz = 0.$$

Find the equations of 45 tritangent planes whose equations depend rationally on the parameters l, m, n, k.

9.18 Show that the polar quadric of a nonsingular cubic surface with respect to an Eckardt point is equal to the union of two planes.

9.19 Show that the equation of the dual of a nonsingular cubic surface can be written in the form $A^3 + B^2 = 0$, where A and B are homogeneous forms of degree 4 and 6, respectively. Show that the dual surface has 27 double lines and a curve of degree 24 of singularities of type A_2.

9.20 Show that any normal cubic surface can be given as the image of a plane under a Cremona transformation of $\mathbb{P}^3$ of degree 3.

9.21 Show that a general cubic surface can be projectively generated by three nets of planes.

9.22 Show that the Eckardt points are singular points of the parabolic curve of a nonsingular cubic surface.

9.23 Show that each line on a nonsingular cubic surface intersects the parabolic curve with multiplicity 2.

9.24 Find an $\mathfrak{A}_5$-invariant determinantal representation of the Clebsch diagonal cubic.

9.25 Use the Hilbert-Burch Theorem to show that any White surface (see Remark 9.1.22) is isomorphic to a determinantal surface W in $\mathbb{P}^n$ of degree $\binom{n}{2}$.

Historical Notes

Good sources for the references here are [286], [385], and [432]. According to [385], the study of cubic surfaces originates from the work of J. Plücker [444] on intersection of quadrics and cubics and L. Magnus [376] on maps of a plane by a linear system of cubics.

However, it is customary to think that the theory of cubic surfaces starts from Cayley's and Salmon's discovery of 27 lines on a nonsingular cubic surface [74], [486] (see the history of discovery in [493], n. 529a, p. 183). Salmon's proof was based on his computation of the degree of the dual surface [485] and Cayley's proof uses the count of tritangent planes through a line, the proof we gave here. It is reproduced in many modern discussions of cubic surfaces (e.g. [457]). The number of tritangent planes was computed by [486] and Cayley [74]. Cayley gives an explicit 4-dimensional family of cubic surfaces with a fixed tritangent plane (see Exercise 9.17). In 1851 J. Sylvester claimed,

without proof, that a general cubic surface can be written uniquely as a sum of five cubes of linear forms [555]. This fact was proven 10 years later by A. Clebsch [105]. In 1854, L. Schläfli discovers 36 double-sixes on a nonsingular cubic surface. This and other results about cubic surfaces were published later in [497]. In 1855, H. Grassmann proved that three collinear nets of planes generate a cubic surface [262]. The fact that a general cubic surface can be obtained in this way (this implies a linear determinantal representation of the surface) has a long history. In 1862, F. August proved that a general cubic surface can be generated by three pencils of planes [18]. L. Cremona deduces from this that a general cubic surface admits Grassmann's generation [144]. In 1904, R. Sturm pointed out that Cremona's proof had a gap. The gap was fixed by C. Segre in [523]. In the same paper Segre proves that any normal cubic surface which does not contain a singularity of type E_6 has a linear determinantal representation. In 1956, J. Steiner introduces what we called Steiner systems of lines [545]. This gives 120 essentially different Cayley-Salmon equations of a nonsingular cubic surface. The existence of which was first shown by Cayley [74] and Salmon [486].

Cubic surfaces with a double line were classified in 1862 by A. Cayley [82] and, via a geometric approach, by L. Cremona [139]. In 1863, L. Schläfli [496] classified singular cubic surfaces with isolated singularities, although most of these surfaces were already known to G. Salmon [486]. The old notations for A_k-singularities are C_2 for A_1 (conic-node), B_{k+1} (biplanar nodes) for $A_k, k, > 1$ and U_{k+1} (uniplanar node) for D_k. The subscript indicates the decrease of the class of the surface. In [88] Cayley gives a combinatorial description of the sets of lines and tritangent planes on singular surfaces. He also gives the equations of the dual surfaces. Even before the discovery of 27 lines, in a paper of 1844 [72], Cayley studied what we now call the Cayley 4-nodal cubic surface. He finds its equation and describes its plane sections which amounts to describing its realization as the image of the plane under the map given by the linear system of cubic curves passing through the vertices of a complete quadrilateral. Schläfli and later F. Klein [337] and L. Cremona [144] also studied the reality of singular points and lines. Benjamino Segre's book [515] on cubic surfaces treats real cubic surfaces with special detail.

In 1866, A. Clebsch proved that a general cubic surface can be obtained as the image of a birational map from the projective plane given by cubics through six points [108]. Using this he showed that Schläfli's notation a_i, b_i, c_{ij} for 27 lines correspond to the images of the exceptional curves, conics through five points and lines through two points. This important result was independently proven by L. Cremona in his memoir [144] of 1868 that got him the prize (shared with R. Sturm) offered by R. Steiner through the Royal Academy of

Sciences of Berlin in 1864 and awarded in 1866. Some of the results from this memoir are discussed from a modern point of view in [184]. Many results from Cremona's memoir are independently proved by R. Sturm [549], and many of them were announced by J. Steiner (who did not provide proofs). In particular, Cremona proved the result, anticipated in the work of Magnus, that any cubic surface can be obtained as the image of a plane under the cubo-cubic birational transformation of $\mathbb{P}^3$. Both of the memoirs had a lengthy discussion of Steiner systems of tritangent planes. We refer to [184] for a historical discussion of Cremona's work on cubic surfaces.

Cremona's hexahedral equations were introduced by L. Cremona in [148]. Although known to T. Reye [463] (in geometric form, no equations can be found in his paper), Cremona was the first who proved that the equations are determined by a choice of a double-six. The invariant theory of Cremona hexaedral equations was studied by A. Coble in [117]. He used the Joubert functions introduced by P. Joubert in [324]. The Segre cubic arose in the work of C. Segre on cubic threefolds with singular points [518]. Its realization as the GIT-quotient space of ordered sets of six points in $\mathbb{P}^1$ is due to Coble. The dual quartic hypersurface was first studied by G. Castelnuovo [**?**] and later by H. Richmond [469]. It was called the Castelnuovo quartic by E. Ciani [98].

F. Eckardt gives a complete classification of cubic surfaces with Eckardt points (called Ovalpoints in [494]) in terms of their Hessian surface [199]. He also considers singular surfaces. A modern account of this work can be found [151]. The Clebsch Diagonalfläche with 10 Eckardt points was first studied by A. Clebsch in [112]. It has an important role in Klein's investigation of the Galois group of a quintic equations [340].

The classification of possible groups of automorphisms of a nonsingular cubic surfaces was initiated by S. Kantor [327]. Some of the mistakes in his classification were later corrected by A. Wiman [602]. However, Wiman had made also a small mistake claiming that in the case VII the group is a dihedral group of order 12. Segre's book [515] contains several mistakes, fore example he missed the case VII. The first complete, purely computational, classification was given in 1997 by T. Hosoh [**?**]. Apparently he was not aware of Wiman's paper.

In 1897, J. Hutchinson showed in [310] that the Hessian surface of a nonsingular cubic surface could be isomorphic to the Kummer surface of the Jacobian of a genus 2 curve. This happens if the invariant $I_8 I_{24} + 8 I_{32}$ vanishes [480]. The group of birational automorphisms of the Hessian of a cubic surface was described only recently [181].

The relationship of the Gosset polynomial 2_{21} to 27 lines on a cubic surface was first discovered in 1910 by P. Schoute [501] (see [571]). The Weyl group

$W(\mathbf{E}_6)$ as the Galois group of 27 lines was first studied by C. Jordan [322]. Together with the group of 28 bitangents of a plane quartic isomorphic to $W(\mathbf{E}_7)$, it is discussed in many classical text-books in algebra (e.g. [596], B. II). S. Kantor [327] realized the Weyl group $W(\mathbf{E}_n), n \leq 8$, as a group of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. A. Coble [117], Part II, was the first who showed that the group is generated by the permutations group and one additional involution. So we should credit him with the discovery of the Weyl groups as reflection groups. Apparently independently of Coble, this fact was rediscovered by P. Du Val [195]. We refer to [52] for the history of Weyl groups, reflection groups and root systems. Note that the realization of the Weyl group as a reflection group in the theory of Lie algebras was obtained by H. Weyl in 1928, ten years later after Coble's work.

As we have already mentioned in the previous chapter, the Gosset polytopes were discovered in 1900 by T. Gosset [257]. The notation n_{21} belongs to him. They had been rediscovered later by E. Elte and H. S. M. Coxeter (see [137]), but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on del Pezzo surfaces of degree $5 - n$ was found by Du Val [195]. This fundamental paper is the origin of a modern approach to the study of del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [163], [377].

Volume 3 of Baker's book [21] contains a lot of information about the geometry if cubic surfaces. Yu. Manin's book [377] is a good source on cubic surfaces with emphasis on the case on a non-algebraically closed base field. It has been used as one of the main sources in the study of arithmetic of del Pezzo surfaces.

10

Geometry of Lines

10.1 Grassmannians of lines

10.1.1 Generalities about Grassmannians

We continue to use E to denote a linear space of dimension $n+1$. Let us recall some basic facts about Grassmann varieties of m-dimensional subspaces of E and fix the notations. We will denote by $G(m, E)$ the set of m-dimensional linear subspaces of E. We will identify it with the set $G_{m-1}(|E|)$ of $m-1$-dimensional planes in the projective space $|E|$ of lines of E. Another identification can be made with the set $G(E, n+1-m)$ of $n+1-m$-dimensional quotients of E. By assigning to $L \in G(m, E)$ the image of $\bigwedge^m L$ in $\bigwedge^m E$, we will identify the set $G(m, E)$ with the set of lines in $\bigwedge^m E$ generated by decomposable m-vectors $v_1 \wedge \ldots \wedge v_m$, where $(v_1, \ldots, v_m)$ is a basis of L. The linear subspace L can be reconstructed from a decomposable m-vector ω via

$$L = \{v \in E : \omega \wedge v = 0\}.$$

In this way $G(m, E)$ acquires a structure of a projective subvariety of $|\bigwedge^m E|$ corresponding to lines $[\omega]$ such that the rank of the linear map

$$E \to \bigwedge^{m+1} E,\ v \mapsto v \wedge \omega,$$

is less than or equal to $n+1-m$.

The tautological embedding of projective varieties $G(m, E) \hookrightarrow |\bigwedge^m E|$ is called the *Plücker embedding*.

By taking the dual subspace $L^\perp \subset E^\vee$, we obtain an isomorphism of projective varieties

$$G(m, E) \cong G(n+1-m, E^\vee), \quad L \mapsto L^\perp.$$

If we fix a basis in E to identify it with $\mathbb{C}^{n+1}$, we write $G(m, n+1)$ or $G_{m-1}(\mathbb{P}^n)$ instead of $G(m, \mathbb{C}^{n+1})$ or $G_{m-1}(|E|)$.

In a more sophisticated way, the Grassmann variety $\mathbb{G} = G(m, E)$ can be defined as the variety representing the functor which assigns to a scheme S the set of rank $n+1-m$ locally free quotients $\mathcal{F}$ of the vector bundle $E \otimes \mathcal{O}_S$. The corresponding morphism $S \to \mathbb{G}$ is defined by assigning to a point $s \in S$ the kernel of the surjection $E \to \mathcal{F}(s)$. The universal object in the sense of representable functors is defined by a vector bundle $E \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q}_{\mathbb{G}}$, where $\mathcal{Q}_{\mathbb{G}}$ is a vector bundle of rank m over $\mathbb{G}$, called the *universal quotient bundle* over $\mathbb{G}$. Its kernel is denoted by $\mathcal{S}_{\mathbb{G}}$ and is called the *universal subbundle* over $\mathbb{G}$. By definition, we have an exact sequence of locally free sheaves (the *tautological exact sequence* on $\mathbb{G}$)

$$0 \to \mathcal{S}_{\mathbb{G}} \to E \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q}_{\mathbb{G}} \to 0, \tag{10.1}$$

and its dual exact sequence

$$0 \to \mathcal{Q}_{\mathbb{G}}^{\vee} \to E^{\vee} \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{S}_{\mathbb{G}}^{\vee} \to 0. \tag{10.2}$$

The Plücker embedding is defined now as the composition of the natural morphisms

$$\mathbb{G} \cong \mathbb{P}(\bigwedge^m \mathcal{S}_{\mathbb{G}}^{\vee}) \to \mathbb{P}(\bigwedge^m E^{\vee} \otimes \mathcal{O}_{\mathbb{G}}) = |\bigwedge^m E| \times \mathbb{G} \to |\bigwedge^m E|.$$

A choice of a basis in E and a choice of a basis $(v_1, \ldots, v_m)$ of $L \in G(m, E)$ defines a matrix A_L of size $m \times (n+1)$ and rank m whose i-th row consists of coordinates of the vector v_i. Two such matrices A and B define the same linear subspace if and only if there exists a matrix $C \in \mathrm{GL}(m)$ such that $CA = B$. In this way $G(m, E)$ can be viewed as the orbit space of the action of $\mathrm{GL}(m)$ on the open subset of $\mathrm{Mat}_{m,n+1}(m)$ of rank m matrices. By the First Fundamental Theorem of Invariant Theory, the orbit space is isomorphic to the projective spectrum of the subring of the polynomial ring in $\binom{n+1}{m}$ variables $X_{ij}, 1 \le i \le m, 1 \le j \le n+1$, generated by the maximal minors of the matrix $X = (X_{ij})$. A choice of an order on the set of maximal minors (we will always use the lexicographic order) defines an embedding of the orbit space in $\mathbb{P}^{\binom{n+1}{m}-1}$. It is isomorphic to the Plücker embedding. In coordinates $(t_0, \ldots, t_n) \in (E^{\vee})^{n+1}$, the maximal minors $X_{i_1 \ldots i_m}$ can be identified with m-vectors $p_{i_1 \ldots i_m} = t_{i_1} \wedge \ldots \wedge t_{i_m} \in \bigwedge^m E^{\vee} = (\bigwedge^m E)^{\vee}$. Considered as coordinates in the vector space $\bigwedge^m E$, they are called the *Plücker coordinates*.

The maximal minors $X_{i_1\dots i_k}$ satisfy the *Plücker equations*

$$\sum_{k=1}^{m+1}(-1)^k p_{i_1,\dots,i_{m-1},j_k}p_{j_1,\dots,j_{k-1},j_{k+1},\dots,j_{m+1}} = 0, \tag{10.3}$$

where $(i_1,\dots,i_{m-1})$ and $(j_1,\dots,j_{m+1})$ are two strictly increasing subsets of $[1,n+1]$. These relations are easily obtained by considering the left-hand-side expression as an alternating $(m+1)$-multilinear function on $\mathbb{C}^m$. It is known that these equations define $G(m,n+1)$ scheme-theoretically in $\mathbb{P}^{\binom{n+1}{m}-1}$ (see, for example, [302], vol. 2).

The open subset $D(p_I)\cap G(m,n+1)$ is isomorphic to the affine space $\mathbb{A}^{m(n-m)}$. The isomorphism is defined by assigning to a matrix A defining L, the point (z_J), where $z_J = |A_J|/|A_I|$ taken in some fixed order. This shows that $G(m,n+1)$ is a smooth rational variety of dimension $m(n+1-m)$.

The surjection $E^\vee\otimes\mathcal{O}_{\mathbb{G}}\to\mathcal{S}^\vee_{\mathbb{G}}$ defines a closed embedding $\mathbb{P}(\mathcal{S}^\vee_{\mathbb{G}})\hookrightarrow\mathbb{P}(E^\vee\otimes\mathcal{O}_{\mathbb{G}}) = |E|\times G$. Its image is the *incidence variety*

$$\mathsf{Z}_{\mathbb{G}} = \{(x,\Pi)\in|E|\times\mathbb{G} : x\in\Pi\}.$$

Let

$$p:\mathsf{Z}_{\mathbb{G}}\to|E|,\quad q:\mathsf{Z}_{\mathbb{G}}\to\mathbb{G}$$

be the corresponding projections. By definition, the projection q is the projective bundle $\mathbb{P}(\mathcal{S}^\vee_{\mathbb{G}}) = |\mathcal{S}_{\mathbb{G}}|$.

The fiber of the projection p over a point $x=[v]\in|E|$ can be canonically identified with $G(m-1,E/\mathbb{C}v)$. Recall that the quotient spaces $E/\mathbb{C}v$, $v\in E$, are the fibres of the quotient sheaf $E\otimes\mathcal{O}_{|E|}/\mathcal{O}_{|E|}(-1)$ which is isomorphic to the twisted tangent sheaf $\mathcal{T}_{|E|}(-1)$ via the Euler exact sequence (the dual of exact sequence (7.50))

$$0\to\mathcal{O}_{|E|}\to\mathcal{O}_{|E|}(1)\otimes E\to\mathcal{T}_{|E|}\to 0.$$

Assume $m=2$, then $G(m-1,E/\mathbb{C}v)\cong|E/\mathbb{C}v|$. This gives

$$\mathsf{Z}_{\mathbb{G}}\cong|\mathcal{T}_{|E|}(-1)| = \mathbb{P}(\Omega^1_{|E|}(1)). \tag{10.4}$$

In the general case, the projection p is the *Grassmann bundle* $G(m-1,\mathcal{T}_{|E|}(-1))$. Since we will not need this fact, we omit the relevant definitions and the proof (see [333]).

Let us compute the canonical sheaf $\omega_{\mathbb{G}}$ of $\mathbb{G}$.

Lemma 10.1.1 *Let $\mathcal{T}_{\mathbb{G}}$ be the tangent bundle of $\mathbb{G}$. There is a natural isomorphism of sheaves*

$$\mathcal{T}_{\mathbb{G}} \cong \mathcal{S}_{\mathbb{G}}^{\vee} \otimes \mathcal{Q}_{\mathbb{G}},$$
$$\omega_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-n),$$

where $\mathcal{O}_{\mathbb{G}}(1)$ is taken with respect to the Plücker embedding.

Proof Let us trivialize $\mathcal{S}_{\mathbb{G}}$ over an open subset U to assume that $\mathcal{S}_{\mathbb{G}} = S \otimes \mathcal{O}_U$. Then U is isomorphic to the quotient of an open subset of $\mathrm{Hom}(S, E)$ by $\mathrm{GL}(S)$. The tangent bundle of U becomes isomorphic to

$$\mathrm{Hom}(S, E)/\mathrm{Hom}(S, S) \cong S^{\vee} \otimes E/S^{\vee} \otimes S \cong S^{\vee} \otimes (E/S).$$

These isomorphisms can be glued together to define a global isomorphism $\mathcal{T}_{\mathbb{G}} \cong \mathcal{S}_{\mathbb{G}}^{\vee} \otimes \mathcal{Q}_{\mathbb{G}}$.

Since $\bigwedge^m V \to \bigwedge^m S_{\mathbb{G}}^{\vee}$ defines the Plücker embedding, we have

$$c_1(\mathcal{S}_{\mathbb{G}}^{\vee}) = c_1(\mathcal{O}_{\mathbb{G}}(1)).$$

Now the second isomorphism follows from a well-known formula for the first Chern class of tensor product of vector bundles (see [282], Appendix A). □

Since $\mathsf{Z}_{\mathbb{G}}$ is a projective bundle over $\mathbb{G}$, we can apply formula (7.51) for the canonical sheaf of a projective bundle to obtain

$$\omega_{Z_{\mathbb{G}}/\mathbb{G}} \cong q^*(\bigwedge^m \mathcal{S}_{\mathbb{G}}^{\vee}) \otimes p^*\mathcal{O}_{|E|}(-m) \cong q^*\mathcal{O}_{\mathbb{G}}(1) \otimes p^*\mathcal{O}_{|E|}(-m),$$
$$\omega_{Z_{\mathbb{G}}} \cong \omega_{Z_{\mathbb{G}}/\mathbb{G}} \otimes q^*(\omega_{\mathbb{G}}) \cong q^*\mathcal{O}_{\mathbb{G}}(-n) \otimes p^*\mathcal{O}_{|E|}(-m),$$

10.1.2 Schubert varieties

Let us recall some facts about the cohomology ring $H^*(\mathbb{G}, \mathbb{Z})$ of $\mathbb{G} = G_r(\mathbb{P}^n)$ (see [231], Chapter 14).

Fix a flag

$$A_0 \subset A_1 \subset \ldots \subset A_r \subset \mathbb{P}^n$$

of subspaces of dimension $a_0 < a_1 < \ldots < a_r$, and define the *Schubert variety*

$$\Omega(A_0, A_1, \ldots, A_r) = \{\Pi \in \mathbb{G} : \dim \Pi \cap A_i \geq i, i = 0, \ldots, r\}.$$

This is a closed subvariety of $\mathbb{G}$ of dimension $\sum_{i=0}^{r}(a_i - i)$. Its homology class $[\Omega(A_0, A_1, \ldots, A_r)]$ in $H_*(\mathbb{G}, \mathbb{Z})$ depends only on $a_0, \ldots, a_r$. It is called

a *Schubert cycle* and is denoted by $(a_0, \ldots, a_r)$. Let $a_0 = N - r - d, a_i = n - r + i, i \geq 1$. The varieties

$$\Omega(A_0) := \Omega(A_0, \ldots, A_r) = \{\Pi \in \mathbb{G} : \Pi \cap A_0 \neq \emptyset\}$$

are called the *special Schubert varieties*. Their codimension is equal to d.

Under the Poincaré Duality $H_*(\mathbb{G}, \mathbb{Z}) \to H^*(\mathbb{G}, \mathbb{Z})$, the cycles $(a_0, \ldots, a_r)$ are mapped to *Schubert classes* $\{\lambda_0, \ldots, \lambda_r\}$ defined in terms of the Chern classes

$$\sigma_s = c_s(\mathcal{Q}_\mathbb{G}) \in H^{2s}(\mathbb{G}, \mathbb{Z}),\ s = 1, \ldots, n - r,$$

by the *determinantal formula*

$$\{\lambda_0, \ldots, \lambda_r\} = \det(\sigma_{\lambda_i + j - i})_{0 \leq i,j \leq r},$$

where $\lambda_i = n - r + i - a_i,\ i = 0, \ldots, r$. The classes σ_s are dual to the classes of special Schubert varieties $\Omega(A_0)$, where $\dim A_0 = n - r - s$.

The tautological exact sequence (10.1) shows that

$$1 = (\sum c_s(\mathcal{Q}_\mathbb{G}))(\sum c_s(\mathcal{S}_\mathbb{G})).$$

In particular,

$$\sigma_1 = -c_1(\mathcal{S}_\mathbb{G}) = c_1(\mathcal{S}_\mathbb{G}^\vee) = c_1(\mathcal{O}_\mathbb{G}(1)).$$

A proof of the following result can be found in [231] or [302], vol. 2.

Proposition 10.1.2 *The cohomology ring $H^*(\mathbb{G}, \mathbb{Z})$ is generated by the special Schubert classes σ_s. The Schubert cycles $(a_0, \ldots, a_r)$ with $\sum_{i=0}^r (a_i - i) = d$ freely generate $H_{2d}(\mathbb{G}, \mathbb{Z})$. The Schubert classes $\{\lambda_0, \ldots, \lambda_r\}$ with $d = \sum_{i=0}^r \lambda_i$ freely generate $H^{2d}(\mathbb{G}, \mathbb{Z})$. In particular,*

$$\mathrm{Pic}(\mathbb{G}) \cong H^2(\mathbb{G}, \mathbb{Z}) = \mathbb{Z}\sigma_1.$$

It follows from the above Proposition that $H^*(\mathbb{G}, \mathbb{Z})$ is isomorphic to the Chow ring $A^*(\mathbb{G})$ of algebraic cycles on $\mathbb{G}$. Under the Poincaré Duality $\gamma \mapsto \alpha_\gamma$, the intersection form on cycles $\langle \gamma, \mu \rangle$ is defined by

$$\langle \gamma, \mu \rangle = \int_\mu \alpha_\gamma = \int_G \alpha_\gamma \wedge \alpha_\mu := \alpha_\gamma \cdot \alpha_\mu.$$

The intersection form on $A^*(\mathbb{G})$ is calculated by using the *Pieri's formulas*

$$\{\lambda_0, \ldots, \lambda_r\} \cdot \sigma_s = \sum \{\mu_0, \ldots, \mu_{r+1}\}, \tag{10.5}$$

where the sum is taken over all $\{\mu\}$ such that

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r \geq \mu_{r+1}$$

and $\sum \lambda_i = s + \sum \mu_i$.

Here are some special cases. We set $\sigma_{s,t} = \{s, t, 0, \dots, 0\}$. Then

$$\begin{aligned} \sigma_1^2 &= \sigma_2 + \sigma_{1,1}, \\ \sigma_1 \cdot \sigma_2 &= \sigma_3 + \sigma_{2,1}, \\ \sigma_1 \cdot \sigma_{1,1} &= \sigma_{2,1}. \end{aligned}$$

For example, the degree of $\mathbb{G}$ is equal to $\sigma_1^{\dim \mathbb{G}}$. We refer to [231], Example 14.7.11, for the following formula computing the degree of $G_r(\mathbb{P}^n)$

$$\deg G_r(\mathbb{P}^n) = \frac{1!2! \dots r! \dim G!}{(n-r)!(n-r+1)! \dots n!}. \tag{10.6}$$

Example 10.1.3 Let us look at the Grassmannian $G_1(\mathbb{P}^3) = G(2, 4)$ of lines in $\mathbb{P}^3$. The Plücker equations are reduced to one quadratic relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{10.7}$$

This is a nonsingular quadric in $\mathbb{P}^5$, often called the *Klein quadric*. The Schubert class of codimension 1 is represented by the special Schubert variety $\Omega(\ell)$ of lines intersecting a given line ℓ. We have two codimension 2 Schubert cycles σ_2 and $\sigma_{1,1}$ represented by the Schubert varieties $\Omega(x)$ of lines containing a given point x and $\Omega(\Pi)$ of lines containing in a given plane Π. Each of these varieties is isomorphic to $\mathbb{P}^2$. In classical terminology, $\Omega(x)$ is an *α-plane* and $\Omega(\Pi)$ is a *β-plane*. We have a 1-dimensional Schubert cycle $\sigma_{2,1}$ represented by the Schubert variety $\Omega(x, \Pi)$ of lines in a plane Π containing a given point $x \in \Pi$. It is isomorphic to $\mathbb{P}^1$. Thus

$$A^*(G(2, 4)) = \mathbb{Z}[G] \oplus \mathbb{Z}\sigma_1 \oplus (\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_{1,1}) \oplus \mathbb{Z}\sigma_{2,1} \oplus \mathbb{Z}[\text{point}].$$

Note that the two Schubert classes of codimension 2 represent two different rulings of the Klein quadric by planes.

We have

$$\sigma_2 \cdot \sigma_{1,1} = 0,\ \sigma_2^2 = 1, \sigma_{1,1}^2 = 1. \tag{10.8}$$

Write $\sigma_1^2 = a\sigma_2 + b\sigma_{1,1}$. Intersecting both sides with σ_2 and $\sigma_{1,1}$, we obtain $a = b = 1$ confirming Pieri's formula (10.5). Squaring σ_1^2, we obtain $\deg G = \sigma_1^4 = 2$, confirming the fact that $G(2, 4)$ is a quadric in $\mathbb{P}^5$.

A surface S in $G_1(\mathbb{P}^3)$ is called a *congruence of lines*. Its cohomology class $[S]$ is equal to $m\sigma_2 + n\sigma_{1,1}$. The number m (resp. n) is classically known as the *order* of S (resp. *class*). It is equal to the number of lines in S passing through a general point in $\mathbb{P}^3$ (resp. contained in a general plane). The sum $m + n$ is equal to $\sigma_1 \cdot [S]$ and hence coincides with the degree of S in $\mathbb{P}^5$.

There are numerous applications of Schubert calculus to enumerative problems in algebraic geometry. Let us prove the following nice result, which can be found in many classical text-books (first proven by L. Cremona [140]).

Proposition 10.1.4 *The number of common secants of two general rational normal curves in $\mathbb{P}^3$ is equal to 10.*

Proof Consider the congruence of lines formed by secants of a rational normal cubic curve in $\mathbb{P}^3$. Through a general point in $\mathbb{P}^3$ passes one secant. In a general plane lie three secants. Thus the order of the congruence is equal to 1 and the class is equal to 3. By using (10.8), we see that the two congruences intersect at 10 points. □

Remark 10.1.5 Let R_1 and R_2 be two general rational normal cubic curves in $\mathbb{P}^3$ and let $\mathcal{N}_i$ be the net of quadrics through R_i. The linear system $\mathcal{W}$ of quadrics in the dual space that is apolar to the linear system $\mathcal{N}$ spanned by $\mathcal{N}_1$ and $\mathcal{N}_2$ is of dimension 3. The Steinerian quartic surface defined by this linear system contains 10 lines, the singular lines of 10 reducible quadrics from $\mathcal{W}$. The dual of these lines are the 10 common secants of R_1 and R_2 (see [465], [386], [121]). Also observe that the 5-dimensional linear system $\mathcal{N}$ maps R_i to a curve C_i of degree 6 spanning the plane Π_i in $\mathcal{N}^\vee$ apolar to the plane $\mathcal{N}_j$. The 10 pairs of intersection points of C_i with the ten common secants correspond to the branches of the ten singular points of C_i.

10.1.3 Secant varieties of Grassmannians of lines

From now on, we will restrict ourselves with the Grassmannian of lines in $\mathbb{P}^n = |E|$. Via contraction, one can identify $\bigwedge^2 E$ with the space of linear maps $u : E^\vee \to E$ such that the transpose map ${}^t u$ is equal to $-u$. Explicitly,

$$v \wedge w(l) = l(v)w - l(w)v.$$

The rank of u is the rank of the map. Since ${}^t u = -u$, the rank takes even values. The Grassmann variety $G(2, E)$ is the set of points $[u]$, where u is a map of rank 2.

After fixing a basis in E, we can identify $\bigwedge^2 E$ with the space of skew-symmetric matrices $A = (p_{ij})$ of size $(n+1) \times (n+1)$. The Grassmann variety $G(2, E)$ is the locus of rank 2 matrices, up to proportionality. The entries $p_{ij}, i < j$, are the Plücker coordinates. In particular, $G(2, E)$ is the zero set of the 4×4 pfaffians of A. In fact, each of the Plücker equations is given

by the 4×4 pfaffian of the matrix (p_{ij})

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \text{Pf} \begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ -p_{ij} & 0 & p_{jk} & p_{jl} \\ -p_{ik} & -p_{jk} & 0 & p_{kl} \\ -p_{il} & -p_{jl} & -p_{kl} & 0 \end{pmatrix}.$$

Another way to look at $G(2, E)$ is to use the decomposition

$$E \otimes E \cong S^2(E) \oplus \bigwedge^2 E.$$

It identifies $G(2, E)$ with the projection of the Segre variety $\mathsf{s}_2(|E| \times |E|) \subset |E \otimes E|$ to $|\bigwedge^2 E|$ from the subspace $|S^2(E)|$.

The formula (10.6) for the degree of the Grassmannian gives in our special case

$$\deg G_1(\mathbb{P}^n) = \frac{(2n-2)!}{(n-1)!n!}. \tag{10.9}$$

One can also compute the degrees of Schubert varieties

$$\deg \Omega(a_0, a_1) = \frac{(a_0 + a_1 - 1)!}{a_0!a_1!}(a_1 - a_0). \tag{10.10}$$

Lemma 10.1.6 *The rank of $u \in \bigwedge^2 E$ is equal to the smallest number k such that ω can be written as a sum $u_1 + \cdots + u_k$ of 2-vectors u_i of rank 2.*

Proof It suffices to show that, for any u of rank $2k \geq 4$, there exists a 2-vector u_1 of rank 2 such that $u - u_1$ is of rank $\leq 2k - 2$. Let R be the kernel of u and $l_0 \notin R$. Choose $v_0 \in E$ such that, for any $l \in R$, $l(v_0) = 0$ and $l_0(v_0) = 1$. By skew-symmetry of u, for any $l, m \in E^\vee$, $m(u(x)) = -l(u(m))$. Consider the difference $u' = u - v_0 \wedge u(l_0)$. For any $l \in R$, we have

$$u'(l) = u(l) - l(v_0)u(l_0) + l(u(l_0))v_0 = l(u(l_0))v_0 = -l_0(u(l))v_0 = 0.$$

This shows that $R \subset \text{Ker}(u)$. Moreover, we have

$$u'(l_0) = u(l_0) + l_0(u(l_0))v_0 - l_0(v_0)u(l_0) = u(l_0) - u(l_0) = 0.$$

This implies that $\text{Ker}(u')$ is strictly larger than $\text{Ker}(u)$. □

This gives the following.

Proposition 10.1.7 *The variety*

$$\mathbb{G}_k =: \{[u] \in |\bigwedge^2 E| : \ u \text{ has rank} \leq 2k + 2\}$$

is equal to the k-secant variety $\text{Sec}_k(\mathbb{G})$ *of* $\mathbb{G}$.

Let $t = [\frac{n-3}{2}]$, then t is the maximal number k such that $\mathrm{Sec}_k(\mathbb{G}) \neq |\bigwedge^2 E|$. So the Plücker space is stratified by the rank of its points and the strata are the following:

$$|E| \setminus \mathbb{G}_t,\ \mathbb{G}_t \setminus \mathbb{G}_{t-1},\ \ldots, \mathbb{G}_1 \setminus \mathbb{G},\ \mathbb{G}. \tag{10.11}$$

It follows from the previous remarks that $\mathbb{G}_k \setminus \mathbb{G}_{k-1}$ is the orbit of a matrix of rank $2k+2$ and size $(n+1) \times (n+1)$ under the action of $\mathrm{GL}(n+1)$. Therefore,

$$\dim \mathbb{G}_k = \dim \mathrm{GL}(n)/H_k,$$

where H_k is the stabilizer of a skew-symmetric matrix of rank $2k+2$. An easy computation gives the following.

Proposition 10.1.8 *Let $0 \leq k \leq t$, then*

$$d_k = \dim \mathbb{G}_k = (k+1)(2n-2k-1) - 1. \tag{10.12}$$

Let $X \subset \mathbb{P}^r$ be a reduced and nondegenerate closed suvariety. The *k-th defect* of X can be defined as

$$\delta_k(X) = \min\big((k+1)\dim X + k, r\big) - \dim \mathrm{Sec}_k(X),$$

which is the difference between the expected dimension of the k-secant variety of X and the effective one. We say that X is *k-defective* if $\mathrm{Sec}_k(X)$ is a proper subvariety and $\delta_k(X) > 0$.

Example 10.1.9 Let $n = 2t+3$, then $\mathbb{G}_t \subset |\bigwedge^2 E|$ is the pfaffian hypersurface of degree $t+2$ in $|\bigwedge^2 E|$ parameterizing singular skew-symmetric matrices (a_{ij}) of size $2t+4$. The expected dimension of $\mathbb{G}_t$ is equal to $4t^2+8t+5$, that is larger than $\dim |\bigwedge^2 E| = \binom{2t+4}{2} - 1$. Thus $d_t(\mathbb{G}) = \dim \mathbb{G}_t + 1$ and $\delta_t(\mathbb{G}) = 1$.

In the special case $n = 5$, the variety $G_1(\mathbb{P}^5)$ is one of the four Severi-Zak varieties.

Using Schubert varieties one can describe the projective tangent space of $\mathbb{G}_k$ at a given point $p = [u] \notin \mathbb{G}_{k-1}$. Let $K = \mathrm{Ker}(u) \subset E^\vee$. Since the rank of u is equal to $2k+2$, the dual subspace $K^\perp \subset E$ defines a linear subspace

$$\Lambda_p = |K^\perp|$$

of $|E|$ of dimension $2k+1$. Let $\Omega(\Lambda_p)$ be the corresponding special Schubert variety and let $\langle \Omega(\Lambda_p) \rangle$ be its linear span in the Plücker space.

Proposition 10.1.10

$$\mathbb{T}_p(\mathbb{G}_k) = \langle \Omega(\Lambda_p) \rangle.$$

Proof Since $\mathbb{G}_k \setminus \mathbb{G}_{k-1}$ is a homogeneous space for $\mathrm{GL}(n+1)$, we may assume that the point p is represented by a 2-vector $u = \sum_{i=0}^{k} e_{2i+1} \wedge e_{2i+2}$, where $(e_1, \dots, e_n)$ is a basis in E. The corresponding subspace $K^\perp$ is spanned by $e_1, \dots, e_{2k+2}$. A line ℓ intersects Λ_p if and only if it can be represented by a bivector $v \wedge w$, where $v \in K^\perp$. Thus $W = \langle \Omega(\Lambda_p) \rangle$ is the span of points $[e_i \wedge e_j]$, where either i or j is less than or equal to $2k+2$. In other words, W is given by vanishing of $\binom{n-2k-1}{2}$ Plücker coordinates p_{ab}, where $a, b > 2k+2$. It is easy to see that this agrees with the formula for $\dim \mathrm{Sec}_k(\mathbb{G})$. So, it is enough to show that W is contained in the tangent space. We know that the equations of $\mathrm{Sec}_k(\mathbb{G})$ are given by pfaffians of size $4k+4$. Recall the formula for the pfaffians from Chapter 2, Exercise 2.1,

$$\mathrm{Pf}(A) = \sum_{S \in \mathcal{S}} \pm \prod_{(ij) \in S} a_{ij},$$

where S is a set of pairs $(i_1, j_1), \dots, (i_{2k+2}, j_{2k+2})$ such that $1 \le i_s < j_s \le 4k+4$, $s = 1, \dots, 2k+2$, $\{i_1, \dots, i_{2k+2}, j_1, \dots, j_{2k+2}\} = \{1, \dots, 4k+4\}$. Consider the Jacobian matrix of $\mathbb{G}_k$ at the point p. Each equation of $\mathbb{G}_k$ is obtained by a choice of a subset I of $\{1, \dots, n\}$ of cardinality $4k+4$ and writing the pfaffian of the submatrix of (p_{ij}) formed by the columns and rows with indices in I. The corresponding row of the Jacobian matrix is obtained by taking the partials of this equation with respect to all p_{ij} evaluated at the point p. If $a, b \le 2k+2$, then one of the factors in the product $\prod_{(ij) \in S} p_{ij}$ corresponds to a pair (i, j), where $i, j > 2k+2$. When we differentiate with respect to p_{ab} its value at p is equal to zero. Thus the corresponding entry in the Jacobian matrix is equal to zero. So, all nonzero entries in a row of the Jacobian matrix correspond to the coordinates of vectors from W that are equal to zero. Thus W is contained in the space of solutions. □

Taking $k = 0$, we obtain the following.

Corollary 10.1.11 *For any $\ell \in \mathbb{G}$,*

$$\mathbb{T}_\ell(\mathbb{G}) = \langle \Omega(\ell) \rangle.$$

Let Λ be any subspace of $\mathbb{P}^n$ of dimension $2k+1$ and

$$P_\Lambda = \{p \in |\bigwedge^2 E| : \Lambda = \Lambda_p\}.$$

This is the projectivization of the linear space of skew-symmetric matrices of rank $2k+2$ with the given nullspace of dimension $2k+2$. An easy computation using formula (10.12) shows that $\dim P_\Lambda = (2k+1)(k+1) - 1$.

Let

$$\gamma_k : \mathbb{G}_k \setminus \mathbb{G}_{k-1} \to G(d_k + 1, \bigwedge^2 E), \quad d_k = \dim \mathbb{G}_k,$$

be the *Gauss map* which assigns to a point its embedded tangent space.

Corollary 10.1.12

$$\gamma_k^{-1}(\langle \Omega(\Lambda) \rangle) = P_\Lambda.$$

In particular, any hyperplane in the Plücker space containing $\Omega(\Lambda)$ *is tangent to* $\mathrm{Sec}_k(\mathbb{G})$ *along the subvariety* P_Λ *of dimension* $(2k+1)(k+1) - 1$.

Example 10.1.13 Let $\mathbb{G} = G(2, 6)$. We have already observed that the secant variety $\mathbb{G}_1$ is a cubic hypersurface X in $\mathbb{P}^{14}$ defined by the pfaffian of the 6×6 skew-symmetric matrix whose entries are Plücker coordinates p_{ij} . The Gauss map is the restriction to X of the polar map $\mathbb{P}^{14} \dashrightarrow (\mathbb{P}^{14})^\vee$ given by the partials of the cubic. The singular locus of X is equal to $G(2, 6)$, it is defined by polars of X. The polar map is a Cremona transformation in $\mathbb{P}^{14}$. This is one of the examples of Cremona transformations defined by Severi-Zak varieties (see Example 7.1.13).

Let X be a subvariety of $\mathbb{G}$, and Z_X be the preimage of X under the projection $q : \mathsf{Z}_\mathbb{G} \to \mathbb{G}$. The image of Z_X in $\mathbb{P}^n$ is the union of lines $\ell \in X$. We will need the description of its set of nonsingular points.

Proposition 10.1.14 *The projection* $p_X : \mathsf{Z}_X \to \mathbb{P}^n$ *is smooth at* (x, ℓ) *if and only if*

$$\dim_\ell \Omega(x) \cap \mathbb{T}_\ell(X) = \dim_{(x,\ell)} p_X^{-1}(x).$$

Proof Let $(x, \ell) \in \mathsf{Z}_X$ and let F be the fiber of $p_X : \mathsf{Z}_X \to \mathbb{P}^n$ passing through the point (x, ℓ) identified with the subset $\Omega(x) \cap X$ under the projection $q : \mathsf{Z}_X \to \mathbb{G}$. Then

$$\mathbb{T}_{x,\ell}(F) = \mathbb{T}_\ell(\Omega(x)) \cap \mathbb{T}_\ell(X) = \Omega(x) \cap \mathbb{T}_\ell(X). \tag{10.13}$$

This proves the assertion. □

Corollary 10.1.15 *Let* $Y = p_X(\mathsf{Z}_X) \subset \mathbb{P}^n$ *be the union of lines* $\ell \in X$. *Assume* X *is nonsingular and* $p_X^{-1}(x)$ *is a finite set. Suppose* $\dim_\ell \Omega(x) \cap \mathbb{T}_\ell(X) = 0$ *for some* $\ell \in X$ *containing* x. *Then* x *is nonsingular point of* Y.

10.2 Linear line complexes

10.2.1 Linear line complexes and apolarity

An effective divisor $\mathfrak{C} \subset \mathbb{G} = G_1(\mathbb{P}^n)$ is called a *line complex*. Since we know that $\mathrm{Pic}(\mathbb{G})$ is generated by $\mathcal{O}_{\mathbb{G}}(1)$ we see that $\mathfrak{C} \in |\mathcal{O}_{\mathbb{G}}(d)|$ for some $d \geq 1$. The number d is called the *degree* of the line complex.

An example of a line complex $\mathfrak{C}$ of degree d in $G_1(\mathbb{P}^n)$ is the *Chow form* of a subvariety $X \subset \mathbb{P}^n$ of codimension 2 (see [239]). It parameterizes lines that have non-empty intersection with X. Its degree is equal to the degree of X. When X is linear, this is of course the special Schubert variety $\Omega(X)$.

A *linear line complex* is a line complex of degree 1, that is a hyperplane section $\mathfrak{C} = H \cap \mathbb{G}$ of $\mathbb{G}$. If no confusion arises we will sometimes identify $\mathfrak{C}$ with the corresponding hyperplane H. A linear line complex is called *special* if it is equal to the special Schubert variety $\Omega(\Pi)$, where Π is a subspace of codimension $n-2$. The corresponding hyperplane is tangent to the Grassmannian at any point ℓ such that $\ell \subset \Pi$. In particular, when $n = 3$, the special linear line complex is isomorphic to a quadric cone.

For any $\omega \in (\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$, let $\mathfrak{C}_\omega$

$$\sum_{0 \leq i < j \leq n} a_{ij} p_{ij} = 0.$$

For example, the line complex $V(p_{ij})$ parameterizes the lines intersecting the coordinate $(n-2)$-plane $t_k = 0, k \neq i, j$, in $\mathbb{P}^n$.

Remark 10.2.1 It follows from the Euler sequence that there is a natural isomorphism

$$H^0(|E|, \Omega^1_{|E|}(2)) \cong \mathrm{Ker}\big(E^\vee \otimes E^\vee \to S^2(E^\vee)\big) \cong \bigwedge^2 E^\vee. \tag{10.14}$$

Also we know from (10.4) that the incidence variety $\mathsf{Z}_{\mathbb{G}}$ is isomorphic to the projective bundle $\mathbb{P}(\Omega^1_{|E|}(1)) \cong \mathbb{P}(\Omega^1_{|E|}(2))$. Thus a linear line complex can be viewed as a divisor in the linear system $|\mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(1)|$, where $p_* \mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(1) \cong \Omega^1_{|E|}(2)$. The fiber of $\mathsf{Z}_{\mathbb{G}}$ over a point $x \in |E|$ is isomorphic to the projectivized tangent space $\mathbb{P}(\Omega^1_{|E|}(x)) \cong |T_x(|E|)|$.

Choose local coordinates $z_1, \ldots, z_n$ in $|E|$ defining the basis $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$ in tangent spaces, then, for any nonzero $\omega \in \bigwedge^2 E^\vee$ the line complex $\mathfrak{C}_\omega \in \bigwedge^2 E^\vee$ is locally given by an expression

$$\sum_{i=1}^{n} A_i(z_1, \ldots, z_n) dz_i = 0.$$

This equation is called the *Pfaff partial differential equation*. More generally,

any line complex of degree d can be considered as the zero set of a section of $\mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(d)$ and can be locally defined by the *Monge's partial differential equation*

$$\sum_{i_1+\cdots+i_n=d} A_{i_1,\ldots,i_n} dz_1^{i_1} \ldots dz_n^{i_n} = 0.$$

We refer to S. Lie's book [362] for the connection between the theory of Pfaff equations and line complexes.

The projective equivalence classes of linear line complexes coincide with the orbits of $\mathrm{GL}(E)$ acting naturally on $|\bigwedge^2 E^\vee|$. The $\mathrm{GL}(E)$-orbit of a linear line complex $\mathfrak{C}_\omega$ is uniquely determined by the *rank* $2k$ of ω. We will identify ω with the associated linear map $E \to E^\vee$. Let

$$\mathsf{S}_\omega = |\mathrm{Ker}(\omega)|. \tag{10.15}$$

It is called the *center* of a linear line complex $\mathfrak{C}_\omega$. We have encountered it in Chapter 2. This is a linear subspace of $|E|$ of dimension $n-2k$, where $2k$ is the rank of ω.

Proposition 10.2.2 *Let $\mathfrak{C}_\omega$ be a linear line complex and S_ω be its center. Then the Schubert variety $\Omega(\mathsf{S}_\omega)$ is contained in $\mathfrak{C}_\omega$ and*

$$G_1(\mathsf{S}_\omega) = \mathrm{Sing}(\mathfrak{C}_\omega).$$

Proof Since $\mathrm{GL}(E)$ acts transitively on the set of linear line complexes of equal rank, we may assume that $\omega = \sum_{i=1}^k e_i^* \wedge e_{k+i}^*$, where $e_1^*, \ldots, e_n^*$ is a basis of $E^\vee$ dual to a basis $e_1, \ldots, e_n$ of E. The linear space $\mathrm{Ker}(\omega)$ is spanned by $e_i, i > 2k$. A line ℓ intersects S_ω if and only if it can be represented by a bivector $v \wedge w \in \bigwedge^2 E$, where $[v] \in \mathsf{S}_\omega$. Therefore, the linear span of the Schubert variety $\Omega(\mathsf{S}_\omega)$ is generated by bivectors $e_i \wedge e_j$, where $i < 2k$. It is obvious that it is contained in the hyperplane $V(\omega) = \langle \mathfrak{C}_\omega \rangle \subset |\bigwedge^2 E|$. This checks the first assertion.

It follows from Corollary 10.1.11 that

$$\ell \in \mathrm{Sing}(\mathfrak{C}_\omega) \Longleftrightarrow \mathbb{T}_\ell(\mathbb{G}) \subset V(\omega) \Longleftrightarrow \Omega(\ell) \subset \mathfrak{C}_\omega.$$

Suppose $\Omega(\ell) \subset \mathfrak{C}_\omega$ but ℓ does not belong to S_ω. We can find a point in ℓ represented by a vector $v = \sum a_i e_i$, where $a_i \neq 0$ for some $i \leq 2k$. Then the line represented by a bivector $v \wedge e_{k+i}$ intersects ℓ but does not belong to $\mathfrak{C}_\omega$ (since $\omega(v \wedge e_{k+i}) = a_i \neq 0$). Thus $\Omega(\ell) \subset \mathfrak{C}_\omega$ implies $\ell \subset \mathsf{S}_\omega$. Conversely, this inclusion implies $\Omega(\ell) \subset \Omega(\mathsf{S}_\omega) \subset \mathfrak{C}_\omega$. This proves the second assertion. □

It follows from the Proposition that any linear line complex is singular unless

its rank is equal to $2[\frac{n+1}{2}]$, maximal possible. Thus the set of hyperplanes in the Plücker space that are tangent to $\mathbb{G}$ can be identified with the set of linear line complexes of rank $\leq 2[\frac{n-3}{2}]$. Consider $G(2, E^\vee)$ in its Plücker embedding in $\mathbb{P}(\bigwedge^2 E)$. Exchanging the roles of E and $E^\vee$, we obtain the following beautiful result.

Corollary 10.2.3 *Let $t = [\frac{n-3}{2}]$, then $\mathrm{Sec}_t(\mathbb{G})$ is equal to the dual variety of the Grassmannian $G(2, E^\vee)$ in $\mathbb{P}(\bigwedge^2 E)$.*

When $n = 4, 5$ we obtain that $G(2, E)$ is dual to $G(2, E^\vee)$. When $n = 6$ we obtain that the dual of $G(2, E^\vee)$ is equal to $\mathrm{Sec}_1(G(2, E))$. This agrees with Example 10.1.13.

For any linear subspace L of E, let

$$L_\omega = \omega(L)^\perp = \{w \in E : \omega(v, w) = 0, \forall v \in L\}.$$

For any subspace $\Lambda = |L| \subset |E|$, let

$$i_\omega(\Lambda) = |L_\omega|.$$

It is clear that $[v \wedge w] \in \mathbb{G}$ belongs to $\mathfrak{C}_\omega$ if and only if $\omega(v, w) = 0$. Thus

$$\mathfrak{C}_\omega = \{\ell \in \mathbb{G} : \ell \subset i_\omega(\ell)\}. \tag{10.16}$$

Clearly $i_\omega(\Lambda)$ contains the center $\mathsf{S}_\omega = |\mathrm{Ker}(\alpha_\omega)|$ of $\mathfrak{C}_\omega$. Its dimension is equal to $n - \dim \Lambda + \dim \Lambda \cap \mathsf{S}_\omega$.

Since ω is skew-symmetric, for any point $x \in |E|$,

$$x \in i_\omega(x).$$

When ω is nonsingular, we obtain a bijective correspondence between points and hyperplanes classically known as a *null-system*.

In the special case when $n = 3$ and $\mathsf{S}_\omega = \emptyset$, this gives the *polar duality* between points and planes. The plane $\Pi(x)$ corresponding to a point x is called the *null-plane* of x. The point t_Π corresponding to a plane Π is called the *null-point* of Π. Note that $x \in \Pi(x)$ and $x_\Pi \in \Pi$. Also in this case the lines ℓ and $i_\omega(\ell)$ are called *polar lines*. We also have a correspondence between lines in $\mathbb{P}^3$

$$i_\omega : G_1(\mathbb{P}^3) \to G_1(\mathbb{P}^3), \quad \ell \mapsto i_\omega(\ell).$$

Note that the lines ℓ and $i_\omega(\ell)$ are always skew or coincide. The set of fixed points of i_ω on $G_1(\mathbb{P}^3)$ is equal to $\mathfrak{C}_\omega$. Since $\mathfrak{C}$ is nonsingular, the pole c of $V(\omega)$ with respect to the Klein quadric $\mathbb{G}$ does not belong to $\mathbb{G}$. It is easy to

see that i_ω is the deck transformation of the projection of $\mathbb{G}$ in $\mathbb{P}^5$ from the point c. Thus

$$\mathbb{G}/(i_\omega) \cong \mathbb{P}^4.$$

The hyperplane $\langle \mathfrak{C}_\omega \rangle$ is the polar hyperplane $P_c(\mathbb{G})$. The ramification divisor of the projection $\mathbb{G} \to \mathbb{P}^4$ is the linear complex $\mathfrak{C}_\omega = P_c(\mathbb{G}) \cap \mathbb{G}$. The branch divisor is a quadric in $\mathbb{P}^4$.

If $\mathfrak{C}$ is singular, then it coincides with the Schuber variety $\Omega(\ell)$, where $\ell = \mathsf{S}_\omega$. For any $\ell \neq \mathsf{S}_\omega$, we have $i_\omega(\ell) = \mathsf{S}_\omega$ and $i_\omega(\mathsf{S}_\omega) = \mathbb{P}^3$.

Proposition 10.2.4 *Let $\mathfrak{C}_\omega$ be a nonsingular linear line complex in $G_1(\mathbb{P}^n)$. Let ℓ be a line in $\mathbb{P}^n$. Then any line $\ell' \in \mathfrak{C}_\omega$ intersecting ℓ also intersects $i_\omega(\ell)$. The linear line complex $\mathfrak{C}_\omega$ consists of lines intersecting the line ℓ and the codimension 2 subspace $i_\omega(\ell)$.*

Proof Let $x = \ell \cap \ell'$. Since $x \in \ell'$, we have $\ell' \subset i_\omega(\ell') \subset i_\omega(x)$. Since $x \in \ell$, we have $i_\omega(\ell) \subset i_\omega(x)$. Thus $i_\omega(x)$ contains ℓ' and $i_\omega(\ell)$. Since $\mathfrak{C}_\omega$ is nonsingular, $\dim i_\omega(x) = n - 1$, hence the line ℓ' intersects the $(n-2)$-plane $i_\omega(\ell)$.

Conversely, suppose ℓ' intersects ℓ at a point x and intersects $i_\omega(\ell)$ at a point x'. Then $x, x' \in i_\omega(\ell')$ and hence $\ell' = \overline{xx'} \subset i_\omega(\ell')$. Thus ℓ' belongs to $\mathfrak{C}_\omega$. □

Definition 10.2.5 *A linear line complex $\mathfrak{C}_\omega$ in $|\bigwedge^2 E|$ is called* apolar *to a linear line complex $\mathfrak{C}_{\omega^*}$ in $|\bigwedge^2 E^\vee|$ if $\omega^*(\omega) = 0$.*

In the case $n = 3$, we can identify $|\bigwedge^2 E|$ with $|\bigwedge^2 E^\vee|$ by using the polarity defined by the Klein quadric. Thus we can speak about apolar linear line complexes in $\mathbb{P}^3$. In Plücker coordinates, this gives the relation

$$a_{12}b_{34} + a_{34}b_{12} - a_{13}b_{24} - a_{24}b_{13} + a_{14}b_{23} + a_{23}b_{14} = 0. \tag{10.17}$$

Lemma 10.2.6 *Let $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ be two nonsingular linear line complexes in $\mathbb{P}^3$. Then $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ are apolar to each other if and only if $g = \omega^{-1} \circ \omega' \in \mathrm{GL}(E)$ satisfies $g^2 = 1$.*

Proof Take two skew lines ℓ, ℓ' in the intersection $\mathfrak{C}_\omega \cap \mathfrak{C}_{\omega'}$. Choose coordinates in E such that ℓ and ℓ' are two opposite edges of the coordinate tetrahedron $V(t_0t_1t_2t_3)$, say $\ell : t_0 = t_2 = 0$, and $\ell' : t_1 = t_3 = 0$. Then the linear line complexes have the following equations in Plücker coordinates

$$\mathfrak{C}_\omega : ap_{12} + bp_{34} = 0; \quad \mathfrak{C}_{\omega'} : cp_{12} + dp_{34} = 0.$$

The condition that $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ are apolar is $ad + bc = 0$. The linear maps

$\omega, \omega' : E \to E^\vee$ are give by the matrices

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix}.$$

This gives

$$A^{-1}B = \begin{pmatrix} c/a & 0 & 0 & 0 \\ 0 & c/a & 0 & 0 \\ 0 & 0 & d/b & 0 \\ 0 & 0 & 0 & d/b \end{pmatrix} = \frac{a}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This shows that $(A^{-1}B)^2$ defines the identical transformation of $|E|$. It is easy to see that conversely, this implies that $ad + bc = 0$. □

In particular, a pair of nonsingular apolar linear line complexes defines an involution of $|E|$. Any pair of linear line complexes defines a projective transformation of $|E|$ as follows. Take a point x, define its null-plane $\Pi(x)$ with respect to ω and then take its null-point y with respect to ω'. For apolar line complexes we must get an involution. That is, the null-plane of y with respect to ω must coincide with the null-plane of x with respect to ω'.

Since any set of nonsingular mutually apolar linear line complexes is linearly independent, we see that the maximal number of mutually apolar linear line complexes is equal to 6. If we choose these line complexes as coordinates z_i in $\bigwedge^2 E$, we will be able to write the equation of the Klein quadric in the form

$$Q = \sum_{i=0}^{5} z_i^2.$$

Since each pair of apolar linear line complexes defines an involution in $|\bigwedge^2 E|$, we obtain 15 involutions. They generate an elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^4$ of projective transformations in $\mathbb{P}^3$. The action of this group arises from a linear representation in $\mathbb{C}^4$ of the non-abelian group $\mathcal{H}_2$ (a *Heisenberg group*) given by a central extension

$$1 \to \mu_2 \to \mathcal{H}_2 \to (\mathbb{Z}/2\mathbb{Z})^4 \to 1.$$

We denote the subgroup of $\mathrm{PGL}(3)$ generated by the 15 involutions defined by six mutually apolar line complexes by $\mathcal{H}_2'$.

An example of six mutually apolar linear line complexes is the set

$$(p_{12} + p_{34}, i(p_{34} - p_{12}), p_{13} - p_{24}, -i(p_{24} + p_{13}), p_{14} + p_{23}, i(p_{23} - p_{14})),$$

where $i = \sqrt{-1}$. These coordinates in the Plücker space are called the *Klein coordinates*.

A set of six mutually apolar linear line complexes defines a symmetric (16_6)-*configuration* of points and planes. It is formed by 16 points and 16 planes in $\mathbb{P}^3$ such that each point is a null-point of 6 planes, each with respect to one of the six line complexes. Also each plane is a null-plane of six points with respect to one of the six line complexes. To construct such a configuration one can start from any point $p_1 = [a_0, a_1, a_2, a_3] \in \mathbb{P}^3$ such that no coordinate is equal to zero. Assume that our six apolar line complexes correspond to Klein coordinates. The first line complex is $p_{12} + p_{34} = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$. It transforms the point p_1 to the plane $-a_1t_0 + a_0t_1 + a_3t_2 - a_2t_3 = 0$. Taking other line complexes we get five more null-planes:

$$\begin{aligned}
a_1t_0 - a_0t_1 + a_3t_2 - a_2t_3 &= 0,\\
a_2t_0 - a_3t_1 - a_0t_2 + a_1t_3 &= 0,\\
a_2t_0 + a_3t_1 - a_0t_2 - a_1t_3 &= 0,\\
a_3t_0 + a_2t_1 - a_1t_2 - a_0t_3 &= 0,\\
-a_3t_0 + a_2t_1 - a_1t_2 + a_0t_3 &= 0.
\end{aligned}$$

Next we take the orbit of p_1 with respect to the Heisenberg group $\mathcal{H}_2$. It consists of 16 points. Computing the null-planes of each point, we find altogether 16 planes forming with the 16 points a (16_6)-configurations. The following table gives the coordinates of the 16 points.

a_0, a_1, a_2, a_3	a_1, a_0, a_3, a_2	$a_0, -a_1, a_2, -a_3$	$a_1, -a_0, a_3, -a_2$
a_2, a_3, a_0, a_1	a_3, a_2, a_1, a_0	$a_2, -a_3, a_0, -a_1$	$a_3, -a_2, a_1, -a_0$
$a_0, a_1, -a_2, -a_3$	$a_1, a_0, -a_3, -a_2$	$a_0, -a_1, -a_2, a_3$	$a_1, -a_0, -a_3, a_2$
$a_2, a_3, -a_0, -a_1$	$a_3, a_2, -a_1, -a_0$	$a_2, -a_3, -a_0, a_1$	$a_3, -a_2, -a_1, a_0$

A point $(\alpha, \beta, \gamma, \delta)$ in this table is contained in six planes $at_0 + bt_1 + ct_2 + dt_3 = 0$, where (a, b, c, d) is one of the following

$$(\delta, -\gamma, \beta, -\alpha),\ (\delta, \gamma, -\beta, -\alpha),\ (\gamma, \delta, -\alpha, -\beta),$$

$$(-\gamma, \delta, \alpha, -\beta),\ (-\beta, \alpha, \delta, -\gamma),\ (\beta, -\alpha, \delta, -\gamma).$$

Dually, a plane $\alpha t_0 + \beta t_1 + \gamma t_2 + \delta t_3 = 0$ contains six points $[a, b, c, d]$, where (a, b, c, d) is as above.

One checks directly that the six null-points of each of the 16 planes of the configuration lie on a conic. So we have a configuration of 16 conics in $\mathbb{P}^3$ each contains six points of the configuration. Also observe that any two conics intersect at 2 points.

There is a nice symbolic way to exhibit the (16_6)-configuration. After we fix

an order on a set of six mutually apolar linear line complexes, we will be able to identify the group $\mathcal{H}_2'$ with the group E_2 defined by 2-element subsets of the set $\{1,2,3,4,5,6\}$ (see Section 5.2.2). A subset of two elements $\{i,j\}$ corresponds to the involution defined by a pair of apolar line complexes. We take the ordered set of apolar linear line complexes defined by the Klein coordinates. First we match the orbit of the point $[a_0,a_1,a_2,a_3]$ from the table from the above with the left-hand side of the following table. To find the six planes that contain a point from the (ij)-th spot we look at the same spot in the right-hand side of the following table. Take the involutions in the i-th row and j-th column but not at the (ij)-spot. These involutions are matched with the planes containing the point. As always we identify a plane $a_0t_0+a_1t_1+a_2t_2+a_3t_3$ with the point $[a_0,a_1,a_2,a_3]$. For example, the point $\emptyset$ is contained in six planes $(15),(13),(26),(46),(24),(35)$. Conversely, take a plane corresponding to the (ij)-th spot in the right-hand side of the table. The point contained in this plane can be found in the same row and the same column in the left-hand side of the table excluding the (ij)-th spot. For example, the plane $\emptyset$ contains the points $(45),(34),(35),(16),(12),(26)$.

$$\begin{array}{cccc} \emptyset & (45) & (34) & (35) \\ (16) & (23) & (25) & (24) \\ (12) & (36) & (56) & (46) \\ (26) & (13) & (15) & (14) \end{array} \qquad \begin{array}{cccc} (14) & (15) & (13) & (26) \\ (46) & (56) & (36) & (12) \\ (24) & (25) & (23) & (16) \\ (35) & (34) & (45) & \emptyset \end{array}.$$

Another way to remember the rule of the incidence is as follows. A point corresponding to an involution (ab) is contained in a plane corresponding to an involution (cd) if and only if

$$(ab)+(cd)+(24)\in\{\emptyset,(16),(26),(36),(46),(56)\}.$$

Consider a regular map $\mathbb{P}^3\to\mathbb{P}^4$ defined by the polynomials

$$t_0^4+t_1^4+t_2^4+t_3^4,\ t_0^2t_3^2+t_1^2t_2^2,\ t_1^2t_3^2+t_0^2t_2^2,\ t_2^2t_3^2+t_0^2t_1^2,\ t_0t_1t_2t_3.$$

Observe that this map is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$. So, it defines a regular map

$$\Phi:\mathbb{P}^3/\mathcal{H}_2'\to\mathbb{P}^4.$$

Proposition 10.2.7 *The map Φ defines an isomorphism*

$$\mathbb{P}^3/\mathcal{H}_2\cong X,$$

where X is a quartic hypersurface in $\mathbb{P}^4$ given by the equation

$$z_0^2z_4^2-z_0z_1z_2z_3+z_1^2z_2^2+z_2^2z_3^2+z_1^2z_3^2-4z_4^2(z_1^2+z_2^2+z_3^2)+z_4^4=0. \quad (10.18)$$

Proof Since the map is given by five polynomials of degree 4, the degree of the map times the degree of the image must be equal to 4^3. We know that its degree must be multiple of 16, this implies that either the image is $\mathbb{P}^3$ or a quartic hypersurface. Since the polynomials are linearly independent the first case is impossible. A direct computation gives the equation of the image. □

Note that the fixed-point set of each nontrivial element of the Heisenberg group $\mathcal{H}_2$ consists of two skew lines. For example, the involution

$$(12) : [a_0, a_1, a_2, a_3] \mapsto [a_0, a_1, -a_2, -a_3]$$

fixes pointwise the lines $t_0 = t_1 = 0$, and $t_2 = t_3 = 0$. Each line has a stabilizer subgroup of index 2. Thus the images of the 30 lines form the set of 15 double lines on X. The stabilizer subgroup acts on the line as the dihedral group D_4. It has six points with nontrivial stabilizer of order 2. Altogether we have $30 \times 6 = 180$ such points which form 15 orbits. These orbits and the double lines form a (15_3)-configuration. The local equation of X at one of these orbits is $v^2 + xyz = 0$.

We will prove later that the orbit space $X = \mathbb{P}^3/\mathcal{H}_2$ is isomorphic to the Castelnuovo-Richmond quartic.

10.2.2 Six lines

We know that any five lines in $\mathbb{P}^3$, considered as points in the Plücker $\mathbb{P}^5$, are contained in a linear line complex. In fact, in a unique linear line complex if the lines are linearly independent. A set of six lines is contained in a linear line complex only if they are linearly dependent. The 6×6-matrix of its Plücker coordinates must have a nonzero determinant. An example of six dependent lines is the set of lines intersecting a given line ℓ. They are contained in the singular line complex which coincides with the Schubert variety $\Omega(\ell)$. We will give a geometric characterization of a set of six linearly dependent lines that contains a subset of five linearly independent lines.

Lemma 10.2.8 *Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution. Then its graph is an irreducible curve $\Gamma_g \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 1)$ such that $\iota(\Gamma_g) = \Gamma_g$, where ι is the automorphism $(x, y) \mapsto (y, x)$. Conversely, any curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with these properties is equal to the graph of some involution.*

Proof This is easy and left to the reader. □

Corollary 10.2.9 *Let σ, τ be two different involutions of $\mathbb{P}^1$. Then there exists a unique common orbit $\{x, y\}$ with respect to σ and τ.*

We will need the following result of M. Chasles.

Theorem 10.2.10 (M. Chasles) *Let Q be a nondegenerate quadric in $\mathbb{P}^3$ and let σ be an automorphism of order 2 of Q which is the identity on one of the rulings. Then the set of lines in $\mathbb{P}^3$ which are either contained in this ruling or intersect an orbit of lines in the second ruling form a linear complex. Conversely, any linear line complex is obtained in this way from some pair (Q, σ).*

Proof Consider the set X of lines defined as in the first assertion of the Theorem. Take a general plane Π and a point $x \in \Pi$. Consider the Schubert variety $\Omega(x, \Pi)$. It is a line in the Plücker space. The plane interesects Q along a conic C. Each line from $\Omega(x, \Pi)$ intersects C at two points. This defines an involution on C. Each line from the second ruling intersects C at one point. Hence σ defines another involution on C. By Corollary 10.2.9 there is a unique common orbit. Thus there is a unique line from $\Omega(x, \Pi)$ which belongs to X. Thus X is a linear line complex.

Let ℓ_1, ℓ_2, ℓ_3 be any three skew lines in a line complex $X = \mathfrak{C}_\omega$. Let Q be a quadric containing these lines. It is obviously nonsingular. The lines belong to some ruling of Q. Take any line ℓ from the other ruling. Its polar line $\ell' = i_\omega(\ell)$ intersects ℓ_1, ℓ_2, ℓ_3 (because it is skew to ℓ or coincides with it). Hence ℓ' lies on Q. Now we have an involution on the second ruling defined by the polarity with respect to X. If $m \in X$ and is not contained in the first ruling, then m intersects a line ℓ from the second ruling. By Proposition 10.2.4, it also intersects ℓ'. This is the description of X from the assertion of the Theorem. □

Remark 10.2.11 Let C be the curve in $G(2, 4)$ parameterizing lines in a ruling of a nonsingular quadric Q. Take a general line ℓ in $\mathbb{P}^3$. Then $\Omega(\ell)$ contains two lines from each ruling, the ones which pass through the points $Q \cap \ell$. This implies that C is a conic in the Plücker embedding. A linear line complex X either intersects each conic at two points and contains two or one line from the ruling, or contains C and hence contains all lines from the ruling.

Lemma 10.2.12 *Let ℓ be a line intersecting a nonsingular quadric Q in $\mathbb{P}^3$ at two different points x, y. Let $\mathbb{T}_x(Q) \cap Q = \ell_1 \cup \ell_2$ and $\mathbb{T}_y(Q) \cap Q = \ell_1' \cup \ell_2'$, where ℓ_1, ℓ_1' and ℓ_2, ℓ_2' belong to the same ruling. Then the polar line $\ell_Q^\perp$ intersects Q at the points $x' = \ell_1 \cap \ell_2'$ and $y' = \ell_2 \cap \ell_1'$.*

Proof Each line on Q is self-polar to itself. Thus $P_x(Q)$ is the tangent plane $\mathbb{T}_x(Q)$ and, similarly, $P_y(Q) = \mathbb{T}_y(Q)$. This shows that $\ell_Q^\perp = \mathbb{T}_x(Q) \cap \mathbb{T}_y(Q) = \overline{x'y'}$. □

Lemma 10.2.13 *let $\ell_1, \ell_2, \ell_3, \ell_4$ be four skew lines in $\mathbb{P}^3$. Suppose not all of them are contained in a quadric. Then there are exactly two lines which intersect all of them. These lines may coincide.*

Proof This is of course well known. It can be checked by using the Schubert calculus since $\sigma_1^4 = \#\cap_{i=1}^4 \Omega(\ell_i) = 2$. A better geometric proof can be given as follows. Let Q be the quadric containing the first three lines. Then ℓ_4 intersects Q at two points p, q which may coincide. The lines through these points belonging to the ruling not containing ℓ_1, ℓ_2, ℓ_3 intersect $\ell_1, \ldots, \ell_4$. Conversely, any line intersecting $\ell_1, \ldots, \ell_4$ is contained in this ruling (because it intersects Q at three points) and passes through the points $\ell_4 \cap Q$.

□

Theorem 10.2.14 *Let $(\ell_1, \ldots, \ell_6)$ be a set of six lines and let $(\ell_1', \ldots, \ell_6')$ be the set of polar lines with respect to some nonsingular quadric Q. Assume that the first five lines are linearly independent in the Plücker space. Then $(\ell_1, \ldots, \ell_6)$ belong to a nonsingular linear line complex if and only if there exists a projective transformation T such that $T(\ell_i) = \ell_i'$. This condition does not depend on the choice of Q.*

Proof First let us check that this condition does not depend on a choice of Q. For each line ℓ let $\ell_Q^\perp$ denote the polar line with respect to Q. Suppose $A(\ell) = \ell_Q^\perp$ for some projective transformation A. Let Q' be another nonsingular quadric. We have to show that $\ell_{Q'}^\perp = B(\ell)$ for some other projective transformation B depending only on A but not on ℓ. Let us identify E with $\mathbb{C}^{n+1}$ and a quadric Q with a nonsingular symmetric matrix. Then $A(\ell) = \ell_Q^\perp$ means that $xQAy = 0$ for any vectors x, y in ℓ. We have to find a matrix B such that $xQ'By = 0$. We have

$$xQAy = xQ'(Q'^{-1}QA)y = xQ'By,$$

where $B = Q'^{-1}QA$. This checks the claim.

Suppose the set $(\ell_1, \ldots, \ell_6)$ is projectively equivalent to $(\ell_1', \ldots, \ell_6')$, where ℓ_i' are polar lines with respect to some quadric Q. Replacing Q with a quadric containing the first three lines ℓ_1, ℓ_2, ℓ_3, we may assume that $\ell_i' = \ell_i, i = 1, 2, 3$. We identify Q with $\mathbb{P}^1 \times \mathbb{P}^1$. If $\ell_j \cap Q = (a_j, b_j), (a_j', b_j')$ for $j = 4, 5, 6$, then, by Lemma 10.2.12, $\ell_j' \cap Q = (a_j, b_j'), (a_j', b_j)$. Suppose $\ell_i' = A(\ell_i)$. Then A fixes three lines in the first ruling hence sends Q to itself. It is also identical on the first ruling. It acts on the second ruling by switching the coordinates $(b_i, b_j'), j = 4, 5, 6$. Thus A^2 has three fixed points on $\mathbb{P}^1$, hence A^2 is the

identity. This shows that $A = \sigma$ as in Chasles' Theorem 10.2.10. Hence the lines $\ell_i, \ell_i',\ i = 1, \dots, 6$, belong to the linear complex.

Conversely, assume $\ell_1, \dots, \ell_6$ belong to a nonsingular linear line complex $X = \mathfrak{C}_\omega$. Applying Lemma 10.2.13, we find two lines ℓ, ℓ' which intersect $\ell_1, \ell_2, \ell_3, \ell_4$ (two *transversals*). By Proposition 10.2.4, the polar line $i_\omega(\ell)$ intersects $\ell_1, \ell_2, \ell_3, \ell_4$. Hence it must coincide with either ℓ or ℓ'. The first case is impossible. In fact, if $\ell = \ell'$, then $\ell \in X$. The pencil of lines through $\ell \cap \ell_1$ in the plane $\langle \ell, \ell_1 \rangle$

Thus we see that ℓ, ℓ' is a pair of polar lines. Now the pair of transversals $\tau, \tau' = i_\omega(\tau)$ of $\ell_1, \ell_2, \ell_3, \ell_5$ is also a pair of polar lines. Consider the quadric Q spanned by ℓ_1, ℓ_2, ℓ_3. The four transversals are the four lines from the second ruling of Q. We can always find an involution σ on Q which preserves the first ruling and such that $\sigma(\ell) = \ell', \sigma(\tau) = \tau'$. Consider the linear line complex X' defined by the pair (Q, σ). Since $\ell_1, \dots, \ell_5$ belong to X, and any line complex is determined by five linearly independent lines, we have the equality $X = X'$. Thus ℓ_6 intersects Q at a pair of lines in the second ruling which are in the involution σ. But σ is defined by the polarity with respect to X (since $\ell_1, \ell_2, \ell_3 \in H$ and the two involutions share two orbits corresponding to the pairs $(\ell, \ell'), (\tau, \tau')$). This implies $(\ell_1, \dots, \ell_6) = \sigma(\ell_1', \dots, \ell_6')$, where $\ell_i' = \ell_i^\perp$.

□

Corollary 10.2.15 *Let $\ell_1, \dots, \ell_6$ be 6 skew lines on a nonsingular cubic surface S. Then they are linearly independent in the Plücker space.*

Proof We first check that any five lines among the six lines are linearly independent. Assume that $\ell_1, \dots, \ell_5$ are linearly dependent. Then one of them, say ℓ_5, lies in the span of $\ell_1, \ell_2, \ell_3, \ell_4$. Let $(\ell_1', \dots, \ell_6')$ is the set of six skew lines which together with $(\ell_1, \dots, \ell_6)$ form a double-six. Then $\ell_1, \ell_2, \ell_3, \ell_4$ lie in the linear line complex $\Omega(\ell_5')$, hence ℓ_5 lies in it too. But this is impossible because ℓ_5 is skew to ℓ_5'.

We know that there exists the unique quadric $\mathcal{Q}$ such that ℓ_i' are polar to $\mathcal{Q}$ with respect to $\mathcal{Q}$ (the Schur quadric). But $(\ell_1', \dots, \ell_6')$ is not projectively equivalent to $(\ell_1, \dots, \ell_6)$. Otherwise, S and its image S' under the projective transformation T will have six common skew lines. It will also have common transversals of each subset of four. Thus the degree of the intersection curve is larger than 9. This shows that the cubic surfaces S and S' coincide and T is an automorphism of S. Its action on $\mathrm{Pic}(S)$ is a reflection with respect to the root corresponding to the double-six. It follows from Section 9.5 that S does not admit such an automorphism. □

Remark 10.2.16 The group $\mathrm{SL}(4)$ acts diagonally on the Cartesian product $\mathbb{G}^6$. Consider the sheaf $\mathcal{L}$ on $\mathbb{G}^6$ defined as the tensor product of the sheaves $p_i^*\mathcal{O}_{\mathbb{G}}(1)$, where $p_i : \mathbb{G}^6 \to \mathbb{G}$ is the i-th projection. The group $\mathrm{SL}(4)$ acts naturally in the space of global sections of $\mathcal{L}$ and its tensor powers. Let

$$R = \bigoplus_{i=0}^{\infty} H^0(\mathbb{G}^6, \mathcal{L}^i)^{\mathrm{SL}(4)}.$$

This is a graded algebra of finite type and its projective spectrum $\mathrm{Proj}(R)$ is the GIT-quotient $\mathbb{G}^6//\mathrm{SL}(4)$. The variety $\mathbb{G}^6$ has an open invariant Zariski subset U which is mapped to $\mathbb{G}^6//\mathrm{SL}(4)$ with fibres equal to $\mathrm{SL}(4)$-orbits. This implies that $\mathbb{G}^6//\mathrm{SL}(4)$ is an irreducible variety of dimension 9. Given six ordered general lines in $\mathbb{P}^3$ their Plücker coordinates make a 6×6-matrix. Its determinant can be considered as a section from the first graded piece R_1 of R. The locus of zeros of this section is a closed subvariety of G^6 whose general point is a 6-tuple of lines contained in a linear line complex. The image of this locus in $\mathbb{G}^6//\mathrm{SL}(4)$ is a hypersurface F. Now the duality of lines by means of a nondegenerate quadric defines an involution on $\mathbb{G}^6$. Since it does not depend on the choice of a quadric up to projective equivalence, the involution descends to an involution of $\mathbb{G}^6//\mathrm{SL}(4)$. The fixed points of this involution is the hypersurface F. One can show that the quotient by the duality involution is an open subset of a certain explicitly described 9-dimensional toric variety X (see [182]).

Finally, observe that a nonsingular cubic surface together with a choice of its geometric marking defines a double-six, which is an orbit of the duality involution in $\mathbb{G}^6//\mathrm{SL}(4)$ and hence a unique point in X that does not belong to the branch locus of the double cover $\mathbb{G}^6//\mathrm{SL}(4) \to X$. Thus we see that the 4-dimensional moduli space of geometrically marked nonsingular cubic surfaces embeds in the 9-dimensional toric variety X.

10.2.3 Linear systems of linear line complexes

Let $W \subset \bigwedge^2 E^\vee$ be a linear subspace of dimension $r+1$. After projectivization and restriction to $G(2, E) \cong G_1(\mathbb{P}^n)$, it defines an r-dimensional linear system $|W|$ of linear line complexes. Let

$$\mathfrak{C}_W = \cap_{\omega \in W} \mathfrak{C}_\omega \subset G(2, E)$$

be the base scheme of $|W|$. It is a subvariety of $G(2, E)$ of dimension $2n-3-r$. Its canonical class is given by the formula

$$\omega_{\mathfrak{C}_W} \cong \mathcal{O}_{\mathfrak{C}_W}(r-n). \tag{10.19}$$

In particular, it is a Fano variety if $r < n$, a Calabi-Yau variety if $r = n$ and a variety of general type if $r > n$.

We also define the *center variety* S_W

$$\mathsf{S}_W = \bigcup_{\omega \in W} \mathsf{S}_\omega.$$

It is also called the *singular variety* of W.

For any $x = [v] \in \mathsf{S}_W$, there exists $\omega \in W$ such that $\omega(v, v') = 0$ for all $v' \in E$, or, equivalently, the line $\ell = \overline{xy}$ is contained in $\mathfrak{C}_\omega$ for all y. This implies that the codimension of $\Omega(x) \cap \mathfrak{C}_W$ in $\Omega(x)$ is $\leq r$, less than expected number $r + 1$. Conversely, since $\Omega(x)$ is irreducible, if the codimension of the intersection $\leq r$, then $\Omega(x)$ must be contained in some $\mathfrak{C}_\omega$, and hence $x \in \mathsf{S}_\omega$. Thus we have proved the following.

Proposition 10.2.17

$$\mathsf{S}_W = \{x \in |E| : \dim \Omega(x) \cap \mathfrak{C}_W \geq n - r - 1\}$$

$$= \{x \in |E| : \Omega(x) \subset \mathfrak{C}_\omega \textit{ for some } \omega \in W\}.$$

For any linear subspace Λ in $|E|$ we can define the *polar subspace* with respect to $|W|$ by

$$i_W(\Lambda) = \bigcap_{\omega \in W} i_\omega(\Lambda).$$

Since $x \in i_\omega(x)$ for any linear line complex $\mathfrak{C}_\omega$, we obtain that, for any $x \in |E|$,

$$x \in i_W(x).$$

It is easy to see that

$$\dim i_W(x) = n - r + \dim |\{\omega \in W : x \in \mathsf{S}_\omega\}|. \tag{10.20}$$

Now we are ready to give examples.

Example 10.2.18 A pencil $|W|$ of linear line complexes in $\mathbb{P}^3 = |E|$ is defined by a line in the Plücker space $\mathbb{P}^5 = |\bigwedge^2 E^\vee|$ which intersects the Klein quadric $G(2, E^\vee)$ at two points or one point with multiplicity 2. The intersection points correspond to special linear line complexes intersecting a given line. Thus, the base locus of a general pencil of linear line complexes consists of lines intersecting two skew lines. It is a nonsingular congruence of lines in $G_1(\mathbb{P}^3)$ of order and degree equal to 1. It is isomorphic to a nonsingular quadric in $\mathbb{P}^3$. It may degenerate to the union of an α-plane and a β-plane if the two lines are coplanar or to a singular quadric if the two lines coincide.

Example 10.2.19 Assume $r = 1$, $n = 2k$ and $|W|$ does not intersect the set of linear line complexes with corank > 1 (the variety of such linear line complexes is of codimension 3 in $|\bigwedge^2 E^\vee|$). Then we have a map $|W| \cong \mathbb{P}^1 \to \mathbb{P}^{2k}$ which assigns to $[\omega] \in |W|$ the center S_ω of $\mathfrak{C}_\omega$. The map is given by the pfaffians of the principal minors of a skew-symmetric matrix of size $(n+1) \times (n+1)$, so the center variety S_W of $|W|$ is a rational curve R_k of degree k in $\mathbb{P}^{2k}$. By Proposition 10.2.17 any secant line of R_k is contained in $\mathfrak{C}_W$. For example, taking $n = 4$, we obtain that the center variety is a conic in a plane contained in $\mathfrak{C}_W$.

Now assume that $r = 2$. We obtain that S_W is a projection of the Veronese surface V_k^2 and the variety of trisecant lines of the surface is contained in $\mathfrak{C}_W$. We have seen it already in the case $k = 2$ (see Subsection 2.1.3).

Example 10.2.20 Let $r = 3$ and $n = 4$ so we have a web $|W|$ of linear line complexes in $\mathbb{P}^9 = |\bigwedge^2 E^\vee|$. We assume that $|W|$ is general enough so that it intersects the Grassmann variety $\mathbb{G}^* = G(2, E^\vee)$ in finitely many points. We know that the degree of $G(2,5)$ is equal to 5, thus $|W|$ intersects G^* at 5 points. Consider the rational map $|W| = \mathbb{P}^3 \dashrightarrow \mathsf{S}_W \subset \mathbb{P}^4$ which assigns to $[\omega] \in |W|$ the center of $\mathfrak{C}_\omega$. As in the previous examples, the map is given by pfaffians of skew-symmetric matrices of size 4×4. They all vanish at the set of five points $p_1, \ldots, p_5$. The preimage of a general line in $\mathbb{P}^4$ is equal to the residual set of intersections of three quadrics, and hence consists of three points. Thus the map is birational map onto a cubic hypersurface. Any line joining two of the five points is blown down to a singular point of the cubic hypersurface. Thus the cubic is isomorphic to the Segre cubic primal. Observe now that $\mathfrak{C}_W$ is a del Pezzo surface of degree 5 and the singular variety of $|W|$ is equal to the projection of the incidence variety $\{(x, \ell) \in \mathbb{P}^4 \times \mathfrak{C}_W : x \in \ell\}$ to $\mathbb{P}^4$. It coincides with the center variety S_W.

One can see the center variety S_W of $|W|$ as the degeneracy locus of a map of rank 3 vector bundles over $|E|$. First, we identify $H^0(|E|, \Omega^1_{|E|}(2))$ with $\bigwedge^2 E^\vee$. To do this, we use the dual Euler sequence twisted by $\mathcal{O}_{|E|}(2)$

$$0 \to \Omega^1_{|E|}(2) \to E^\vee \otimes \mathcal{O}_{|E|}(1) \to \mathcal{O}_{|E|}(2) \to 0. \tag{10.21}$$

Passing to the global sections, we obtain an isomorphism

$$H^0(|E|, \Omega^1_{|E|}(2)) \cong \mathrm{Ker}(E^\vee \otimes E^\vee \to S^2 E^\vee) \cong \bigwedge^2 E^\vee.$$

The composition of the inclusion map $W \to \bigwedge^2 E^\vee$ and the evaluation map $\bigwedge^2 E^\vee \to \Omega^1_{|E|}(2)$ defines a morphism of vector bundles

$$\sigma : W \otimes \mathcal{O}_{|E|} \to \Omega^1_{|E|}(2).$$

The degeneracy locus of this morphism consists of points $x \in |E|$ such that the composition of $E \otimes \mathcal{O}_{|E|}(-1) \to \mathcal{T}^1_{|E|}(-2)$ and the dual map $\sigma^\vee : \mathcal{T}^1_{|E|}(-2) \to W^\vee \otimes \mathcal{O}_{|E|}$ is not of full rank at x. For any $x = [v] \in |E|$, the map of fibres $\phi(x)$ sends a vector v' to the linear function on W defined by $\omega \mapsto \omega(v, v')$. This linear function is equal to zero if and only if the line $\overline{[v][v']}$ intersects $\mathfrak{C}_W$. Applying Proposition 10.2.17, we obtain that the degeneracy locus of point $x = [v]$ for which the rank of $\phi(x)$ is smaller than $r + 1$ must be equal to S_W.

If we choose coordinates and take a basis of W defined by $r + 1$ skew-symmetric bilinear forms $\omega_k = \sum a^{(k)}_{ij} dt_i \wedge dt_j$, then the matrix is

$$\begin{pmatrix} \sum_{s=0}^n a^{(1)}_{1,s} t_s & \dots & \sum_{s=0}^n a^{(1)}_{n,s} t_s \\ \vdots & \vdots & \vdots \\ \sum_{s=0}^n a^{(r+1)}_{1,s} t_s & \dots & \sum_{s=0}^n a^{(r+1)}_{n,s} t_s \end{pmatrix},$$

where $a^k_{ij} = -a^k_{ji}$.

The expected dimension of the degeneracy locus is equal to $n - r$. Assume that this is the case. It follows from Example 14.3.2 in [231] that

$$\deg \mathsf{S}_W = \deg c_{n-r}(\Omega^1_{|E|}(2)) = \sum_{i=0}^{n-r} (-1)^i \binom{n-i}{r}. \tag{10.22}$$

Example 10.2.21 Assume $n + 1 = 2k$. If $\omega \in W$ is nondegenerate, then $\mathsf{S}_\omega = \emptyset$. Otherwise, $\dim \mathsf{S}_\omega \geq 1$. Thus the varieties S_W are ruled by linear subspaces. For a general W of dimension $1 < r < n$, the dimensions of these subspaces is equal to 1 and each point in S_W is contained in a unique line S_ω. In other words, S_W is a scroll with 1-dimensional generators parameterized by the subvariety B of $|W|$ of degenerate ω's. Thus B is equal to the intersection of $|W|$ with a pfaffian hypersurface of degree k in $|\Lambda^2 E^\vee|$. The scrolls S_W are called *Palatini scrolls*. If $n = 3$, the only Palatini scroll is a quadric in $\mathbb{P}^3$ and B is a conic. In $\mathbb{P}^5$ we get a 3-dimensional Palatini scroll of degree 7 defined by a web $|W|$ of linear line complexes. The family of generators B is a cubic surface in $|W|$. We refer to [418] for the study of this scroll. There is also a Palatini ruled surface of degree 6 defined by a net of linear line complexes. Its generators are parameterized by a plane cubic curve. If we take W with $\dim W = 5$, we get a quartic hypersurface in $\mathbb{P}^5$.

10.3 Quadratic line complexes

10.3.1 Generalities

Recall that a quadratic line complex $\mathfrak{C}$ is the intersection of the Grassmannian $\mathbb{G} = G(2, E) \subset |\bigwedge^2 E|$ with a quadric hypersurface Q. Since $\omega_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-n-1)$, by the adjunction formula

$$\omega_{\mathfrak{C}} \cong \mathcal{O}_{\mathfrak{C}}(1-n).$$

If $\mathfrak{C}$ is nonsingular, i.e. the intersection is transversal, we obtain that $\mathfrak{C}$ is a Fano variety of index $n-1$.

Consider the incidence variety $\mathsf{Z}_{\mathfrak{C}}$ together with its natural projections $p_{\mathfrak{C}} : \mathsf{Z}_{\mathfrak{C}} \to \mathbb{P}^n$ and $q_{\mathfrak{C}} : \mathsf{Z}_{\mathfrak{C}} \to K$. For each point $x \in \mathbb{P}^n$ the fiber of $p_{\mathfrak{C}}$ is isomorphic to the intersection of the Schubert variety $\Omega(x)$ with Q. We know that $\Omega(x)$ is isomorphic to $\mathbb{P}^{n-1}$ embedded in $|\bigwedge^2 E|$ as a linear subspace. Thus the fiber is isomorphic to a quadric in $\mathbb{P}^{n-1}$. This shows that $\mathfrak{C}$ admits a structure of a *quadric bundle*, i.e. a fibration with fibres isomorphic to a quadric hypersurface. The important invariant of a quadric bundle is its *discriminant locus*. This is the set of points of the base of the fibration over which the fiber is a singular quadric or the whole space. In our case we have the following classical definition.

Definition 10.3.1 *The* singular variety Δ *of a quadratic line complex is the set of points $x \in \mathbb{P}^n$ such that $\Omega(x) \cap Q$ is a singular quadric in $\Omega(x) = \mathbb{P}^{n-1}$ or $\Omega(x) \subset Q$.*

We will need the following fact from linear algebra that is rarely found in modern text-books on the subject.

Lemma 10.3.2 *Let $A = (a_{ij}), B = (b_{ij})$ be two matrices of sizes $k \times m$ and $m \times k$ with $k \leq m$. Let $|A_I|, |B_I|, I = (i_1, \dots, i_k), 1 \leq i_1 < \dots < i_k \leq m$, be maximal minors of A and B. For any $m \times m$-matrix $G = (g_{ij})$*

$$|A \cdot G \cdot B| = \sum_{I,J} g_{IJ} |A_I||B_J|,$$

where $g_{IJ} = g_{i_1 j_1} \cdots g_{i_k j_k}$.

Proof Consider the product of the following block-matrices

$$\begin{pmatrix} A \cdot B & A \\ 0_{mk} & I_m \end{pmatrix} \cdot \begin{pmatrix} I_k & 0_{km} \\ -B & I_m \end{pmatrix} = \begin{pmatrix} 0_{kk} & A \\ -B & I_m \end{pmatrix}, \tag{10.23}$$

where 0_{ab} is the zero matrix of size $a \times b$ and I_a is the identity matrix of size $a \times a$. The determinant of the first matrix is equal to $|A \cdot B|$, the determinant

of the second matrix is equal to 1. Applying the Laplace formula, we find that the determinant of the product is equal to $\sum |A_I||B_I|$. Now we apply (10.23), replacing A with $A \cdot G$,. Write a j-th column of $A \cdot G$ as the sum $\sum_{i=1}^{m} g_{ij}A_i$, where A_i are the columns of A. Then

$$|(A \cdot G)_{j_1,\ldots,j_k}| = \sum_{1 \le i_1 < \ldots < i_k \le m} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_k j_k} |A_{i_1,\ldots,i_k}|.$$

This proves the assertion. □

Suppose we have a bilinear form $b : E \times E \to \mathbb{K}$ on a vector space E over a field $\mathbb{K}$. Let $G = \big(b(e_i, e_j)\big)$ be the matrix of the bilinear form with respect to a basis $e_1, \ldots, e_m$. Let L be a subspace of E with basis $f_1, \ldots, f_k$. Then the matrix $G_L = \big(b(f_i, f_j)\big)$ is equal to the product ${}^tA \cdot G \cdot A$, where the $f_j = \sum a_{ij}e_i$. It follows from the previous Lemma that $|G_L| = \sum_{I,J} g_{IJ}|A_I||A_J|$. If we extend b to $\bigwedge^k E$ by the formula

$$b(v_1, \ldots, v_k; w_1, \ldots, w_k) = \det\big(b(v_i, w_j)\big),$$

then the previous formula gives an explicit expression for $b(f_1 \wedge \ldots \wedge f_k, f_1 \wedge \ldots \wedge f_k)$. If $E = \mathbb{R}^n$ and we take b to be the Euclidean inner-product, we get the well-known formula for the area of the parallelogram spanned by vectors $f_1, \ldots, f_k$ in terms of the sum of squares of maximal minors of the matrix with columns equal to f_j. If $m = 3$ this is the formula for the length of the cross-product of two vectors.

Proposition 10.3.3 *Δ is a hypersurface of degree $2(n-1)$.*

Proof Consider the map

$$i : |E| \to G(n, \bigwedge^2 E),\ x \mapsto \Omega(x). \tag{10.24}$$

If $x = [v_0]$, the linear subspace of $\bigwedge^2 E$ corresponding to $\Omega(x)$ is the image of E in $\bigwedge^2 E$ under the map $v \mapsto v \wedge v_0$. This is an n-dimensional subspace $\Lambda(x)$ of $\bigwedge^2 E$. Hence it defines a point in the Grassmann variety $G(n, \bigwedge^2 E)$. If we write $v_0 = \sum_{i=0}^{n} a_i e_i$, where we assume that $a_n \neq 0$, then $\Lambda(x)$ is spanned by the vectors $e_i \wedge v_0 = \sum_{j \neq i} a_j e_i \wedge e_j, i = 0, \ldots, n$. Thus the rows of the matrix of Plücker coordinates of the basis are linear functions in coordinates of v_0. Its maximal minors are polynomials of order $n+1$. Observe now that each (in)-th column contains a_n in the i-th row and has zero elsewhere. This easily implies that all maximal minors are divisible by a_n. Thus the Plücker coordinates of $\Lambda(x)$ are polynomials of degree $n-1$ in coordinates of v_0. We see now that the map i is given by a linear system of divisors of degree $n-1$. Fix a quadric Q in $|\bigwedge^2 E|$ that does not vanish on $\mathbb{G}$. For any $n-1$-dimensional linear subspace L

of $|\bigwedge^2 E|$, the intersection of Q with L is either a quadric or the whole L. Let us consider the locus D of L's such that this intersection is a singular quadric. We claim that this is a hypersurface of degree 2.

Let $b : E \times E$ be a nondegenerate symmetric bilinear form on a vector space E of dimension $n+1$. The restriction of b to a linear subspace $L \subset E$ with a basis $(f_1, \dots, f_k)$ is a degenerate bilinear form if and only if the determinant of the matrix $\big(b(f_i, f_j)\big)$ is equal to zero. If we write $f_i = \sum a_{ij} e_j$ in terms of a basis in E, we see that this condition is polynomial of degree $2k$ in coefficients a_{ij}. By the previous Lemma, this polynomial can be written as a quadratic polynomial in maximal minors of the matrix (a_{ij}). Applying this to our situation we interpret the maximal minors as the Plücker coordinates of L and obtain that D is a quadric hypersurface.

It remains for us to use the fact that $\Delta = i^{-1}(D)$, where i is given by polynomials of degree $n-1$. □

Let

$$\Delta_k = \{x \in \Delta : \operatorname{corank} Q \cap \Omega(x) \geq k\}.$$

These are closed subvarieties of Δ_k.

Let

$$\tilde{\Delta} = \{(x, \ell) \in \mathsf{Z}_{\mathfrak{C}} : \operatorname{rank} d_{p_{\mathfrak{C}}}(x, \ell) < n\}. \tag{10.25}$$

In other words, $\tilde{\Delta}$ is the locus of points in $\mathsf{Z}_{\mathfrak{C}}$ where the projection $p_{\mathfrak{C}} : \mathsf{Z}_{\mathfrak{C}} \to \mathbb{P}^n$ is not smooth. This set admits a structure of a closed subscheme of $Z_{\mathfrak{C}}$ defined locally by vanishing of the maximal minors of the Jacobian matrix of the map $p_{\mathfrak{C}}$. Globally, we have the standard exact sequence of the sheaves of differentials

$$0 \to p_{\mathfrak{C}}^* \Omega^1_{\mathbb{P}^n} \xrightarrow{\delta} \Omega_{\mathsf{Z}_{\mathfrak{C}}} \to \Omega^1_{\mathsf{Z}_{\mathfrak{C}}/\mathbb{P}^n} \to 0, \tag{10.26}$$

and the support of $\tilde{\Delta}$ is equal to the set of points where $\Omega^1_{\mathsf{Z}_{\mathfrak{C}}/\mathbb{P}^n}$ is not locally free. Locally the map δ is given by a matrix of size $n \times (2n-2)$. Thus $\tilde{\Delta}$ is given locally by $n \times n$ minors of this matrix and is of dimension n.

Tensoring (7.31) with the residue field $\kappa(p)$ at a point $p = (x, \ell) \in \mathsf{Z}_{\mathfrak{C}}$, we see that $\tilde{\Delta}$ is equal to the degeneracy locus of points where the map $\delta_p : (p_{\mathfrak{C}}^* \Omega^1_{\mathbb{P}^n})_p \to (\Omega^1_{\mathsf{Z}_{\mathfrak{C}}})_p$ is not injective. Using the *Thom-Porteous formula* (see [231], 14.4), we can express the class of $\tilde{\Delta}$ in $H^*(Z_{\mathfrak{C}}, \mathbb{Z})$.

Definition 10.3.4 *Let $\mathfrak{C}$ be a line complex of degree d in $G_1(\mathbb{P}^n)$. A line ℓ in $\mathfrak{C}$ is called* singular *if ℓ is a singular point of the intersection $\Omega(x) \cap \mathfrak{C}$ for some $x \in \mathbb{P}^n$ or any point on $\Omega(x)$ if $\Omega(x) \subset \mathfrak{C}$. The locus $S(\mathfrak{C})$ of singular lines is called the* singular variety *of $\mathfrak{C}$.*

Proposition 10.3.5 *Assume $n+1 = 2k$ and $\mathfrak{C}$ is nonsingular. Then the singular variety $S(\mathfrak{C})$ of $\mathfrak{C}$ is equal to the intersection of $\mathfrak{C}$ with a hypersurface of degree $k(d-1)$.*

Proof Let ℓ be a singular line of $\mathfrak{C} = \mathbb{G} \cap X$, where X is a hypersurface of degree d. We have $\Omega(x) \subset \mathbb{T}_\ell(\mathfrak{C}) = \mathbb{T}_\ell(\mathbb{G}) \cap \mathbb{T}_\ell(X)$. Thus $\Omega(x) \subset \mathbb{T}_\ell(X) \cap \mathbb{G}$. By Proposition 10.2.4, the linear line complex $\mathbb{T}_\ell(X) \cap \mathbb{G}$ consists of lines intersecting a line and its polar $(n-2)$-plane unless it is singular. Since $\Omega(x)$ is not contained in the Schubert variety of lines intersecting a codimension 2 linear subspace, we obtain that $\mathbb{T}_\ell(X) \cap \mathbb{G}$ is singular. This shows that the singular variety $S(\mathfrak{C})$ of $\mathfrak{C}$ consists of lines in $\mathfrak{C}$ such that $\mathbb{T}_\ell(X)$ coincides with a tangent hyperplane of $\mathbb{G}$. In other words,

$$S(\mathfrak{C}) = \gamma^{-1}(\mathbb{G}^\vee), \tag{10.27}$$

where $\gamma : \mathfrak{C} \to (\mathbb{P}^n)^\vee$ is the restriction of the Gauss map $X \to (\mathbb{P}^n)^\vee$ to $\mathfrak{C}$. Since $\mathfrak{C}$ is nonsingular, X is nonsingular at any point of $X \cap \mathbb{G}$, and hence γ is well-defined. It remains for us to use that γ is given by polynomials of degree $d-1$, the partials of X. □

Let $n = 3$ and let $\mathfrak{C}$ be a line complex defined by a hypersurface $X = V(\Phi)$ of degree d in the Plücker space. The equation of the singular surface $S(\mathfrak{C})$ in Plücker coordinates is easy to find. Let $\Phi_{ij} = \frac{\partial \Phi}{\partial p_{ij}}(l)$, where $[l] = \ell$. The tangent hyperplane to X at the point ℓ is given by the equation

$$\sum_{1 \le i < j \le 4} \Phi_{ij}(l) p_{ij} = 0.$$

Since the dual quadric $\mathbb{G}^*$ is given by the same equation as $\mathbb{G}$, we obtain the equation of $S(\mathfrak{C})$ in $\mathbb{G}$:

$$\Phi_{12}\Phi_{34} - \Phi_{13}\Phi_{24} + \Phi_{14}\Phi_{23} = 0.$$

10.3.2 Intersection of two quadrics

Let Q_1, Q_2 be two quadrics in $\mathbb{P}^n$ and $X = Q_1 \cap Q_2$. We assume that X is nonsingular. It follows from the proof of Theorem 8.6.2 that this is equivalent to the condition that the pencil $\mathcal{P}$ of quadrics spanned by Q_1, Q_2 has exactly $n+1$ singular quadrics of corank 1. This set can be identified with a set of $n+1$ points $p_1, \dots, p_{n+1}$ in $\mathbb{P}^1 \cong \mathcal{P}$.

If $n = 2g+1$, we get the associated nonsingular hyperelliptic curve C of genus g, the double cover of $\mathbb{P}^1$ branched at $p_1, \dots, p_{2g+2}$.

The variety X is of degree 4 in $\mathbb{P}^n, n \ge 3$, of dimension $n-2$. Its canonical

class is equal to $-(n-3)h$, where h is the class of a hyperplane section. When $n = 4$ it is a quartic del Pezzo surface.

Theorem 10.3.6 (A. Weil) *Assume $n = 2g + 1$. Let $F(X)$ be the variety of $g-1$-dimensional linear subspaces contained in X. Then $F(X)$ is isomorphic to the Jacobian variety of the curve C and also to the intermediate Jacobian of X.*

Proof We will restrict ourselves only to the case $g = 2$, leaving the general case to the reader. For each $\ell \in F(X)$ consider the projection map $p_\ell : X' = X \setminus \ell \to \mathbb{P}^3$. For any point $x \in X$ not on ℓ, the fiber over $p_\ell(x)$ is equal to the intersection of the plane $\ell_x = \langle \ell, x \rangle$ with X'. The intersection of this plane with a quadric Q from the pencil $\mathcal{P}$ is a conic containing ℓ and another line ℓ'. If we take two nonsingular generators of $\mathcal{P}$, we find that the fiber is the intersection of two lines or the whole $\ell' \in F(X)$ intersecting ℓ. In the latter case, all points on $\ell' \setminus \ell$ belong to the same fibre. Since all quadrics from the pencil intersect the plane $\langle \ell, \ell' \rangle$ along the same conic $\ell \cup \ell'$, there exists a unique quadric $Q_{\ell'}$ from the pencil which contains $\langle \ell, \ell' \rangle$. The plane belongs to one of the two rulings of planes on $Q_{\ell'}$ (or a unique family if the quadric is singular). Note that each quadric from the pencil contains at most one plane in each ruling which contains ℓ (two members of the same ruling intersect along a subspace of even codimension). Thus we can identify the following sets:

- pairs (Q, r), where $Q \in \mathcal{P}$, r is a ruling of planes in Q,
- $B = \{\ell' \in F(X) : \ell \cap \ell' \neq \emptyset\}$.

If we identify $\mathbb{P}^3$ with the set of planes in $\mathbb{P}^5$ containing ℓ, then the latter set is a subset of $\mathbb{P}^3$. Let D be the union of ℓ''s from B. The projection map p_ℓ maps D to B with fibres isomorphic to $\ell' \setminus \ell \cap \ell'$.

Extending p_ℓ to a morphism $f : \bar{X} \to \mathbb{P}^3$, where $\bar{X}$ is the blow-up of X with center at ℓ, we obtain that f is an isomorphism outside B and that the fibres over points in B are isomorphic to $\mathbb{P}^1$. Observe that $\bar{X}$ is contained in the blow-up $\bar{\mathbb{P}}^5$ of $\mathbb{P}^5$ along ℓ. The projection f is the restriction of the projection $\bar{\mathbb{P}}^5 \to \mathbb{P}^3$ which is a projective bundle of relative dimension 2. The crucial observation now is that B is isomorphic to our hyperelliptic curve C. In fact, consider the incidence variety

$$\mathcal{X} = \{(Q, \pi) \in \mathcal{P} \times G_2(\mathbb{P}^5) : \pi \subset Q\}.$$

Its projection to $\mathcal{P}$ has fiber over Q isomorphic to the rulings of planes in Q. It consists of two connected components outside of the set of singular quadrics and one connected component over the set of singular quadrics. Taking the

Stein factorization, we get a double cover of $\mathcal{P} = \mathbb{P}^1$ branched along six points. It is isomorphic to C.

Now the projection map p_ℓ maps each line ℓ' intersecting ℓ to a point in $\mathbb{P}^3$. We will identify the set of these points with the curve B. A general plane in $\mathbb{P}^3$ intersects B at $d = \deg B$ points. The preimage of the plane under the projection $p_\ell : X \dashrightarrow \mathbb{P}^3$ is isomorphic to the complete intersection of two quadrics in $\mathbb{P}^4$. It is a del Pezzo surface of degree 4, hence it is obtained by blowing up five points in $\mathbb{P}^2$. Thus $d = 5$. An easy argument using Riemann-Roch shows that B lies on a unique quadric $\mathcal{Q} \subset \mathbb{P}^3$. Its preimage under the projection $\bar{X} \to \mathbb{P}^3$ is the exceptional divisor E of the blow-up $\bar{X} \to X$. It follows from Subsection 7.1.5 that the normal bundle of ℓ in X is trivial, so $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathcal{Q}$ is a nonsingular quadric. Thus (X, ℓ) defines a biregular model $B \subset \mathbb{P}^3$ of C such that B is of degree 5 and lies on a unique nonsingular quadric $\mathcal{Q}$. One can show that the latter condition is equivalent to the fact that the invertible sheaf $\mathcal{O}_B(1) \otimes \omega_B^{-2}$ is not effective. It is easy to see that B is of bidegree $(2, 3)$.

Let us construct an isomorphism between $\mathrm{Jac}(C)$ and $F(X)$. Recall that $\mathrm{Jac}(C)$ is birationally isomorphic to the symmetric square $C^{(2)}$ of the curve C. The canonical map $C^{(2)} \to \mathrm{Pic}^2(C)$ defined by $x + y \mapsto [x + y]$ is an isomorphism over the complement of one point represented by the canonical class of C. Its fiber over K_C is the linear system $|K_C|$. Also note that $\mathrm{Pic}^2(C)$ is canonically identified with $\mathrm{Jac}(C)$ by sending a divisor class ξ of degree 2 to the class $\xi - K_C$.

Each line ℓ' skew to ℓ is projected to a secant line of B. In fact, $\langle \ell, \ell' \rangle \cap X$ is a quartic curve in the plane $\langle \ell, \ell' \rangle \cong \mathbb{P}^3$ that contains two skew line components. The residual part is the union of two skew lines m, m' intersecting both ℓ and ℓ'. Thus ℓ' is projected to the secant line joining two points on C which are the projections of the lines m, m'. If $m = m'$, then ℓ' is projected to a tangent line of B. Thus the open subset of lines in X skew to ℓ is mapped bijectively to an open subset of $C^{(2)}$ represented by "honest" secants of C, i.e. secants which are not 3-secants. Each line $\ell' \in F(X) \setminus \{\ell\}$ intersecting ℓ is projected to a point b of B. The line f of the ruling of $\mathcal{Q}$ intersecting B with multiplicity 3 and passing through a point $b \in B$ defines a positive divisor D of degree 2 such that $f \cap B = b + D$. The divisor class $[D] \in \mathrm{Pic}^2(C)$ is assigned to ℓ'. So we see that each trisecant line of B (they are necessarily lie on $\mathcal{Q}$) defines three lines passing through the same point of ℓ. By taking a section of X by a hyperplane tangent to X at a point $x \in X$, we see that x is contained in four lines (taken with some multiplicity). Finally, the line ℓ itself corresponds to K_C. This establishes an isomorphism between $\mathrm{Pic}^2(C)$ and $F(X)$.

□

Note that we have proved that X is a rational variety by constructing an explicit rational map from X to $\mathbb{P}^3$. This map becomes a regular map after we blow up a line ℓ on X. The image of the exceptional divisor is a quadric. This map blows down the union of lines on X that intersect ℓ to a genus 2 curve C of degree 5 lying on the quadric. The inverse map $\mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^5$ is given by the linear system of cubic hypersurfaces through the curve C. It becomes a regular map after we blow-up C. Since any trisecant line of C defined by one of the rulings of the quadric blows down to a point, the image of the proper transform of the quadric is the line ℓ on X. The exceptional divisor is mapped to the union of lines on X intersecting ℓ.

10.3.3 Kummer surfaces

We consider the case $n = 3$. The quadratic line complex $\mathfrak{C}$ is the intersection of two quadrics $\mathbb{G} \cap Q$. We shall assume that $\mathfrak{C}$ is nonsingular. Let C be the associated hyperelliptic curve of genus 2.

First let us look at the singular surface Δ of $\mathfrak{C}$. By Proposition 10.3.3, it is a quartic surface. For any point $x \in \Delta$, the conic $C_x = \mathfrak{C} \cap \Omega(x)$ is the union of two lines. A line in $\mathbb{G}$ is always equal to a 1-dimensional Schubert variety. In fact, $\mathbb{G}$ is a nonsingular quadric of dimension 4, and hence contains two 3-dimensional families of planes. These are the families realized by the Schubert planes $\Omega(x)$ and $\Omega(\Pi)$. Hence a line must be a pencil in one of these planes, which shows that $C_x = \Omega(x, \Pi_1) \cup \Omega(x, \Pi_2)$ for some planes Π_1, Π_2 in $\mathbb{P}^3$. Any line in $\mathfrak{C}$ is equal to some $\Omega(x, A)$ and hence is equal to an irreducible component of the conic C_x. Thus we see that any line in $\mathfrak{C}$ is realized as an irreducible component of a conic $C_x, x \in \mathfrak{C}$. It follows from Theorem 10.3.6 that the variety of lines $F(\mathfrak{C})$ in $\mathfrak{C}$ is isomorphic to the Jacobian variety of C.

Proposition 10.3.7 *The variety $F(\mathfrak{C})$ of lines in $\mathfrak{C}$ is a double cover of the quartic surface Δ. The cover ramifies over the set Δ_1 of points such that the conic $C_x = p_{\mathfrak{C}}^{-1}(x)$ is a double line.*

Let $x \in \Delta$ and $C_x = \Omega(x, \Pi_1) \cup \Omega(x, \Pi_2)$. A singular point of C_x is called a *singular line* of $\mathfrak{C}$. If $x \notin \Delta_1$, then C_x has only one singular point equal to $\Omega(x, \Pi_1) \cap \Omega(x, \Pi_2)$. Otherwise, it has the whole line of them.

Let $S = S(\mathfrak{C})$ be the singular surface of $\mathfrak{C}$. By Proposition 10.3.5, S is a complete intersection of three quadrics.

By the adjunction formula, we obtain $\omega_S \cong \mathcal{O}_S$. The assertion that S is nonsingular follows from its explicit equations (10.28) given below. Thus S is a K3 surface of degree 8.

Theorem 10.3.8 *The set of pairs (x, ℓ), where ℓ is a singular line containing x is isomorphic to the variety $\tilde{\Delta} \subset Z_{\mathfrak{C}}$, the locus of points where the morphism $p_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathbb{P}^3$ is not smooth. It is a nonsingular surface with trivial canonical class. The projection $p_{\mathfrak{C}} : \tilde{\Delta} \to \Delta$ is a resolution of singularities. The projection $q_{\mathfrak{C}} : \tilde{\Delta} \to S$ is an isomorphism. The surface S is equal to $\mathfrak{C} \cap \mathcal{Q}$, where $\mathcal{Q}$ is a quadric in $\mathbb{P}^5$.*

Proof The first assertion is obvious since the fibres of $p_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathbb{P}^3$ are isomorphic to the conics C_x. To see that $q_{\mathfrak{C}}$ is one-to-one we have to check that a singular line ℓ cannot be a singular point of two different fibres C_x and C_y. The planes $\Omega(x)$ and $\Omega(y)$ intersect at one point $\ell = \overline{xy}$ and hence span $\mathbb{P}^4$. If Q is tangent to both planes at the same point ℓ, then the two planes are contained in $\mathbb{T}_\ell(Q) \cap \mathbb{T}_\ell(G)$, hence $\mathfrak{C} = Q \cap G$ is singular at ℓ. This contradicts our assumption on $\mathfrak{C}$. Thus the projection $\tilde{\Delta} \to S$ is one-to-one. Since the fibres of $q_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathfrak{C}$ are projective lines, this easily implies that the restriction of $q_{\mathfrak{C}}$ to $\tilde{\Delta}$ is an isomorphism onto S.

□

Theorem 10.3.9 *The set Δ_1 consists of 16 points, and each point is an ordinary double point of the singular surface Δ.*

Proof Let $A = F(\mathfrak{C})$ be the variety of lines in $\mathfrak{C}$. We know that it is a double cover of Δ ramified over the set Δ_1. Since Δ is isomorphic to S outside Δ_1, we see that A admits an involution with a finite set F of isolated fixed points such that the quotient is birationally isomorphic to a K3 surface. The open set $A \setminus F$ is an unramified double cover of the complement of $s = \#F$ projective lines in the K3 surface S. For any variety Z we denote by $e_c(Z)$ the topological Euler characteristic with compact support. By the additivity property of e_c, we get $e_c(A - S) = e(A) - s = 2(e_s(S) - 2s) = 48 - 4s$. Thus $e(A) = 48 - 3s$. Since $A \cong \mathrm{Jac}(C)$, we have $e(A) = 0$. This gives $s = 16$. Thus Δ has 16 singular points. Each point is resolved by a (-2)-curve on S. This implies that each singular point is a rational double point of type A_1, i.e. an ordinary double point. □

Definition 10.3.10 *For any abelian variety A of dimension g the quotient of A by the involution $a \mapsto -a$ is denoted by* $\mathrm{Kum}(A)$ *and is called the* Kummer variety *of A.*

Note that $\mathrm{Kum}(A)$ has 2^{2g} singular point locally isomorphic to the cone over the Veronese variety V_g^{g-1}. In the case $g = 2$ we have 16 ordinary double points. It is easy to see that any involution with this property must coincide

with the negation involution (look at its action in the tangent space, and use that A is a complex torus). This gives the following.

Corollary 10.3.11 *The singular surface of $\mathfrak{C}$ is isomorphic to the Kummer surface of the Jacobian variety of the hyperelliptic curve C of genus 2.*

The Kummer variety of a Jacobian variety of a nonsingular curve is called a *jacobian Kummer variety*.

Proposition 10.3.12 *The surface S contains two sets of 16 disjoint lines.*

Proof The first set is formed by the lines $q_{\mathfrak{C}}(p_{\mathfrak{C}}^{-1}(z_i))$, where $z_1, \dots, z_{16}$ are the singular points of the singular surface. The other set comes from the dual picture. We can consider the dual incidence variety

$$\check{\mathsf{Z}}_{\mathfrak{C}} = \{(\Pi, \ell) \in (\mathbb{P}^3)^\vee \times \mathfrak{C} : \ell \subset \Pi\}.$$

The fibres of the projection to $(\mathbb{P}^3)^\vee$ are conics. Again we define the singular surface $\check{\Delta}$ as the locus of planes such that the fiber is the union of lines. A line in the fiber is a pencil of lines in the plane. These pencils form the set of lines in $\mathfrak{C}$. The lines are common to two pencils if lines are singular lines of $\mathfrak{C}$. Thus we see that the surface S can be defined in two ways by using the incidence $Z_{\mathfrak{C}}$ or $\check{Z}_{\mathfrak{C}}$. As before, we prove that $\check{\Delta}$ is the quotient of the abelian surface A and is isomorphic to the Kummer surface of C. The lines in S corresponding to singular points of $\check{\Delta}$ is the second set of 16 lines. □

Choosing six mutually apolar linear line complexes we write the equation of the Klein quadric as a sum of squares. The condition of nondegeneracy allows one to reduce the quadric Q to the diagonal form in these coordinates. Thus the equation of the quadratic line complex can be written in the form

$$\sum_{i=0}^{5} t_i^2 = \sum_{i=0}^{5} a_i t_i^2 = 0. \tag{10.28}$$

Since $\mathfrak{C}$ is nonsingular, $a_i \neq a_j,\ i \neq j$. The parameters in the pencil corresponding to six singular quadrics are $(t_0, t_1) = (-a_0, 1), i = 0, \dots, 5$. Thus the hyperelliptic curve C has the equation

$$t_2^2 = (t_1 + a_0 t_0) \cdots (t_1 + a_5 t_0),$$

which has to be considered as an equation of degree 6 in the weighted plane $\mathbb{P}(1, 1, 3)$.

To find the equation of the singular surface S of $\mathfrak{C}$, we apply (10.27). The dual of the quadric $V(\sum a_i t_i^2)$ is the quadric $V(\sum a_i^{-1} u_i^2)$. Its preimage under

the Gauss map defined by the quadric $V(\sum t_i^2)$ is the quadric $V(\sum a_i^{-1}x_i^2)$. After scaling $t_i \mapsto a_i t_i$, we obtain that the surface S, a nonsingular model, of the Kummer surface, is given by the equations

$$\sum_{i=0}^{5} t_i^2 = \sum_{i=0}^{5} a_i t_i^2 = \sum_{i=0}^{5} a_i^2 t_i^2 = 0. \tag{10.29}$$

We know that the surface given by the above equations contains 32 lines. Consider six lines ℓ_i in $\mathbb{P}^2$ given by the equations

$$X_0 + a_i X_1 + a_i^2 X_2 = 0, \quad i = 0, \dots, 5. \tag{10.30}$$

Since the points $(1, a_i, a_i^2)$ lie on the conic $X_0X_2 - X_1^2 = 0$, the lines ℓ_i are tangent to the conic.

Lemma 10.3.13 *Let $X \subset \mathbb{P}^{2k-1}$ be a variety given by complete intersection of k quadrics*

$$q_i = \sum_{j=0}^{2k-1} a_{ij} t_j^2 = 0, \quad i = 1, \dots, k.$$

Consider the group G of projective transformations of $\mathbb{P}^{2k-1}$ that consists of transformations

$$[t_0, \dots, t_{2k-1}] \mapsto [\epsilon_0 t_0, \dots, \epsilon_{2k-1} t_{2k-1}],$$

where $\epsilon_i = \pm 1$ and $\epsilon_0 \cdots \epsilon_{2k-1} = 1$. Then X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the union of $2k$ hyperplanes with equations explicitly given below in (10.31).

Proof Let $R = \mathbb{C}[t_0, \dots, t_{2k-1}]/(q_1, \dots, q_k)$ be the ring of projective coordinates of X. Then the subring of invariants R^G is generated by the cosets of $t_0^2, \dots, t_{2k-1}^2$ and $t_0 \cdots t_{2k-1}$. Since $(t_0 \cdots t_{2k-1})^2 = t_0^2 \cdots t_{2k-1}^2$, we obtain that

$$R^G \cong \mathbb{C}[t_0, \dots, t_{2k-1}, t]/I,$$

where I is generated by

$$\sum_{j=0}^{2k-1} a_{ij} t_j, \ i = 1, \dots, k, \quad t^2 - t_0 \cdots t_{2k-1}.$$

Let $A = (a_{ij})$ be the matrix of the coefficients a_{ij}. Its rank is equal to k.

Choose new coordinates t'_i in $\mathbb{C}^{2k}$ such that $t'_{i+k-1} = \sum_{j=0}^{2k-1} a_{ij}t_j,\ i = 1,\dots,k$. Write

$$t_i = \sum_{j=0}^{k-1} b_{ij}t'_j \mod (t'_k,\dots,t'_{2k-1}),\ i = 0,\dots,2k-1.$$

Then

$$X/G \cong \text{Proj } R^G \cong \text{Proj}(\mathbb{C}[t'_0,\dots,t'_{k-1},t])/\Big(t^2 - \prod_{i=0}^{2k-1}\sum_{j=0}^{k-1} b_{ij}t'_j]\Big).$$

Thus X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$\sum_{j=0}^{k-1} b_{ij}z_j = 0,\ j = 0,\dots,2k-1. \tag{10.31}$$

□

Corollary 10.3.14 *Suppose the set of $2k$ points*

$$[a_{00},\dots,a_{k0}],\dots,[a_{0\ 2k-1},\dots,a_{k\ 2k-1}]$$

in $\mathbb{P}^{k-1}$ is projectively equivalent to an ordered set of points on a Veronese curve of degree $k-1$. Then X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$a_{0j}z_0 + \cdots + a_{k-1j}z_{k-1} = 0, \quad i = 0,\dots,2k-1.$$

Proof Choose coordinates such that the matrix $A = (a_{ij})$ has the form

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{2k} \\ \vdots & \vdots & \vdots\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_{2k}^{k-1} \end{pmatrix}.$$

Let

$$D_j = \prod_{1\le i<j\le k} (\alpha_j - \alpha_i)$$

and

$$f(x) = (x-\alpha_1)\cdots(x-\alpha_k) = a_0 + a_1x + \cdots + a_kx^k,$$

$$f_j(x) = \frac{f(x)}{D_j(x-\alpha_j)} = a_{0j} + a_{1j}x + \cdots + a_{k-1j}x^{k-1},\ j = 1,\dots,k.$$

We have

$$B = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \vdots & \vdots & \vdots\!\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \end{pmatrix}^{-1} = \begin{pmatrix} a_{01} & a_{11} & \dots & a_{k-11} \\ a_{02} & a_{12} & \dots & a_{k-12} \\ \vdots & \vdots & \vdots & \\ a_{0k} & a_{1k} & \dots & a_{k-1k} \end{pmatrix}.$$

Multiplying A by B on the left we obtain

$$B \cdot A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & f_1(\alpha_{k+1}) & \dots & f_1(\alpha_{2k}) \\ 0 & 1 & 0 & \dots & 0 & f_2(\alpha_{k+1}) & \dots & f_2(\alpha_{2k}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & f_k(\alpha_{k+1}) & \dots & f_k(\alpha_{2k}) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\alpha_1) & \dots & f_1(\alpha_k) & f_1(\alpha_{k+1}) & \dots & f_1(\alpha_{2k}) \\ f_2(\alpha_1) & \dots & f_2(\alpha_k) & f_2(\alpha_{k+1}) & \dots & f_2(\alpha_{2k}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_k(\alpha_1) & \dots & f_k(\alpha_k) & f_k(\alpha_{k+1}) & \dots & f_k(\alpha_{2k}) \end{pmatrix}.$$

The polynomials $f_1(x), \dots, f_k(x)$ form a basis in the space of polynomials of degree $\leq k-1$. Thus we see that the columns of the matrix $B \cdot A$ can be taken as the projective coordinates of the images of points $[1, \alpha_1], \dots, [1, \alpha_{2k}] \in \mathbb{P}^1$ under a Veronese map. Under the projective transformation defined by the matrix B, the ordered set of columns of matrix A is projectively equivalent to the set of points defined by the column of the matrix $B \cdot A$. Write the matrix $B \cdot A$ in the block-form $[I_k \; C]$. Then the null-space of this matrix is the columns space of the matrix $[-C \; I_k]$. It defines the same set of points up to a permutation. □

The following Lemma is due to A. Verra.

Lemma 10.3.15 *Let X be the base locus of a linear system $\mathcal{N}$ of quadrics of dimension $k-1$ in $\mathbb{P}^{2k-1}$. Suppose that*

- *$\mathcal{N}$ contains a nonsingular quadric;*
- *X contains a linear subspace Λ of dimension $k-2$;*
- *X is not covered by lines intersecting Λ.*

Then X is birationally isomorphic to the double cover of $\mathcal{N}$ branched over the discriminant hypersurface of $\mathcal{N}$.

Proof Let Λ be a linear subspace of dimension $k-2$ contained in X. Take a general point $x \in X$ and consider the span $\Pi = \langle \Lambda, x \rangle$. By our assumption

x is not contained in any line. The restriction of the linear system $\mathcal{N}$ to Π is a linear system of quadrics in $\Pi \cong \mathbb{P}^{k-1}$ containing Λ and x in its base locus. The residual components of these quadrics are hyperplanes in Π containing x. The base locus of this linear system of hyperplanes consists only of x, because otherwise x will be contained in a line on X intersecting Λ. Our assumption excludes this. Thus the dimension of the restriction of $\mathcal{N}$ to Π is equal to $k-2$. This implies that there exists a unique quadric in $\mathcal{N}$ containing Π. This defines a rational map $X \dashrightarrow \mathcal{N}$. A general member of $\mathcal{N}$ is a nonsingular quadric in $\mathbb{P}^{2k-1}$. It contains two rulings of $(k-1)$-planes. Our $(k-1)$-plane Π belongs to one of the rulings. The choice of a ruling to which Π belongs defines a rational map to the double cover $Y \to \mathcal{N}$ branched along the discriminant variety of $\mathcal{N}$ parameterizing singular quadrics. The latter is constructed by considering the second projection of the incidence variety

$$\{(\Pi, Q) \in G_k(\mathbb{P}^{2k-1}) \times \mathcal{N} : \Pi \in \mathcal{N}\}$$

and applying the Stein factorization. Now we construct the inverse rational map $Y \dashrightarrow X$ as follows. Take a nonsingular quadric $Q \in \mathcal{N}$ and choose a ruling of $(k-1)$-planes in Q. If $Q = V(q)$, then $\Pi = |L|$, where L is an isotropic k-dimensional linear subspace of the quadratic form q, hence it can be extended to a unique maximal isotropic subspace of q in any of the two families of such subspaces. Thus Λ is contained in a unique $(k-1)$-plane Π from the chosen ruling. The restriction of $\mathcal{N}$ to Π is a linear system of quadrics of dimension $k-2$ with Λ contained in the base locus. The free part of the linear system is a linear system of hyperplanes through a fixed point x. This point belongs to all quadrics in $\mathcal{N}$, hence belongs to X. So this point is taken to be the value of our map at the pair Q plus a ruling. □

Applying this Lemma to the case when the linear system of quadrics consists of diagonal quadrics, we obtain that the discriminant hypersurface in $\mathcal{N}$ is the union of hyperplanes

$$\sum_{i=0}^{k} a_{ij} t_i = 0,\ j = 0, \dots, 2k+1.$$

This shows that in the case when the hyperplanes, considered as points in the dual space, lie on a Veronese curve, the base locus X of $\mathcal{N}$ is birationally isomorphic to the quotient X/G.

This applies to our situation, and gives the following.

Theorem 10.3.16 *The surface S given by Equation (10.29) is birationally isomorphic to the double cover of $\mathbb{P}^2$ branched along the six lines $\ell_i = V(z_0 +$*

$a_i z_1 + a_i^2 z_2$). *It is also birationally isomorphic to the quotient S/G, where G consists of involutions $[t_0, \dots, t_5] \mapsto [\epsilon_0 t_0, \dots, \epsilon_5 t_5]$ with $\epsilon_0 \cdots \epsilon_5 = 1$.*

Remark 10.3.17 Consider the double cover F of $\mathbb{P}^2$ branched over six lines $\ell_1, \dots, \ell_6$ tangent to an irreducible conic C. It is isomorphic to a hypersurface in $\mathbb{P}(1,1,1,3)$ given by the equation $z^2 - f_6(t_0, t_1, t_2)$, where $V(f_6)$ is the union of 6 lines. The restriction of f_6 to the conic C is the divisor $2D$, where D is the set of points where the lines are tangent to C. Since $C \cong \mathbb{P}^1$ we can find a cubic polynomial $g(t_0, t_1, t_2)$ which cuts out D in C. Then the preimage of C in F is defined by the equation $z^2 - g_3^2 = 0$ and hence splits into the union of two curves $C_1 = V(z - g_3)$ and $C_2 = V(z + g_3)$ each isomorphic to C. These curves intersect at six points. The surface F has 15 ordinary double points over the points $p_{ij} = \ell_i \cap \ell_j$. Let $\bar{F}$ be a minimal resolution of F. It follows from the adjunction formula for a hypersurface in a weighted projective space that the canonical class of F is trivial. Thus $\bar{F}$ is a K3 surface. Since C does not pass through the points p_{ij} we may identify C_1, C_2 with their preimages in $\bar{F}$. Since $C_1 \cong C_2 \cong \mathbb{P}^1$, we have $C_1^2 = -2$. Consider the divisor class h on $\bar{F}$ equal to $C_1 + L$, where L is the preimage of a line in $\mathbb{P}^2$. We have

$$h^2 = C_1^2 + 2C_1 \cdot L + L^2 = C_1^2 + (C_1 + C_2) \cdot L + L^2 = -2 + 4 + 2 = 4.$$

We leave it to the reader to check that the linear system $|H|$ maps $\bar{F}$ to a quartic surface in $\mathbb{P}^3$. It blows down all 15 exceptional divisors of $\bar{F} \to F$ to double points and blows down C_1 to the sixteenth double point.

Conversely, let Y be a quartic surface in $\mathbb{P}^3$ with 16 ordinary double points. Projecting the quartic from a double point q, we get a double cover of $\mathbb{P}^2$ branched along a curve of degree 6. It is the image of the intersection R of Y with the polar cubic $P_q(Y)$. Obviously, R the singular points of Y are projected to 15 singular points of the branch curve. A plane curve of degree 6 with 15 singular points must be the union of six lines $\ell_1, \dots, \ell_6$. The projection of the tangent cone at q is a conic everywhere tangent to these lines.

Theorem 10.3.18 *A Kummer surface is projectively isomorphic to a quartic surface in $\mathbb{P}^3$ with equation*

$$\begin{aligned} &A(x^4 + y^4 + z^4 + w^4) + 2B(x^2y^2 + z^2w^2) \\ &+2C(x^2z^2 + y^2w^2) + 2D(x^2w^2 + z^2y^2) + 4Exyzw = 0, \end{aligned} \tag{10.32}$$

where

$$A(A^2 + E^2 - B^2 - C^2 - D^2) + 2BCD = 0. \tag{10.33}$$

The cubic hypersurface defined by the above equation is isomorphic to the Segre cubic primal.

Proof Choosing apolar linear line complexes, we transform the Klein quadric to the form $t_1^2 + \ldots + t_6^2 = 0$. Consider the Heisenberg group with nonzero elements defined by involutions associated to a pair of apolar linear line complexes. The Heisenber group is induced by transformations of $\mathbb{P}^3$ listed in Subsection 10.2.1. In these coordinates the equation of the Kummer surface must be invariant with respect to these transformations. It is immediately checked that this implies that the equation must be as in (10.32). It remains for us to check the conditions on the coefficients. We know that a Kummer surface contains singular points. Taking the partials, we find

$$Ax^3 + x(By^2 + Cz^2 + Dw^2) + Eyzw = 0,$$

$$Ay^3 + y(Bx^2 + Cw^2 + Dz^2) + Exzw = 0,$$

$$Az^3 + z(Bw^2 + Cx^2 + Dy^2) + Exyw = 0,$$

$$Aw^3 + w(Bz^2 + Cy^2 + Dx^2) + Exyz = 0.$$

Multiplying the first equation by y and the second equation by x, and adding up the two equations, we obtain

$$(A+B)(x^2+y^2) + (C+D)(z^2+w^2) = \alpha\frac{x^2+y^2}{x^2y^2}, \tag{10.34}$$

where $\alpha = -Exyzw$. Similarly, we get

$$(C+D)(x^2+y^2) + (A+B)(z^2+w^2) = \alpha\frac{z^2+w^2}{z^2w^2}. \tag{10.35}$$

Dividing the first equation by x^2+y^2, the second equation by z^2+w^2, and adding up the results, we obtain

$$2(A+B) + (C+D)\Big(\frac{z^2+w^2}{x^2+y^2} + \frac{z^2+w^2}{x^2+y^2}\Big) = \alpha\Big(\frac{1}{x^2y^2} + \frac{1}{z^2w^2}\Big). \tag{10.36}$$

Multiplying both sides of equations (10.34) and (10.35), and dividing both sides by $(x^2+y^2)(z^2+w^2)$, we obtain

$$(A+B)^2 + (C+D)^2 + (A+B)(C+D)\Big(\frac{z^2+w^2}{x^2+y^2} + \frac{z^2+w^2}{x^2+y^2}\Big) = E^2.$$

Now, we multiply Equation (10.35) by $A+B$, and, after subtracting Equation (10.36) from the result, we obtain

$$(A+B)^2 - (C+D)^2 + E^2 = \alpha(A+B)\Big(\frac{1}{x^2y^2} + \frac{1}{z^2w^2}\Big).$$

Similarly, we get

$$(A-B)^2-(C-D)^2+E^2=-\alpha(A-B)(\frac{1}{x^2y^2}+\frac{1}{z^2w^2}),$$

hence,

$$\frac{(A+B)^2-(C+D)^2+E^2}{(A-B)^2-(C-D)^2+E^2}+\frac{A+B}{A-B}=0.$$

From this we easily derive (10.33).

Equation (10.33) defines a cubic hypersurface in $\mathbb{P}^4$ isomorphic to the Segre cubic primal S_3 given by Equation (9.47). After substitution

$$\begin{aligned} A &= z_0+z_3, \\ B &= z_0+2z_2+2z_4+z_3, \\ C &= z_0+2z_1+2z_4+z_3, \\ D &= -z_0-z_1-2z_2-z_3, \\ E &= -2z_0+2z_3, \end{aligned} \tag{10.37}$$

we obtain the equation

$$z_0^3+z_1^3+z_2^3+z_3^3+z_4^3-(z_0+z_1+z_2+z_3+z_4)^3=0.$$

Since Kummer surfaces depend on three parameters, and the Segre cubic is irreducible, we obtain that a general point on the Segre cubic corresponds to a Kummer surface. □

Let $V=H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(4))^{\mathcal{H}_2}\cong\mathbb{C}^5$ with coordinates A,B,C,D,E. The linear system $|V|\subset|\mathcal{O}_{\mathbb{P}^3}(4)|$ defines a map $\Phi:\mathbb{P}^3\to|V|^\vee\cong\mathbb{P}^4$ whose image is isomorphic to the orbit space $X=\mathbb{P}^3/\mathcal{H}_2$ from (10.18). The preimage of a hyperplane in $\mathbb{P}^4$ is singular if and only if it does not intersect X transversally. This implies that the dual of the hypersurface X is equal to the Segre primal cubic S_3, and, by Proposition 9.4.18, it is isomorphic to the Castelnuovo-Richmond quartic.

A tangent hyperplane of CR_4 at its nonsingular point is a quartic surface with 16 nodes, 15 come from the 15 singular lines of the hypersurface and one more point is the tangency point. It coincides with the surface corresponding to the point on the dual hypersurface. In this way we see a moduli-theoretical interpretation of the set of nonsingular points of CR_4. They correspond to the Kummer surfaces of abelian surfaces equipped with some additional data. Recall that that we have chosen Klein coordinates in the Plücker space that allowed us to write the equation of a Kummer surface in $\mathcal{H}_2$-invariant form. The double plane construction model of the Jacobian Kummer surface comes with

the order on the set of six lines defining the branch curve. This is the same as the order on six Weierstrass points of the corresponding curve of genus 2. As we saw in Chapter 5, the order on the Weierstrass point is equivalent to a choice of a symplectic basis in the group of 2-torsion points of the Jacobian variety. In this way we see that a Zariski open subset of CR_4 can be identified with the moduli space of Jacobian abelian surfaces with full level 2 structure defined by a choice of a symplectic basis in the group of 2-torsion points. It turns out that the whole hypersurface CR_4 is isomorphic to a certain natural compactification $\overline{\mathcal{A}_2(2)}$ of the moduli space of abelian surface with full level 2 structure. This was proven by J. Igusa in [314], who gave an equation of the quartic CR_4 in different coordinates. The quartic hypersurface isomorphic to CR_4 is often referred to in modern literature as an *Igusa quartic* (apparently, reference [176] is responsible for this unfortunate terminology).

The 16 singular points of the Kummer surface Y given by (10.32) form an orbit of $\mathcal{H}_2$. As we know this orbit defines a (16_6)-configuration. A plane containing a set of six points cuts out on Y a plane quartic curve with 6 singular points, no three of them lying on a line. This could happen only if the plane is tangent to the surface along a conic. This conic, or the corresponding plane, is called a *trope*. Again this confirms the fact that in any general $\mathcal{H}_2$-orbit a set of coplanar six points from the (16_6)-configuration lies on a conic.

On a nonsingular model of Y isomorphic to the octavic surface S in $\mathbb{P}^5$ the exceptional curves (the singular lines of the quadratic complex) of the 16 singular points and the proper transforms of 16 tropes form the (16_6)-configuration of lines.

Consider the Gauss map from Y to its dual surface $Y^\vee$ given by cubic partials. Obviously, it should blow down each trope to a singular point of $Y^\vee$. Thus $Y^\vee$ has at least 16 singular points. It follows from the Plücker-Teissier formulas (1.2.7) that each ordinary double point decreases the degree of the dual surface by 2. Thus the degree of the dual surface $Y^\vee$ is expected to be equal to $36 - 32 = 4$. In fact we have the following beautiful fact.

Theorem 10.3.19 *A Kummer surface is projectively isomorphic to its dual surface.*

Proof In the proof of Theorem 10.3.18 we had computed the partial cubics of Equation (10.32). The linear system of the partial cubics is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$ and defines an isomorphism of projective representations. If we choose a basis appropriately, we will be able to identify $\mathcal{H}_2$-equivariantly the dual of the linear system with the original space $\mathbb{P}^3$. We know that the image of the surface is a quartic surface with

16 singular points. Since the tropes of the original surfaces are mapped to singular points of the dual surface, we see that the two surfaces share the same configurations of nodes and tropes. Thus they share 16 conics, and hence coincide (since the degree of intersection of two different irreducible surfaces is equal to 16). □

Remark 10.3.20 One can see the duality also from the duality of the quadratic line complexes. If we identify the space $E = \mathbb{C}^4$ with its dual space by means of the standard basis e_1, e_2, e_3, e_4 and its dual basis $e_1^*, e_2^*, e_3^*, e_4^*$, then the Plücker coordinates $p_{ij} = e_i^* \wedge e_j^*$ in $\bigwedge^2 E$ can be identified with the Plücker coordinates $p_{ij}^* = e_i \wedge e_j$ in $\bigwedge^2 E^\vee$. The Klein quadrics could be also identified. Now the duality isomorphism $G(2, E) \to G(2, E^\vee), \ell \mapsto \ell^\perp$, becomes compatible with the Plücker embeddings. The quadratic line complex given in Klein coordinates by two diagonal quadrics (10.28) is mapped under the duality isomorphism to the quadratic line complex given by two diagonal quadrics $\sum y_i^2 = 0, \sum a_i^{-1} y_i^2 = 0$, the dual quadrics. However, the intersection of these two pairs of quadrics is projectively isomorphic under the scaling transformation $y_i \mapsto \sqrt{a_i} y_i$. This shows that, under the duality isomorphism, the singular surfaces of the quadratic line complex and its dual are projectively isomorphic. It follows from the definition of the duality that the tropes of the Kummer surface correspond to β-planes that intersect the quadratic line complex along the union of two lines.

The Kummer surface admits an infinite group of birational automorphisms. For a general one, the generators of this group have been determined in modern works of J. Keum [332] and S. Kondō [345]. We give only examples of some automorphisms.

- Projective automorphisms defined by the Heisenberg group. They correspond to translations by 2-torsion points on the abelian surface cover.
- Involutions defined by projections from one of 16 nodes.
- Switches defined by choosing a duality automorphism and composing it with elements of the Heisenberg group.
- Cubic transformations given in coordinates used in Equation (10.32) by

$$(x, y, z, w) \mapsto (yzw, xzw, xyw, xyz)$$

  .
- Certain automorphisms induces by Cremona transformations of degree 7.

10.3.4 Harmonic complex

Consider a pair of irreducible quadrics Q_1 and Q_2 in $\mathbb{P}^n$. A *harmonic line complex* or a *Battaglini complex* is the closure in $G_1(\mathbb{P}^n)$ of the locus of lines which intersect Q_1 and Q_2 at two harmonically conjugate pairs. Let us see that this is a quadratic line complex and find its equation.

Let $A = (a_{ij}), B = (b_{ij})$ be two symmetric matrices defining the quadrics. Let $\ell = \overline{xy}$, where $x = [v], y = [w]$ for some $v, w \in \mathbb{C}^4$. Let $\ell = [sv + tw]$ be a parametric equation of ℓ. Then the restriction of Q_1 to ℓ is a binary form in s, t defined by $(vAv)s^2 + 2(vAw)st + (wAw)t^2$ and the restriction of Q_2 to ℓ is defined by the bilinear form $(vBw)s^2 + 2(vBw)st + (wBw)t^2$. By definition, the two roots of the binary forms are harmonically conjugate if and only if

$$(vAv)(wBw) + (wAw)(vBv) - 2(vAw)(vBw) = 0.$$

Let $[vw]$ be the matrix with two columns equal to the coordinate vectors of v and w. We can rewrite the previous expression in the form

$$\det\big({}^t[vw][AvBw]\big) + \det\big({}^t[v,w][BvAw]\big) = 0. \tag{10.38}$$

The expression is obviously a quadratic form on $\bigwedge^2 \mathbb{C}^{n+1}$ and also a symmetric bilinear form on the space of symmetric matrices. Take the standard basis $E_{ij} + E_{ji}, E_{ii}, 1 \le i \le j \le n+1$, of the space of symmetric matrices and compute the coefficients of the symmetric bilinear forms in terms of coordinates of v and w. We obtain

$$a_{ij;kl} = 4(x_i x_j y_k y_l + x_k x_l y_i y_j) - 2(x_k y_l + x_l y_k)(x_j y_i + x_i y_j)$$

$$= 2(p_{ik}p_{jl} + p_{il}p_{jk}),$$

where $p_{ab} = -p_{ba}$ if $a > b$. Thus (10.38) is equal to

$$\sum (a_{ij}b_{kl} + a_{kl}b_{ij})(p_{ik}p_{jl} + p_{il}p_{jk}) = 0. \tag{10.39}$$

This is an equation of a quadratic complex. If we assume that $a_{ij} = b_{ij} = 0$ if $i \neq j$, then the equation simplifies

$$\sum (a_{ii}b_{jj} + a_{jj}b_{ii})p_{ij}^2 = 0. \tag{10.40}$$

Consider the pencil $\mathcal{P}$ of quadrics $\lambda Q_1 + \mu Q_2$. Let us assume, for simplicity, that the equations of the quadrics can be simultaneously diagonalized. Then a line ℓ is tangent to a quadric from $\mathcal{P}$ if and only if

$$\sum (\lambda a_{ii} + \mu b_{ii})(\lambda a_{jj} + \mu b_{jj})p_{ii}^2$$

$$= \sum(\lambda^2 a_{ii}a_{jj} + \lambda\mu(a_{ii}b_{jj} + a_{jj}b_{ii}) + \mu^2 b_{ii}b_{jj})p_{ij}^2 = 0.$$

The restriction of the pencil to ℓ is a linear series g_2^1 unless ℓ has a base point in which case the line intersects the base locus of the pencil. The two quadrics which touch ℓ correspond to the points $[\lambda, \mu] \in \mathcal{P}$ which satisfy the equation in above. Denote by $A, 2B, C$ the coefficients at $\lambda^2, \lambda\mu, \mu^2$. The map

$$G_1(\mathbb{P}^n) \to \mathbb{P}^2,\ \ell \mapsto [A, B, C]$$

is a rational map defined on the complement of codimension 3 subvariety of $G_1(\mathbb{P}^n)$ given by the equations $A = B = C = 0$. Its general fiber is the loci of lines which touch a fixed pair of quadrics in the pencil. It is given by intersection of two quadratic line complexes. In case $n = 2$, we recognize a well-known fact that two conics have four common tangents. The preimage of a line $At_0 + 2Bt_1 + Ct_2 = 0$ with $AC - B^2 = 0$ is a line complex such that there is only one quadric in the pencil which touches the line. Hence it equals the Chow form of the base locus, a hypersurface of degree 4 in $G(2, n)$.

Let us consider the case $n = 3$. In this case a harmonic line complex is a special case of a quadratic line complex given by two quadrics

$$\begin{aligned}\mathcal{Q}_1 &:= p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0,\\ \mathcal{Q}_2 &:= a_{12}p_{12}^2 + \cdots + a_{34}p_{34}^2 = 0.\end{aligned}$$

We assume that $\mathcal{Q}_2$ is a nonsingular quadric, i.e. all $a_{ij} \neq 0$. It is easy to see that the pencil $\lambda\mathcal{Q}_1 + \mu\mathcal{Q}_2 = 0$ has six singular quadrics corresponding to the parameters

$$[1, \pm\sqrt{a_{12}a_{34}}],\ [1, \pm\sqrt{a_{13}a_{24}}],\ [1, \pm\sqrt{a_{14}a_{23}}].$$

Thus we diagonalize both quadrics to reduce the equation of the quadratic line complex to the form

$$t_0^2 + \cdots + t_5^2 = 0,$$

$$k_1(t_0^2 - t_1^2) + k_2(t_2^2 - t_3^2) + k_3(t_4^2 - t_5^2) = 0.$$

The genus 2 curve corresponding to the intersection of the two quadrics is a special one. Its branch points are $[1, \pm k_1], [1, \pm k_2], [1, \pm k_3]$. The involution of $\mathbb{P}^1$ defined by $[t_0, t_1] \mapsto [t_0, -t_1]$ leaves the set of branch points invariant and lifts to an involution of the genus 2 curve. It follows from the description of binary forms invariant under a projective automorphism of finite order given in Section 8.8.4 that there is only one conjugacy class of involutions of order 2 and each binary sextic whose set of zeros is invariant with respect to an involution can be reduced to the form $(t_0^2 - t_1^2)(t_0^2 - \alpha t_1^2)(t_0^2 - \beta t_1^2)$. Thus

we see that the harmonic line complexes form a hypersurface in the moduli space of smooth complete intersections of two quadrics in $\mathbb{P}^5$. It is isomorphic to the hypersurface in $\mathcal{M}_2$ formed by isomorphism classes of genus 2 curves admitting two commuting involutions.

Proposition 10.3.21 *The singular surface of a harmonic line complex is projectively isomorphic to a quartic surface given by Equation* (10.32) *with coefficient E equal to 0.*

Proof We use that, in Klein coordinates, our quadratic line complex has additional symmetry defined by the transformation

$$(t_0, t_1, t_2, t_3, t_4, t_5) \mapsto (-it_1, it_0, -it_3, it_2, -it_5, it_4).$$

Here we may assume that $t_0 = i(p_{14} - p_{23}), t_1 = p_{14} + p_{23}$, etc. The transformation of $\mathbb{P}^3$ that induces this transformation is defined by $[x, y, z, w] \mapsto [-x, y, z, w]$. Equation (10.32) shows that the Kummer surface is invariant with respect to this transformation if and only if the coefficient E is zero. □

Note that under the isomorphism from the cubic (10.33) to the Segre cubic primal given by formulas (10.37), the coefficient E is equal to $-z_0 + z_3$. This agrees with a remark before Lemma 9.4.11.

Consider the Kummer surface S given by Equation (10.32) with $E = 0$. Intersecting the surface with the plane $x = 0$ we obtain the plane quartic with equation $Q(x^2, y^2, z^2) = 0$ where $Q = A(s^2 + u^2 + v^2) + 2Bsu + 2Csv + 2Duv$. Its discriminant is equal to $A(A^2-B^2-C^2-D^2)+2BCD$. Comparing it with Equation (10.33), we find that the quadratic form is degenerate. Thus the plane section of the Kummer surface is the union of two conics with equations $(ax^2 + by^2 + cz^2)(a'x^2 + b'y^2 + c'z^2) = 0$. The four intersection points of these conics are singular points of S. This easily follows from the equations of the derivatives of the quartic polynomial defining S. Thus we see that the 16 singular points of the Kummer surface lie by four in the coordinate planes $x, y, z, w = 0$. Following A. Cayley [73], a Kummer surface with this property is called a *Tetrahedroid*.

Note the obvious symmetry of the coordinate hyperplane sections. The coordinates of 16 nodes can be put in the following symmetric matrix:

$$\begin{pmatrix} 0 & \pm a_{12} & \pm a_{13} & \pm a_{14} \\ \pm a_{21} & 0 & \pm a_{23} & \pm a_{24} \\ \pm a_{31} & \pm a_{32} & 0 & \pm a_{34} \\ \pm a_{41} & \pm a_{42} & \pm a_{43} & 0 \end{pmatrix}.$$

The complete quadrangle formed by four nodes $p_1, \ldots, p_4$ in each coordinate

plane has the property that the lines $\overline{p_ip_j}$ and $\overline{p_kp_l}$ with $\{i,j,\}\cap\{k,l\}=\emptyset$ intersect at the vertices of the coordinate tetrahedron. One can also find the 16 tropes. Take a vertex of the coordinate tetrahedron. There will be two pairs of nodes, not in the same coordinate plane, each pair lying on a line passing through the vertex. For example,

$$[0,a_{12},a_{13},a_{14}],[0,a_{12},-a_{13},a_{14}],[0,a_{21},0,a_{23},a_{24}],[0,a_{21},0,-a_{23},a_{24}].$$

The plane containing the two pairs contains the third pair. In our example, the third pair is $[a_{41},-a_{42},a_{43},0],[a_{41},-a_{42},-a_{43},0]$. This is one of the 16 tropes. Its equation is $a_{24}x+a_{14}y-a_{12}w=0$. Similarly, we find the equations of all 16 tropes

$$\begin{aligned}\pm a_{34}y\pm a_{42}z\pm a_{23}w&=0,\\ \pm a_{34}x\pm a_{41}z\pm a_{13}w&=0,\\ \pm a_{24}x\pm a_{41}y\pm a_{12}w&=0,\\ \pm a_{23}x\pm a_{31}y\pm a_{12}z&=0.\end{aligned}$$

Remark 10.3.22 For experts on K3 surfaces, let us compute the Picard lattice of a general Tetrahedroid. Let $\sigma:\tilde{S}\to S$ be a minimal resolution of S. Denote by h the class of the preimage of a plane section of S and by $e_i, i=1,\dots,16$, the classes of the exceptional curves. Let c_1 and c_2 be the classes of the proper transforms of the conics C_1,C_2 cut out by one of the coordinate plane, say $x=0$. We have

$$c_1+c_2=h-e_1-e_2-e_3-e_4.$$

Obviously, $c_1\cdot c_2=0$ and $h\cdot c_i=2$ and $c_i^2=-2$. Consider another coordinate plane and another pair of conics. We can write

$$c_3+c_4=h-e_5-e_6-e_7-e_8.$$

This shows that the classes of the eight conics can be expressed as linear combinations of classes h,e_i and $c=c_1$. It is known that the Picard group of a general Kummer surface is generated by the classes e_i and the classes of tropes t_i satisfying $2t_i=h-e_{i_1}-\cdots-e_{i_6}$. The Picard group of a Tetrahedroid acquires an additional class c.

The Jacobian variety of a genus 2 curve C with two commuting involutions contains an elliptic curve, the quotient of C by one of the involutions. In the symmetric product $C^{(2)}$ it represents the graph of the involution. Thus it is isogenous to the product of two elliptic curves.

Note that the pencil of quadrics passing through the set of eight points $(C_1 \cap C_2) \cup (C_3 \cap C_4)$ defines a pencil of elliptic curves on $\tilde{S}$ with the divisor class

$$2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 = c_1 + c_2 + c_3 + c_4.$$

Since $c_1 \cdot c_2 = c_3 \cdot c_4 = 0$, Kodaira's classification of fibres of elliptic fibrations shows that c_1, c_2, c_3, c_4 are the classes of irreducible components of a fiber of type I_4. This implies that the four intersection points $(C_1 \cup C_2) \cap (C_3 \cup C_4)$ lie on the edges of the coordinate tetrahedron.

The parameters A, B, C, D used to parameterize Tetrahedroid surfaces have be considered as points on the cubic surface

$$A(A^2 - B^2 - C^2 - D^2) + 2BCD = 0.$$

One can write an explicit rational parameterization of this surface using the formulas

$$A = 2abc,\ B = a(b^2 + c^2),\ C = b(a^2 + c^2),\ D = c(a^2 + b^2).$$

The formulas describe a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ of degree 2 given by the linear system of plane cubics with three base points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]$. It extends to a degree 2 map from a del Pezzo surface of degree 6 onto a 4-nodal cubic surface. In fact, if one considers the standard Cremona involution $[a, b, c] \mapsto [a^{-1}, b^{-1}, c^{-1}]$, then we observe that the map factors through the quotient by this involution. It has four singular points corresponding to the fixed points

$$[a, b, c] = [1, 1, 1],\ [-1, 1, 1],\ [1, -1, 1],\ [1, 1, -1].$$

of the Cremona involution. The corresponding singular points are the points $[1, 1, 1, 1],\ [1, 1, -1, -1],\ [1, -1, 1, -1],\ [1, -1, -1, 1]$.

If we change the variables $X^2 = bcx^2, Y^2 = acy^2, X^2 = abx^2, W = w$, the equation

$$A(x^4 + y^4 + z^4 + w^4) + 2B(x^2w^2 + y^2z^2) + 2C(y^2w^2 + x^2z^2)$$

$$+2D(z^2w^2 + x^2y^2) = 0$$

is transformed to the equation

$$(X^2 + Y^2 + Z^2)(a^2X^2 + b^2Y^2 + c^2Z^2) -$$

$$[a^2(b^2+c^2)X^2W^2 + b^2(c^2+a^2)Y^2W^2 + c^2(a^2+b^2)Z^2W^2] + a^2b^2c^2W^4 = 0,$$

or, equivalently,

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2w^2}+\frac{b^2y^2}{x^2+y^2+z^2-b^2w^2}+\frac{c^2z^2}{x^2+y^2+z^2-c^2w^2}=0. \tag{10.41}$$

When a, b, c are real numbers, the real points $(x, y, z, 1) \in \mathbb{P}^3(\mathbb{R})$ on this surface describe the propagation of light along the interface between two different media. The real surface with Equation (10.41) is called a *Fresnel's wave surface*. It has four real nodes

$$\Big(\pm c\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm a\sqrt{\frac{b^2-c^2}{a^2-c^2}}, 1\Big),$$

where we assume that $a^2 > b^2 > c^2$. It has four real tropes given by planes $\alpha x + \beta y + \gamma z + w = 0$, where

$$(\alpha, \beta, \gamma, 1) = \Big(\pm\frac{c}{b^2}\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm\frac{a}{b^2}\sqrt{\frac{b^2-c^2}{a^2-c^2}}, 1\Big).$$

One of the two conics cut out on the surface by coordinate planes is a circle. On the plane $w = 0$ at infinity one of the conics is the ideal conic $x^2+y^2+z^2 = 0$.

10.3.5 The tangential line complex

When we considered the harmonic line complex defined by two quadrics Q_1 and Q_2 we did not need to assume that the quadrics are different. In the case when $Q_1 = Q_2 = Q$, the definition of a harmonic self-conjugate pair implies that the two points in the pair coincide, i.e. the line is tangent to the quadric. This is a special case of the harmonic complex, the locus of tangent lines to a quadric.

Equation (10.39) gives us the equation of the tangential line complex of a quadric Q defined by a symmetric matrix $A = (a_{ij})$:

$$\sum a_{ij}a_{kl}(p_{ik}p_{jl} + p_{il}p_{jk}) = 0. \tag{10.42}$$

Proposition 10.3.23 *The tangential quadratic line complex X_Q associated to a nonsingular quadric surface Q in $\mathbb{P}^n$ is singular along the variety* $\mathrm{OG}(2, Q)$ *of lines contained in Q.*

Proof It is easy to see that a line $\mathcal{P}$ in $\mathbb{G} = G_1(\mathbb{P}^n)$ is a pencil of lines in some plane Π in $\mathbb{P}^n$. The plane Π intersects Q in a conic. If the line is general, then the conic is nonsingular, and the pencil $\mathcal{P}$ contains two points represented by lines in Π that are tangent to the conic. This confirms that X_Q

is a quadratic complex. Now, assume that ℓ is contained in Q. A general line $\mathcal{P}$ in $\mathbb{G}$ containing ℓ contains only one point represented by a line in $\mathbb{P}^n$ tangent to Q, namely the line ℓ. This shows that $\mathcal{P}$ is tangent to X_Q at the point ℓ. Since $\mathcal{P}$ was a general line in $\mathbb{G}$, it shows that the tangent space of X_Q at ℓ coincides with the tangent space of $\mathbb{G}$ at ℓ. This implies that X_Q is singular at ℓ. Since X_Q is a quadratic complex, ℓ is point of multiplicity 2 in X_Q. □

Let $\mathcal{T}_Q$ be the tangent bundle of Q and let $\sigma : |\mathcal{T}_Q| \to Q$ be its projectivization. The fiber of $|\mathcal{T}_Q|$ at a point $x \in Q$ consists of lines tangent to Q at x. This defines a natural birational morphism

$$\pi : \mathbb{P}(\mathcal{T}_Q^\vee) \to X_Q$$

which is a resolution of singularities of the tangential complex. It is easy to see that $\mathrm{OG}(2, Q)$ is of codimension 2 in X_Q. Thus the exceptional divisor of π is isomorphic to a $\mathbb{P}^1$-bundle over $\mathrm{OG}(2, Q)$.

Remark 10.3.24 One can identify $\bigwedge^2 \mathbb{C}^{n+1}$ with the Lie algebra $\mathfrak{so}(n+1)$ of the special orthogonal group $\mathrm{SO}(n+1)$ of the space $\mathbb{C}^{n+1}$ equipped with the dot-product symmetric bilinear form and the associated quadratic form q. The orthogonal group $\mathrm{SO}(n+1)$ the only orbit contained in the boundary of its closure. Considered as linear operators, points in $\mathrm{OG}(2, \mathcal{Q})$ are operators A of rank 2 satisfying $A^2 = 0$, and points of X_Q are operators of rank 2 satisfying $A^3 = 0$ (see, for example, [32]). In particular, we see that the variety $\mathrm{OG}(2, Q)$ can be given by quadratic equations expressing the condition that the square of the matrix (p_{ij}) is equal to 0.

Thus both orbits are *nilpotent orbits*, i.e. they are contained in the subvariety of nilpotent linear operators. We refer for the classification of nilpotent orbits to [123]. For classical Lie algebras $\mathfrak{sl}_{n+1}, \mathfrak{so}_{n+1}, \mathfrak{so}_{n+1}$, the nilpotent orbits are classified by partition of $n+1$ defining the Jordan form of the linear operator. Thus the minimal orbit $\mathrm{OG}(2, Q)$ corresponds to the partition $(2, 2, 1, \ldots, 1)$ and the supminimal orbit corresponds to the partition $(3, 1, \ldots, 1)$.

Replacing the Lie algebra $\mathfrak{so}(n+1)$ by any simple complex lie algebra $\mathfrak{g}$ we obtain an analog of the tangential line complex X_Q and its singular locus $\mathrm{OG}(2, \mathcal{Q})$. The latter becomes the unique minimal adjoint orbit in $|\mathfrak{g}|$, the former becomes the unique supminimal adjoint orbit. Both of these orbits are nilpotent orbits, i.e. they are contained in the subvariety of nilpotent elements of the Lie algebra. An algebraic variety isomorphic to a minimal adjoint orbit for some simple Lie algebra $\mathfrak{g}$ is called a *adjoint variety*. The adjoint varieties and, in particular, the line complexes $\mathrm{OG}(2, \mathcal{Q})$ of lines in a nonsingular quadric, are Fano contact varieties. Recall that a complex manifold M is called a *contact manifold* if its tangent bundle T_M contains a corank one subbundle

F such that the bilinear form $F \times F \to T_M/F$ defined by the Lie bracket is nondegenerate. It is conjectured that any Fano contact variety is isomorphic to an adjoint variety (see [35]).

10.3.6 Tetrahedral line complex

Consider the union of four planes in $\mathbb{P}^3$ which define a coordinate tetrahedron in the space. Let q_1, q_2, q_3, q_4 be its vertices, $\ell_{ij} = \overline{q_i q_j}$ be its edges and $\pi_i = \overline{q_j q_k q_l}$ be its faces. Let $[A, B] \in \mathbb{P}^1$ and $\mathfrak{C}$ be the closure of the set of lines in $\mathbb{P}^3$ intersecting the four faces at four distinct points with the cross ratio equal to $[A, B]$. Here we assume that the vertices of the tetrahedron are ordered in some way. It is easy to see that $\mathfrak{C}$ is a line complex. It is called a *tetrahedral line complex*.

Proposition 10.3.25 *A tetrahedral line complex $\mathfrak{C}$ is of degree 2. If p_{ij} are the Plücker coordinates with respect to the coordinates defined by the tetrahedron, then $\mathfrak{C}$ is equal to the intersection of the Grassmannian with the quadric*

$$Ap_{12}p_{34} - Bp_{13}p_{24} = 0. \tag{10.43}$$

Conversely, this equation defines a tetrahedral line complex.

Proof Let ℓ be a line spanned by the points $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4]$. It intersects the face π_i at the point corresponding to the coordinates on the line $[s, t] = [b_i, -a_i], i = 1, \dots, 4$. We assume that ℓ does not pass through one of the vertices. Then ℓ intersects the faces at four points not necessarily distinct with cross ratio equal to $[p_{12}p_{34}, p_{13}p_{24}]$, where p_{ij} are the Plücker coordinates of the line. So, the equation of the tetrahedral line complex containing the line is $[p_{12}p_{34}, p_{13}p_{24}] = [a, b]$ for some $[a, b] \in \mathbb{P}^1$. □

Note that any tetrahedral line complex $\mathfrak{C}$ contains the set of points in $G(2, 4)$ satisfying $p_{is} = p_{it} = p_{ik} = 0$ (the lines in the coordinate plane $t_i = 0$). Also, any line containing a vertex satisfies $p_{ij} = p_{jk} = p_{ik} = 0$ and hence also is contained in $\mathfrak{C}$. Thus we obtain that $\mathfrak{C}$ contains four planes from one ruling of the Klein quadric and four planes from another ruling. Each plane from one ruling intersects three planes from another ruling along a line and does not intersect the fourth plane.

Observe that the tetrahedral line complex is reducible if and only if the corresponding cross ratio is equal to $0, 1, \infty$. In this case it is equal to the union of two hyperplanes representing lines intersecting one of the two opposite edges. An irreducible tetrahedral line complex has six singular points corresponding to the edges of the coordinate tetrahedron. Their Plücker coordinates are all equal to zero except one.

Proposition 10.3.26 *The singular surface of an irreducible tetrahedral line complex $\mathfrak{C}$ is equal to the union of the faces of the coordinate tetrahedron.*

Proof We know that the degree of the singular surface is equal to 4. So, it suffices to show that a general point in one of the planes of the tetrahedron belongs to the singular surface. The lines in this plane belong to the complex. So, a line in the plane passing through a fixed point p_0 is an irreducible component of the conic $\Omega(p_0) \cap \mathfrak{C}$. This shows that p_0 belongs to the singular surface of $\mathfrak{C}$. □

From now on we consider only irreducible tetrahedral line complexes. There are different geometric ways to describe a tetrahedral complex.

First we need the following fact, known as *von Staudt's Theorem* (see [540].

Theorem 10.3.27 (G. von Staudt) *Let ℓ be a line belonging to a tetrahedral line complex $\mathfrak{C}$ defined by the cross ratio R. Assume that ℓ does not pass through the vertices and consider the pencil of planes through ℓ. Then the cross ratio of the four planes in the pencil passing through the vertices is equal to R.*

Proof Let e_1, e_2, e_3, e_4 be a basis in $E = \mathbb{C}^4$ corresponding to the vertices of the tetrahedron. Choose the volume form $\omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and consider the *star-duality* in $\bigwedge^2 E$ defined by $(\alpha, \beta) = (\alpha \wedge \beta)/\omega$. Under this duality $(e_i \wedge e_j, e_k \wedge e_l) = 1(-1)$ if (i, j, k, l) is an even (odd) permutation of $(1, 2, 3, 4)$ and 0 otherwise. Let $\gamma = \sum_{1 \le i < j \le 4} p_{ij} e_i \wedge e_j$ be the 2-form defining the line ℓ and let $\gamma^* = \sum p'_{ij} e_i \wedge e_j$ define the dual line ℓ^*, where $e_i \wedge e_j$ is replaced with $(e_i \wedge e_j, e_k \wedge e_l) e_k \wedge e_i$, where i, j, k, l are all distinct. The line ℓ (resp. ℓ^*) intersects the coordinate planes at the points represented by the columns of the matrix

$$A = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}, \text{ resp. } B = \begin{pmatrix} 0 & p_{34} & -p_{24} & p_{23} \\ -p_{34} & 0 & p_{14} & -p_{13} \\ p_{24} & -p_{14} & 0 & p_{12} \\ -p_{23} & p_{13} & -p_{12} & 0 \end{pmatrix}.$$

We have $A \cdot B = B \cdot A = 0$. It follows from the proof of the previous Proposition that the cross ratio of the four points on ℓ^* is equal to $(p'_{13}p'_{24}, p'_{12}p'_{34}) = (p_{24}p_{13}, p_{24}p_{13})$. Thus ℓ and ℓ^* belong to the same tetrahedral line complex. Now a plane containing ℓ can be identified with a point on ℓ^* equal to the intersection point. A plane containing e_1 and ℓ is defined by the 3-form

$$e_1 \wedge \gamma = p_{23} e_1 \wedge e_2 \wedge e_3 + p_{24} e_1 \wedge e_2 \wedge e_4 + p_{34} e_1 \wedge e_3 \wedge e_4$$

and we check that $e_1 \wedge \gamma \wedge (-p_{34}e_2 + p_{24}e_3 - p_{23}e_4) = 0$ since $B \cdot A = 0$. This means that the plane containing e_1 intersects ℓ^* at the first point on ℓ^*

defined by the first column. Thus, under the projective map from the pencil of planes through ℓ to the line ℓ^*, the plane containing e_1 is mapped to the intersection point of ℓ^* with the opposite face of the tetrahedron defined by $t_0 = 0$. Similarly, we check that the planes containing other vertices correspond to intersection points of ℓ^* with the opposite faces. This proves the assertion. □

Proposition 10.3.28 *A tetrahedral line complex is equal to the closure of the set of secants of rational cubic curves in $\mathbb{P}^3$ passing through the vertices of the coordinate tetrahedron.*

Proof Let R be one of those curves and $x \in R$. Projecting from x we get a conic C in the plane with four points, the projections of the vertices. Let $\ell = \overline{xy}$ be a secant of R. The projection $\bar{y}$ of y is a point on the conic C and the pencil of lines through $\bar{y}$ is projectively equivalent to the pencil of planes through the secant ℓ. Under this equivalence, the planes passing through the vertices of the tetrahedron correspond to the lines connecting their projection with $\bar{y}$. Applying von Staudt's Theorem, we conclude the proof. □

Consider the action of the torus $T = (\mathbb{C}^*)^4$ on $\mathbb{P}^3$ by scaling the coordinates in $E = \mathbb{C}^4$. Its action on $\bigwedge^2 E$ is defined by

$$(t_1, t_2, t_3, t_4) : (p_{12}, \dots, p_{34}) \mapsto (t_1 t_2 p_{12}, \dots, t_3 t_4 p_{34}).$$

It is clear that the Klein quadric is invariant with respect to this action. This defines the action of T on the Grassmannian of lines. It is also clear that the equations of a tetrahedral line complex $\mathfrak{C}$ are also invariant with respect to this action, so T acts on a tetrahedral complex. If $\ell \in \mathfrak{C}$ has nonzero Plücker coordinates (a general line), then the stabilizer of ℓ is equal to the kernel of the action of T in $\mathbb{P}^3$, i.e. equal to the diagonal group of $(z, z, z, z), z \in \mathbb{C}^*$. Hence the orbit of ℓ is 3-dimensional, and since $\mathfrak{C}$ is irreducible and 3-dimensional, it is a dense Zariski subset of $\mathfrak{C}$. Thus we obtain that $\mathfrak{C}$ is equal to the closure of a general line in $G(2, 4)$ under the torus action. Since any general line belongs to a tetrahedral line complex, we get an equivalent definition of a tetrahedral line complex as the closure of a torus orbit of a line with nonzero Plücker coordinates.

Here is another description of a tetrahedral complex. Consider a projective automorphism $\phi : \mathbb{P}^3 \to \mathbb{P}^3$ with four distinct fixed points and let $\mathfrak{C}$ be the closure of lines $\overline{x\phi(x)}$, where x is not a fixed point of ϕ. Let us see that $\mathfrak{C}$ is an irreducible tetrahedral complex. Choose the coordinates in $\mathbb{C}^4$ such that the matrix of ϕ is a diagonal matrix with four distinct eigenvalues λ_i. Then $\mathfrak{C}$ is the closure of lines defined by 2-vectors $\gamma = A \cdot v \wedge v, v \in \mathbb{C}^4$. A straightforward computation shows that the Plücker coordinates of γ are equal

to $p_{ij} = t_i t_j(\lambda_i - \lambda_j)$, where $(t_1, \dots, t_4)$ are the coordinates of the vector v. Thus, if we take v with nonzero coordinates, we obtain that $\mathfrak{C}$ contains the torus orbit of the vector with nonzero Plücker coordinates $p_{ij} = \lambda_i - \lambda_j$. As we explained in above, $\mathfrak{C}$ is an irreducible tetrahedral complex.

It is easy to see that the map which assigns to a point $x \in \mathbb{P}^3$ the line $\overline{x\phi(x)}$ defines a birational transformation $\Phi : \mathbb{P}^3 \dashrightarrow \mathfrak{C}$ with fundamental points at the fixed points of ϕ. It is given by quadrics. The linear system of quadrics through four general points in $\mathbb{P}^3$ is of dimension 5 and defines a rational map from $\mathbb{P}^3$ to $\mathbb{P}^5$. The preimage of a general plane is equal to the intersection of three general quadrics in the linear system. Since there are four base points, we obtain that the residual intersection consists of four points. This implies that the linear system defines a map of degree 2 onto a quadric in $\mathbb{P}^5$ or a degree 1 map onto a threefold of degree 4. Since a tetrahedral line complex is obtained in this way and any four general points in $\mathbb{P}^3$ are projectively equivalent, we see that the image must be projectively isomorphic to a tetrahedral complex. Observe that the six lines joining the pairs of fixed points of ϕ are blown down to singular points of the tetrahedral complex. Also, we see the appearance of eight planes; four of these planes are the images of the exceptional divisors of the blow-up of $\mathbb{P}^3$ at the fixed points, and the other four are the images of the planes spanned by three fixed points. We see that the blow-up of $\mathbb{P}^3$ is a small resolution of the tetrahedral complex.

There is another version of the previous construction. Take a pencil $\mathcal{Q}$ of quadrics with a nonsingular base curve. Consider a rational map $\mathbb{P}^3 \dashrightarrow G_1(\mathbb{P}^3)$ which assigns to a point $x \in \mathbb{P}^3$ the intersection of the polar planes $P_x(Q), Q \in \mathcal{Q}$. This is a line in $\mathbb{P}^3$ unless x is a singular point of one of quadrics in $\mathcal{Q}$. Under our assumption on the pencil, there are exactly four such points which we can take as the points $[1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1]$. Thus we see that the rational map is of the same type as in the previous construction and its image is a tetrahedral complex.

10.4 Ruled surfaces

10.4.1 Scrolls

A *scroll* or a *ruled variety* is an irreducible subvariety S of $\mathbb{P}^n$ such that there exists an irreducible family X_0 of linear subspaces of dimension r sweeping S such that a general point of S lies in unique subspace from this family. We will also assume that each point is contained only in finitely many linear subspaces. Following classical terminology, the linear subspaces are called *generators*.

Note that the condition that any point lies in finitely many generators excludes cones.

We identify X_0 with its image in the Grassmann variety $\mathbb{G} = G_r(\mathbb{P}^n)$. For any $x \in X_0$ let Λ_x denote the generator defined by the point x. The universal family

$$\{(x, p) \in X_0 \times \mathbb{P}^n : p \in \Lambda_x\}$$

is isomorphic to the incidence variety Z_{X_0} over X_0. The projection $Z_{X_0} \to \mathbb{P}^n$ is a finite morphism of degree 1 which sends the fibres of the projective bundle $\mathsf{Z}_{X_0} \to X_0$ to generators. For any finite morphism $\nu : X \to X_0$ of degree 1, the pull-back $\mathcal{E} = \nu^*(\mathcal{S}^\vee_{X_0})$ defines the projective bundle $\mathbb{P}(\mathcal{E})$ and a finite morphism $\tilde{\nu} : \mathbb{P}(\mathcal{E}) \to \mathsf{Z}_{X_0}$ such that the composition $f : \mathbb{P}(\mathcal{E}) \to \mathsf{Z}_{X_0} \to S$ is a finite morphism sending the fibres to generators. Recall that the projection $\mathsf{Z}_G \to \mathbb{P}^n = |E|$ is defined by a surjection of the locally free sheave $\alpha : E^\vee \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{S}^\vee_{\mathbb{G}}$. Thus the morphism $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$ is defined by a surjection

$$\nu^*(\alpha) : E^\vee \otimes \mathcal{O}_X \to \mathcal{E}.$$

In particular, the morphism f is given by a linear system $|E^\vee| \subset |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

Thus we see that any scroll is obtained as the image of a birational morphism

$$f : \mathbb{P}(\mathcal{E}) \to |E|$$

defined by a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. The linear system can be identified with the image of $E^\vee \to H^0(X, \mathcal{E})$ under the surjective map $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}$. This map also gives a finite map $\nu : X \to X_0 \subset \mathbb{G}$. The base X of the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ can be always assumed to be a normal variety. Then $\nu : X \to X_0$ is the normalization map.

A scroll defined by the complete linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ is a linearly normal subvariety of $\mathbb{P}^n$. It is called a *normal scroll*. Any scroll is a projection of a normal scroll. Note that, in many text-books, a normal scroll is assumed to be a nonsingular variety. We have already classified smooth rational normal scrolls of dimension 2 in Chapter 8.

A surjective map of locally free sheaves $\mathcal{E} \to \mathcal{F}$ defines a closed embedding $\mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$. If rank $\mathcal{F} = r' + 1$, the image of $\mathbb{P}(\mathcal{F})$ under the map $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ is an r'*-directrix* of the scroll, a closed subvariety intersecting each generator along an r'-plane. If $r' = 0$, we get a section of $\mathbb{P}(\mathcal{E})$. Its image is *directrix* of the scroll, a closed subvariety of the scroll that intersects each generator at one point. Note that not every directrix comes from a section; for example, a generator could be a directrix.

Suppose $\mathcal{E} \to \mathcal{E}_1$ and $\mathcal{E} \to \mathcal{E}_2$ are two surjective maps of locally free sheaves

on a smooth curve X. Let $\mathcal{E} \to \mathcal{E}_1 \oplus \mathcal{E}_2$ be the direct sum of the maps and let $\mathcal{E}'$ be the image of this map which is locally free since X is a smooth curve. Assume that the quotient sheaf $(\mathcal{E}_1 \oplus \mathcal{E}_2)/\mathcal{E}'$ is a skyscraper sheaf. Then the surjection $\mathcal{E} \to \mathcal{E}'$ corresponds to a closed embedding $j : \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E})$. We call the projective bundle $\mathbb{P}(\mathcal{E}')$ the *join* of $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_2)$. We will denote it by $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$. The compositions $\mathcal{E} \to \mathcal{E}' \to \mathcal{E}_i$ are surjective maps, hence the projections $\mathcal{E}' \to \mathcal{E}_i$ are surjective and therefore define closed embedding $\mathbb{P}(\mathcal{E}_i) \hookrightarrow \langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$.

It follows from (1.33) that

$$\omega_{\mathbb{P}(\mathcal{E})/X} \cong \pi^*(\det \mathcal{E})(-r-1). \tag{10.44}$$

If X admits a canonical sheaf ω_X, we get

$$\omega_{\mathbb{P}(\mathcal{E})} \cong \pi^*(\omega_X) \otimes \pi^* \det \mathcal{E})(-r-1). \tag{10.45}$$

Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Recall that the Chern classes $c_i(\mathcal{E})$ can be defined by using the identity in $H^*(\mathbb{P}(\mathcal{E}), \mathbb{Z})$ (see [282], Appendix A):

$$(-\xi)^{r+1} + \pi^*(c_1(\mathcal{E}))(-\xi)^r + \cdots + \pi^*(c_{r+1}(\mathcal{E})) = 0.$$

Let $d = \dim X$. Multiplying the previous identity by ξ^{d-1}, we get

$$\xi^{d+r} = \sum_{i=1}^{r+1} (-1)^i \pi^*(c_1(\mathcal{E}))\xi^{d+r-i}. \tag{10.46}$$

Assume that $d = \dim X = 1$. Then $c_i(\mathcal{E}) = 0$ for $i > 1$ and $c_1(\mathcal{E})$ can be identified with the degree of $\det \mathcal{E}$ (the degree of $\mathcal{E}$). Since ξ intersects the class of a general fiber with multiplicity 1, we obtain

$$\xi^{r+1} = \deg \mathcal{E}. \tag{10.47}$$

Since $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ gives a finite map of degree 1, the degree of the scroll $S = f(\mathbb{P}(\mathcal{E}))$ is equal to ξ^{r+1}. Also $\mathcal{E} = \nu^*(\mathcal{S}_G^\vee)$, hence

$$\deg \mathcal{E} = \nu^*(c_1(\mathcal{S}_G^\vee)) = \nu^*(\sigma_1) = \deg \nu(X) = \deg X_0,$$

where the latter degree is taken in the Plücker embedding of G. This gives

$$\deg S = \deg X_0. \tag{10.48}$$

The formula is not anymore true if $d = \dim X > 1$. For example, if $d = 2$, we get the formula

$$\deg S = \xi^{r+2} = \pi^*(c_1(\mathcal{E}))\xi^{r+1} - \pi^*(c_2(\mathcal{E}))\xi^r$$

$$= \pi^*(c_1(\mathcal{E})^2 - c_2(\mathcal{E}))\xi^r = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) = \nu^*(\sigma_2),$$

where σ_2 is the special Schubert class.

Example 10.4.1 Exercise 19.13 from [278] asks us to show that the degree of S_X may not be equal to $\deg X_0$ if $\dim X_0 > 1$. An example is the scroll S of lines equal to the Segre variety $\mathsf{s}_{2,1}(\mathbb{P}^2 \times \mathbb{P}^1) \subset \mathbb{P}^5$. Its degree is equal to 3. If we identify the space $\mathbb{P}^5$ with the projective space of 1-dimensional subspaces of the space of matrices of size 2×3, the Segre variety is the subvariety of matrices of rank 1. If we take homogeneous coordinates t_0, t_1, t_2 in $\mathbb{P}^2$ and homogeneous coordinates z_0, z_1 in $\mathbb{P}^1$, then S is given by

$$\operatorname{rank} \begin{pmatrix} t_0 z_0 & t_1 z_0 & t_2 z_0 \\ t_0 z_1 & t_1 z_1 & t_2 z_1 \end{pmatrix} \leq 1.$$

When we fix (t_0, t_1, t_2), the parametric equation of the corresponding line in $\mathbb{P}^5$ is $z_0[t_0, t_1, t_2, 0, 0, 0] + z_1[0, 0, 0, t_0, t_1, t_2]$. The Plücker coordinates of the line are equal to $p_{i4+j} = t_i t_j, 0 \leq i \leq j \leq 2$, with other coordinates equal to zero. Thus we see that the variety X parameterizing the generators of S spans a subspace of dimension 5 in $\mathbb{P}^9$ and is isomorphic to a Veronese surface embedded in this subspace by the complete linear system of quadrics. This shows that the degree of X is equal to 4.

From now on we shall assume that $X = C$ is a smooth curve C so that the map $\nu : C \to C_0 \subset G_r(\mathbb{P}^n)$ is the normalization map of the curve C_0 parameterizing generators.

Let S_1 and S_2 be two scrolls in $|E|$ corresponding to vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ of ranks r_1 and r_2 and surjections $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}_1$ and $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}_2$. Let $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2) \rangle$ be the join in $\mathbb{P}(E^\vee \otimes \mathcal{O}_X) = X \times |E|$ and let S be the projection of the join to $|E| = \mathbb{P}(E^\vee)$. It is a scroll in $|E|$ whose generators are the joins of the corresponding generators of S_1 and S_2. Let $\{x_1, \ldots, x_m\}$ be the support of the sheaf $\mathcal{E}_1 \oplus \mathcal{E}_2/\mathcal{E}'$ and let h_i be the dimension of the quotient at the point x_i. Two generators corresponding to a point $x \notin \{x_1, \ldots, x_m\}$ span a linear subspace of expected dimension $r_1 + r_2 + 1$. The generators corresponding to a point x_j span a subspace of dimension $r_1 + r_2 - h_1$. The scroll S is denoted by $\langle S_1, S_2 \rangle$ and is called the *join* of scrolls S_1 and S_2. Since $\deg \mathcal{E}' = \deg \mathcal{E}_1 + \deg \mathcal{E}_2$, we obtain

$$\deg \langle S_1, S_2 \rangle = \deg S_1 + \deg S_2 - \sum_{i=1}^{m} h_i. \tag{10.49}$$

Let us consider some special examples.

Example 10.4.2 Let $E_i^\vee \otimes \mathcal{O}_C \to \mathcal{E}_i$ define scrolls S_i in $|E_i|, i = 1, 2$. Consider the surjection $E^\vee \otimes \mathcal{O}_C = (E_1^\vee \oplus E_2^\vee) \otimes \mathcal{O}_C \to \mathcal{E}_1 \oplus \mathcal{E}_2$. It defines the scroll equal to the join of the scroll $S_1 \subset |E_1| \subset |E|$ and the scroll $S_2 \subset |E_2| \subset |E|$. Its degree is equal to $\deg S_1 + \deg S_2$. For example, let $\mathcal{E}_i$ be

an invertible sheaf on C defining a closed embedding $\tau_i : C \subset |E_i|$ so that $S_i = \tau_i(C)$ are curves of degree a_i spanning E_i. Then the join of S_1 and S_2 is a surface of degree $a_1 + a_2$ with generators parameterized by C. Specializing further, we take $C = \mathbb{P}^1$ and $\mathcal{E}_i = \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_1 \leq a_2$. The scroll $\langle S_1, S_2\rangle$ is the rational normal scroll S_{a_1, a_1+a_2-1}. Iterating this construction we obtain rational normal scrolls $S_{a_1,\dots,a_k,n} \subset \mathbb{P}^n$, where $n = a_1 + \cdots + a_k - k + 1$.

Example 10.4.3 Suppose we have two scrolls S_1 and S_2 in $\mathbb{P}^n = |E|$ defined by surjections $\alpha_i : E^\vee \otimes \mathcal{O}_{C_i} \to \mathcal{E}_i$, where rank $\mathcal{E}_i = r_i + 1$. Let $\Gamma_0 \subset C_1 \times C_2$ be a correspondence of bidegree (α_1, α_2) and let $\mu : \Gamma \to \Gamma_0$ be its normalization map. Let $p_i : \Gamma \to C_i$ be the composition of μ and the projection maps $C_1 \times C_2 \to C_i$. Consider the surjections $p_i^*(\alpha_i) : E^\vee \otimes \mathcal{O}_\Gamma \to p_i^*\mathcal{E}_i$. Let $\langle \mathbb{P}(p_1^*\mathcal{E}_1), \mathbb{P}(p_1^*\mathcal{E}_1)\rangle$ be the corresponding join. Let S be the image of the join in $|E|$. We assume that it is a scroll whose generators are parameterized by an irreducible curve $C_0 \subset G_{r_1+r_2-1}(|E|)$ equal to the closure of the image of the map $\phi : \Gamma \to G_{r_1+r_2-1}(|E|)$ defined by $\phi(z) = \overline{\nu_1(p_1(z))\nu_2(p_1(z))}$. Let a be the degree of this map. Then

$$\deg S = \frac{1}{a}(\alpha_1 \deg S_1 + \alpha_2 \deg S_2 - h),$$

where

$$h = h^0(\mathrm{Coker}(\mu^*(E^\vee \otimes \mathcal{O}_\Gamma \to p_1^*\mathcal{E}_1 \oplus p_2^*\mathcal{E}_2))).$$

Here are some special examples. We can take for S_1 and S_2 two isomorphic curves in $\mathbb{P}^n$ of degrees d_1 and d_2 intersecting transversally at m points $x_1, \dots, x_m$. Let Γ be the graph of an isomorphism $\sigma : S_1 \to S_2$. Let h be the number of points $x \in S_1$ such that $\sigma(x) = x$. Obviously, these points must be among the points x_i's. Assume that x_1 and $\sigma(t_1)$ do not lie on a common trisecant for a general point $x_1 \in S_1$. Then $h^0 = 1$ and the scroll S is a scroll of lines of degree $d_1 + d_2 - h$. We could also take $S_1 = S_2$ and σ be an automorphism of S_1 with h fixed points. Then the degree of the scroll S is equal to $2d - h$ if σ^2 is not equal to the identity and $\frac{1}{2}(2d - h)$ otherwise.

10.4.2 Cayley-Zeuthen formulas

From now on, until the end of this Chapter, we will be dealing only with scrolls with 1-parameter family C_0 of generators. A 2-dimensional scroll is called a *ruled surface*. This classical terminology disagrees with the modern one, where a ruled surface means a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth projective curve (see [282]). Our ruled surfaces are their images under a degree 1 morphism given by a linear system in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

Let $\nu : C \to C_0$ be the normalization map and $\mathcal{E} = \nu^*(\mathcal{S}_{C_0}^\vee)$. The projective bundle $\mathbb{P}(\mathcal{E})$ is isomorphic to the normalization of the ruled surface S defined by the curve $C_0 \subset G_1(\mathbb{P}^n)$.

Let us remind ourselves some known facts about projective 1-bundles $X = \mathbb{P}(\mathcal{E})$ over smooth curves that can be found in [282], Chapter V, §2.

After tensoring $\mathcal{E}$ with an appropriate invertible sheaf we may assume that $\mathcal{E}$ is *normalized* in the sense that $H^0(C, \mathcal{E}_0) \neq \{0\}$ but $H^0(C, \mathcal{E}_0 \otimes \mathcal{L}) = \{0\}$ for any invertible sheaf $\mathcal{L}$ of negative degree. In this case the integer $e = -\deg \mathcal{E} \geq 0$ is an invariant of the surface and the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is isomorphic to $\mathcal{O}_X(E_0)$, where $E_0^2 = -e$. If $e < 0$ the curve E_0 is the unique curve on X with negative self-intersection. It is called the *exceptional section*.

Let $\sigma_0 : C \to X$ be the section of $\pi : X \to C$ with the image equal to E_0. Then $\sigma_0^* \mathcal{O}_X(E_0) \cong \mathcal{O}_C(\mathfrak{e})$. If we identify E_0 and C by means of σ_0, then $\mathcal{O}_C(\mathfrak{e}) \cong \mathcal{O}_X(E_0) \otimes \mathcal{O}_{E_0}$ so that $\deg \mathfrak{e} = e$. A section $\sigma : C \to X$ is equivalent to a surjection of locally free sheaves $\mathcal{E} \to \mathcal{L} \cong \sigma^* \mathcal{O}_X(\sigma(C))$. In particular, $\deg \mathcal{L} = \sigma(C)^2$. The canonical class of X is given by the formula

$$K_X \sim -2E_0 + \pi^*(K_C + \mathfrak{e}), \tag{10.50}$$

which is a special case of (1.33).

Let $|H|$ be a complete linear system of dimension $N > 2$ on $\mathbb{P}(\mathcal{E})$ defined by an ample section H. Since $\pi_*(\mathcal{O}_X(H)) = \mathcal{E} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$, we can write

$$H \sim E_0 + \pi^*(\mathfrak{a})$$

for some effective divisor class $\mathfrak{a}$ on C of degree a. Since H is irreducible, intersecting both sides with E_0 we find that $a \geq e$. Using the Moishezon-Nakai criterion of ampleness it is easy to see that H is ample if and only if $a > e$. We shall assume that H is ample. Assume also that $\mathfrak{a}$ is not special in the sense that $H^1(C, \mathcal{O}_C(\mathfrak{a})) = 0$ and $|\mathfrak{e} + \mathfrak{a}|$ has no base points on C. Then the exact sequence

$$0 \to \mathcal{O}_X(\pi^*(\mathfrak{a})) \to \mathcal{O}_X(H) \to \mathcal{O}_{E_0}(H) \to 0$$

shows that the restriction of $|H|$ to E_0 is a complete linear system without base points. It is clear that any possible base point of $|H|$ must lie on E_0, hence under the above assumptions $|H|$ has no base points. It defines a finite map $f : X \to S \subset \mathbb{P}^N$. The surface S is a linearly normal surface in $\mathbb{P}^N$ swept by lines, the images of fibres. The family of lines is defined by the image of C in $G_1(\mathbb{P}^N)$. The next Proposition shows that the map is of degree 1, hence S is a ruled surface.

Proposition 10.4.4 *Let H be an ample section on $X = \mathbb{P}(\mathcal{E})$ as in the above and $|V|$ be a linear system in $|H|$ that defines a finite map $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$. Then the degree of the map is equal to 1.*

Proof Suppose $f(x) = f(y)$ for some general points $x, y \in X$. Let F_x and F_y be the fibres containing x and y. Since $|H|$ has no base points, its restriction to any fiber is a linear system of degree 1 without base points. Suppose the degree of the map is greater than 1. Take a general fiber F; then, for any general point $x \in F$, there is another fiber F_x such that $f(F_x)$ and $f(F)$ are coplanar. This implies that there exists a divisor $H(x) \in |H - F_x - F|$. We can write $H(x) = F_x + F + R(x)$ for some curve $R(x)$ such that $R(x) \cdot F_x = R(x) \cdot F = 1$. When we move x along F we get a pencil of divisors $H(x)$ contained in $|H - F|$. The divisors of this pencil look like $F_x + R(x)$ and hence all have a singular point at $R(x) \cap F_x$. Since the fiber F_x moves with x, we obtain that a general member of the pencil has a singular point that is not a base point of the pencil. This contradicts Bertini's Theorem on singular points [282], Chapter 3, Corollary 10.9. □

Corollary 10.4.5 *Let S be an irreducible surface in $\mathbb{P}^N$ containing a 1-dimensional irreducible family of lines. Suppose S is not a cone. Then S is a ruled surface equal to the image of projective bundle $\mathbb{P}(\mathcal{E})$ over a smooth curve C under a birational morphism given by a linear subsystem in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.*

Proof Let $C_0 \subset G_1(\mathbb{P}^n)$ be the irreducible curve parameterizing the family of lines and let $\nu : C \to C_0$ be its normalization. The preimage of the universal family $Z_{C_0} \to C_0$ is a projective bundle $\mathbb{P}(\mathcal{S}^\vee_{C_0})$ over C. Since S is not a cone, the map $f : \mathbb{P}(\mathcal{E}) \to S$ is a finite morphism. The map is given by a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Since f is a finite morphism, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = f^*(\mathcal{O}_{\mathbb{P}(\mathcal{S}_{C_0})^\vee}(1))$ is ample. It remains for us to apply the previous Proposition. □

An example of a nonsingular quadric surface seems contradicts the previous statement. However, the variety of lines on a nonsingular quadric surface is not irreducible and consists of two projective lines embedded in $G_1(\mathbb{P}^3)$ as the union of two disjoint conics. So the surface has two systems of rulings, and it is a 2-way scroll.

It follows from (10.48) that the degree of the ruled surface $S = f(\mathbb{P}(\mathcal{E}))$ is equal to the degree of C in the Plücker space. It is also equal to the self-intersection H^2 of the tautological line bundle on $\mathbb{P}(\mathcal{E})$. The latter is equal to $H^2 = (E_0 + aF)^2 = 2a - e$. The genus of C is called the *genus* of the ruled surface.

Proposition 10.4.6 *Let $S = f(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}^n$ be a projection of a minimal ruled surface $\mathbb{P}(\mathcal{E})$ embedded in projective space by a linear system $|H|$, where $H \sim E_0 + \pi^*(\mathfrak{a})$. Let D be a directrix on S that is not contained in the singular locus of S. Then*

$$\deg \mathsf{D} \geq \deg \mathfrak{a} - e.$$

The equality takes place if and only if the preimage of D on $\mathbb{P}(\mathcal{E})$ is in the same cohomology class as E_0.

Proof The assumption on D implies that $\deg \mathsf{D} = H \cdot E$, where E is the preimage of D on $\mathbb{P}(\mathcal{E})$. Intersecting with H we get $H \cdot E = E \cdot E_0 + a$. If $E \neq E_0$, then $H \cdot E \geq a$, if $[E] = [E_0]$, then $E \cdot E_0 = a - e$. The equality takes place if and only if $E \cdot E_0 = 0$ and $e = 0$. Since E is a section, we can write $[E] = [E_0] + m[F]$, and intersecting with E_0, we get $m = 0$. □

Since $f : \mathbb{P}(\mathcal{E}) \to S$ is of degree 1, the ruled surface is non-normal at every point over which the map is not an isomorphism.

Recall the *double-point formula* from [231], 9.3. Let $f : X \to Y$ be a morphism of nonsingular varieties of dimensions m and n, respectively. Let Z be the blow-up of the diagonal of $X \times X$ and let R be the exceptional divisor. We think about points in R as points in X together with a tangent direction t_x at x. Let $\tilde{D}(f)$ be the proper transform in Z of the fibered product $X \times_Y X \subset X \times X$. One can view points in $\tilde{D}(f)$ either as points $x \in X$ such that there exists $x' \neq x$ with $f(x) = f(x')$, or as points (x, t_x) such that $df_x(t_x) = 0$. Let $D(f)$ be the image of $\tilde{D}(f)$ under one of the projections $X \times_Y X \to X$. This is called the *double-point set* of the morphism f. Define the *double point class*

$$\mathbb{D}(f) = f^* f_*[X] - (c(f^*\mathcal{T}_Y)c(\mathcal{T}_X)^{-1})_{n-m} \cap [X] \in H^{n-m}(X, \mathbb{Q}), \quad (10.51)$$

where c denotes the total Chern class $[X] + c_1 + \cdots + c_m$ of a vector bundle. In case $D(f)$ has the expected dimension equal to $2m - n$, we have

$$\mathbb{D}(f) = [D(f)] \in H^{n-m}(X, \mathbb{Z}).$$

Assume now that $f : X \to S$ is the normalization map and S is a surface in $\mathbb{P}^3$. Since S is a hypersurface, it does not have isolated non-normal points. This implies that $D(f)$ is either empty, or is of expected dimension $2m - n = 1$. The double-point class formula applies, and we obtain

$$[D(f)] = f^*(S) + f^*(K_{\mathbb{P}^3}) - K_X. \quad (10.52)$$

In fact, it follows from the theory of adjoints (see [355]) that the linear equivalence class of $D(f)$ is expressed by the same formula.

We say that a non-normal surface S in $\mathbb{P}^n$ has *ordinary singularities* if its singular locus is a double curve Γ on S. This means that the completion of the local ring of S at a general point of Γ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1 z_2)$. The curve Γ may have also *pinch point* locally isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 - z_2^2 z_3)$ and also triple points locally isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1 z_2 z_3)$. The curve Γ is nonsingular outside triple points, the curve $D(f)$ is nonsingular outside the preimages of the triple points. It has three double points over each triple point.

Under these assumptions, the map $\tilde{D}(f) \to D(f) \to \Gamma$ is of degree 2. It is ramified at pinch points only, and the preimage of a triple point consists of six points.

Assume that S is a surface in $\mathbb{P}^3$ with ordinary singularities. Let $f : X \to S$ be the normalization map, and Γ be the double curve of S. The degree of any curve on X, is the degree with respect to $f^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Let us introduce the following numerical invariants in their classical notation:

- $\mu_0 =$ the degree of S;
- $\mu_1 =$ the rank of S, the class of a general plane section of S;
- $\mu_2 =$ the class of S;
- $\nu_2 =$ the number of pinch-points on S;
- $t =$ the number of triple points on S;
- $\epsilon_0 = \deg \Gamma$;
- $\epsilon_1 =$ the rank of Γ, the number of tangents to Γ intersecting a general line in $\mathbb{P}^3$;
- $\rho =$ the *class of immersion* of Γ equal to the degree of the image of $D(f)$ under the Gauss map $G : X \to S \xrightarrow{\gamma} (\mathbb{P}^3)^\vee$, where γ is the Gauss map.;
- $g(\Gamma) =$ the genus of Γ;
- $c =$ the number of connected components of Γ;
- $\kappa =$ the degree of the ramification divisor $p : X \to \mathbb{P}^2$, where p is the composition of f and the general projection of S.

The following Theorem summarizes different relations between the listed invariants of S. These relations are called the *Cayley-Zeuthen formulas*.

Theorem 10.4.7 *The following relations hold:*

(i) $\mu_1 = \mu_0(\mu_0 - 1) - 2\epsilon_0$;
(ii) $\epsilon_0(\mu_0 - 2) = \rho + 3t$;
(iii) $\mu_1(\mu_0 - 2) = \kappa + \rho$;
(iv) $2g(\Gamma) - 2c = \epsilon_1 - 2\epsilon_0$;
(v) $\nu_2 = 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1$;
(vi) $2\rho - 2\epsilon_1 = \nu_2$;

(vii) $\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t - 2\nu_2$;

(viii) $2\nu_2 + \mu_2 = \mu_1 + \kappa$.

Proof (i) A general plane section of S is a plane curve of degree μ_0 with ϵ_0 ordinary double points as singularities. Thus (i) is just the Plücker formula. Note also that μ_1 is equal to the degree of the *contact curve*, the closure of smooth points $p \in S$ such that a general point $q \in \mathbb{P}^3$ is contained in $\mathbb{T}_p(S)$, or, equivalently, the residual curve to Γ of the intersection of S and the first polar $P_q(S)$. Taking a general plane H and a general point $q \in H$, we obtain that $\deg \Delta$ is equal to the class of $H \cap S$.

(ii) The number ρ is equal to the number of tangent planes to S at points in Γ which pass through a general point q in $\mathbb{P}^3$. Here a tangent plane to a singular point $p \in \Gamma$ means the tangent plane to one of the two branches of S at q, or, equivalently, the image of a preimage of p on X under the Gauss map. Consider the intersection of the second polar P_{q^2} with the contact curve Γ. It follows from subsection 1.1.3 that $P_{q^2}(S) \cap S$ is equal to the locus of points p such that the line $\overline{pq}$ intersects S at p with multiplicity ≥ 3. This means that $P_{q^2}(S) \cap \Gamma$ consists of t triple points and points such that q belongs to a tangent plane of S at p. The latter number is equal to ρ. As we observed in subsection 1.1.3, P_{q^2} has a point of multiplicity 3 at p, hence each triple point enters with multiplicity 3 in the intersection of $\overline{pq}$ with Γ. It remains for us to use that the degree of the second polar is equal to $\mu_0 - 2$.

(iii) Now let us consider the intersection of the second polar $P_{q^2}(S)$ with the contact curve Δ. This intersection consists of the lines $\overline{qp}$ such that p is either one of κ ramification points of the projection of the surface from q or p is one of ρ points on $\Gamma \cap \Delta$, where the tangent plane contains p. In fact, these points lie on the intersection of Δ and Γ.

(iv) - (vi) Let $\pi = h^1(\mathcal{O}_{D(f)})$ be the arithmetic genus of the curve $D(f)$ and let s be the number of connected components of $D(f)$. Applying (10.52), we get

$$-2\chi(D(f), \mathcal{O}_{D(f)}) = 2\pi - 2c = (D(f) + K_X) \cdot D(f)$$

$$= (\mu_0 - 4) \deg D(f) = 2\epsilon_0(\mu_0 - 4).$$

The curve $D(f)$ has $3t$ ordinary double points and the projection from the normalization of $D(f)$ to Γ is a degree 2 cover ramified at ν_2 points. Applying the Hurwitz formula, we obtain $2\pi - 2c - 6t = 2(2g(\Gamma) - 2c) + \nu_2$. Projecting Γ from a general line defines a degree ϵ_0 map from the normalization of Γ to $\mathbb{P}^1$. The number of ramification points is equal to ϵ_1. Applying the Hurwitz

formula again, we get $2g(\Gamma) - 2c = -2\epsilon_0 + \epsilon_1$. This gives (iv) and also gives

$$\nu_2 = 2\epsilon_0(\mu_0 - 4) - 6t - 2(2g(\Gamma) - 2c) = 2\epsilon_0(\mu_0 - 4) - 6t - 2\epsilon_1 + 4\epsilon_0$$

$$= 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1.$$

This is equality (v). It remains for us to use (ii) to get (vi).

(vii) The formula for the class of a non-normal surface with ordinary singularities has a modern proof in [231], Example 9.3.8. In our notation, it gives (vii).

(viii) We have

$$\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t.$$

Using this and (i), we get

$$\mu_2 + 2\nu_2 = (\mu_0 - 1)(\mu_1 + 2\epsilon_0) + 4\epsilon_0 - 3\mu_0\epsilon_0$$

$$= \mu_0\mu_1 - \mu_1 + 2\epsilon_0 + \rho - \epsilon_0\mu_0 + 3t.$$

It remains for us to use (ii) and (iii). □

Corollary 10.4.8 *Let S be a surface in $\mathbb{P}^3$ with ordinary singularities and let X be its normalization. Then*

(i) $K_X^2 = \mu_0(\mu_0 - 4)^2 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2$;
(ii) $c_2(X) = \mu_0(\mu_0^2 - 4\mu_0 + 6) - (3\mu_0 - 8)\epsilon_0 + 3t - 2\nu_2$;
(iii) $\chi(X, \mathcal{O}_X) = 1 + \binom{\mu_0 - 1}{3} - \frac{1}{2}(\mu_0 - 4)\epsilon_0 + \frac{1}{2}t - \frac{1}{4}\nu_2$.

Proof (i) Applying (10.52), we get

$$K_X = (\mu_0 - 4)H - D(f), \tag{10.53}$$

where $H \in |f^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. The first polar of S with respect to a general point cuts out on S the union of Γ and Δ. Taking the preimages on X, we get

$$(\mu_0 - 1)H = D(f) + f^*(\Delta).$$

It follows from the local computation that Γ and Δ intersect simply at ν_2 pinch points and ρ additional points (see the proof of (iii) in Theorem 10.4.7). This gives

$$D(f)^2 = (\mu_0 - 1)H \cdot D(f) - \rho - \nu_2 = 2\epsilon_0(\mu_0 - 1) - \rho - \nu_2$$

$$= 2\epsilon_0(\mu_0 - 1) - \epsilon_0(\mu_0 - 2) + 3t - \nu_2 = \epsilon_0(\mu_0 - 2) + 3t - \nu_2.$$

Hence

$$K_X^2 = (\mu_0 - 4)^2\mu_0 - 4(\mu_0 - 4)\epsilon_0 + D(f)^2$$

$$= (\mu_0 - 4)^2\mu_0 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2.$$

(ii) The preimage of a pinch point on X is a point in X such that the rank of the tangent map $\mathcal{T}_X \to f^*(\mathcal{T}_{\mathbb{P}^3})$ drops by 2. According to the modern theory of degeneracy loci (see [231]), this set is given by the relative second Chern class $c_2(f^*(\mathcal{T}_{\mathbb{P}^3})/\mathcal{T}_X)$. Computing this Chern class, we find

$$\nu_2 = c_1(X)^2 - c_2(X) + 4K_X \cdot H + 6\mu_0.$$

Applying (10.53), we get

$$\nu_2 = K_X^2 - c_2(X) + 4(\mu_0 - 4)\mu_0 - 8\epsilon_0 + 6\mu_0. \tag{10.54}$$

Together with (i) we get (ii). Formula (iii) follows from the *Noether formula*

$$12\chi(X, \mathcal{O}_X) = K_X^2 + c_2(X).$$

□

We know that μ_0 is equal to the degree d of C_0 in its Plücker embedding. The next Theorem shows that all the numerical invariants can be expressed in terms of μ_0 and g.

Theorem 10.4.9 *Let S be a ruled surface in $\mathbb{P}^3$ of degree μ_0 and genus g. Assume that S has only ordinary singularities. Then*

(i) $\epsilon_0 = \frac{1}{2}(\mu_0 - 1)(\mu_0 - 2) - g$;
(ii) $\nu_2 = 2(\mu_0 + 2g - 2)$;
(iii) $\mu_1 = 2\mu_0 - 2 + 2g$;
(iv) $\mu_2 = \mu_0 = \mu_0$;
(v) $\kappa = 3(\mu_0 + 2g - 2)$;
(vi) $\rho = (\mu_0 - 2)(2\mu_0 - 5) + 2g(\mu_0 - 5)$;
(vii) $t = \frac{1}{6}(\mu_0 - 4)[(\mu_0 - 2)(\mu_0 - 3) - 6g]$;
(viii) $\epsilon_1 = 2(\mu_0 - 2)(\mu_0 - 3) + 2g(\mu_0 - 6)$;
(ix) $2g(\Gamma) - 2s = (\mu_0 - 5)(\mu_0 + 2g - 2)$.

Proof A general plane section of S is a plane curve of degree d with $k = \deg \Gamma$ ordinary singularities. This gives (i).

The canonical class formula gives

$$K_{\mathbb{P}(\mathcal{E})} = -2H + \pi^*(K_{\bar{C}} + \mathfrak{d}), \tag{10.55}$$

where $\mathcal{O}_{\bar{C}}(\mathfrak{d}) \cong \nu^*(\mathcal{O}_{C_0}(1))$ is of degree $d = \mu_0$.

Comparing it with formula (10.50), we find that

$$H \sim E_0 + \pi^*(\mathfrak{f}), \tag{10.56}$$

where $2\mathfrak{f} = \mathfrak{d} - \mathfrak{e}$. In particular, $e + d$ is always an even number.

Applying (10.55), we get $K^2_{\mathbb{P}(\mathcal{E})} = 4\mu_0 - 4(2g - 2 + \mu_0)$. Topologically, $\mathbb{P}(\mathcal{E})$ is the product of $\mathbb{P}^1$ and C. This gives $c_2(X) = 2(2 - 2g)$. Applying (10.54), we find

$$\nu_2 = 4\mu_0 - 4(2g-2+\mu_0) - 2(2-2g) + 4(\mu_0-4)\mu_0 - 4(\mu_0-1)(\mu_0-2) - 8g + 6\mu_0$$

$$= 2(\mu_0 + 2g - 2).$$

From (i) of Theorem 10.4.7, we get (iii).

To prove (iv) we have to show that the degree of S is equal to the degree of its dual surface. The image of a generator of S under the Gauss map is equal to the dual line in the dual $\mathbb{P}^3$, i.e. the set of hyperplanes containing the line. Since S has only finitely many torsor generators, the Gauss map is a birational map, this shows that S^* is a ruled surface. If S is defined by the vector bundle $\mathcal{E} = \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$, then the dual ruled surface is defined by the bundle $\mathcal{Q}_G \otimes \mathcal{O}_{C_0}$, where $\mathcal{O}_G$ is the universal quotient bundle. The exact sequence (10.1) shows that $\det \mathcal{Q}_G \otimes \mathcal{O}_{C_0} \cong \det \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$. In particular, the degrees of their inverse images under $\nu : C \to C_0$ are equal. Thus the degrees of S and S^* are equal.

Now (i) and (viii) of Theorem 10.4.7 and our formula (i) give (v). Using (iii) and (ii) of the same Theorem, we get formulas (vi) and (vii). Finally, (vi) of the Theorem gives formulas (viii) and (ix). □

The double-point formula gives

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D(f)) \sim \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu_0 - 2) \otimes \pi^*(\omega_C(1)).$$

A general point of Γ is contained in two rulings and formula (10.52) implies that a general ruling intersects $\mu_0 - 2$ other rulings. Consider a symmetric correspondence on C defined by

$$T = \{(x, y) \in C \times C : |H - \ell_x - \ell_y| \neq \emptyset\}.$$

A point $(x, x) \in T$ corresponds to a generator that is called a *torsal generator*. The plane in $\mathbb{P}^3$ cutting out this generator with multiplicity ≥ 2 is tangent to the ruled surface at any smooth point of the generator. For a general point x, we have $\#T(x) = d - 2$. Since all generators $\ell_y, y \in T(x)$, intersect the same line ℓ_x the points $y \in T(x)$ lie in the tangent hyperplane of $G_1(\mathbb{P}^3)$ at the point x. This implies that the divisor $2x + T(x)$ belongs to the linear system $|\mathcal{O}_C(1)|$ and, in particular, T has valence equal to 2. Applying the Cayley-Brill formula from Corollary 5.5.2, we obtain the following.

Proposition 10.4.10 *The number of torsal generators of a ruled surface in* $\mathbb{P}^3$ *with ordinary singularities is equal to* $2(\mu_0 + 2g - 2)$.

Comparing with Theorem 10.4.9, we find that the number of torsal generators is equal to the number ν_2 of pinch points.

When $n = 4$ we expect that a ruled surface has only finitely many singular non-normal points and for $n = 5$, we expect that it is nonsingular.

We have already encountered in Example 7.2.6 with a ruled surface S of degree 8 with a triple curve C as its singular curve. A general plane section of this surface is a plane curve of degree 8 of genus 3 with six triple points. Applying formula (10.52) we see that the linear equivalence class of the curve $D(f)$ is equal to $2H - \pi^*(K_C + \mathfrak{d})$ for some divisor $\mathfrak{d}$ of degree d. However, the curve $D(f)$ comes with multiplicity 2, so the curve C in S is the image of a curve $\tilde{C}$ on Z_C from the linear system $|H - \pi^*(\mathfrak{f})|$, where $2\mathfrak{f} \sim K_C + \mathfrak{d}$. So, each generator intersects it at three points, as expected. One can show that $\mathfrak{d} \sim K_C + 2\mathfrak{a}$ so that $\mathfrak{f} \sim K_C + \mathfrak{a}$. Note that the curve $\tilde{C}$ defines a $(3, 3)$-correspondence on the curve C with the projections p_C and q_C to C. Its genus is equal to 19 and each projection is a degree 3 cover ramified at 24 points. In the case when the divisor $\mathfrak{a}$ is an even theta characteristic, the curve $\tilde{C}$ is the Scorza correspondence which we studied in section 5.5.2.

Next example shows that the double curve of a ruled surface may be disconnected.

Example 10.4.11 Consider three nonsingular nondegenerate curves $X_i, i = 1, 2, 3$, in $\mathbb{P}^3$ with no two having common points. Let S be the set of lines intersecting each curve. Let us show that these lines sweep a ruled surface of degree $2d_1d_2d_3$, where $d_i = \deg C_i$. Recall that the set of lines intersecting a curve X of degree X is a divisor in $G_1(\mathbb{P}^3)$ of degree d. This is the *Chow form* of C (see [239]). Thus the set of lines intersecting three curves is a complete intersection of three hypersurfaces in $G_1(\mathbb{P}^3)$, hence a curve of degree $2d_1d_2d_3$. If we assume that the curves are general enough so that the intersection is transversal, we obtain that the ruled surface must be of degree $2d_1d_2d_3$. The set of lines intersecting two curves X_1 and X_2 is a surface W in $G_1(\mathbb{P}^3)$ of degree $2d_1d_2$. Its intersection with the Schubert variety $\Omega(\Pi)$, where Π is a general plane, consists of d_1d_2 lines. It follows from the intersection theory on $G_1(\mathbb{P}^3)$ that the intersection of W with the α-plane $\Omega(p)$ is of degree d_1d_2. Thus we expect that, in a general situation, the number of generators of S passing through a general point on X_3 is equal to d_1d_2. This shows that a general point of X_3 is a singular point of multiplicity d_1d_2. Similarly, we show that X_1 is a singular curve of multiplicity d_2d_3 and X_2 is a singular curve of multiplicity d_1d_3.

Remark 10.4.12 According to [149], the double curve Γ is always connected if $\mu_0 \geq g + 4$. If it is disconnected, then it must be the union of two lines.

10.4.3 Developable ruled surfaces

A ruled surface is called *developable* if the tangent planes at nonsingular points of any ruling coincide. In other words, any generator is a torsal generator. One expects that the curve of singularities is a cuspidal curve. In this subsection we will give other characterizations of developable surfaces.

Recall the definition of the vector bundle of *principal parts* on a smooth variety X. Let Δ be the diagonal in $X \times X$ and let $\mathcal{J}_\Delta$ be its sheaf of ideals. Let p and q be the first and the second projections to X from the closed subscheme Δ^m defined by the ideal sheaf $\mathcal{J}_\Delta^{m+1}$. For any invertible sheaf $\mathcal{L}$ on X one defines the sheaf of m-th principal parts $\mathcal{P}^m(\mathcal{L})$ of $\mathcal{L}$ as the sheaf $\mathcal{P}^m_X(\mathcal{L}) = p_*q^*(\mathcal{L})$ on X. Recall that the m-th tensor power of the sheaf of 1-differentials Ω^1_X can be defined as $p_*(\mathcal{J}^m_\Delta/\mathcal{J}^{m+1}_\Delta)$ (see [282]). The exact sequence

$$0 \to \mathcal{J}^m_\Delta/\mathcal{J}^{m+1}_\Delta \to \mathcal{O}_{X\times X}/\mathcal{J}^{m+1}_\Delta \to \mathcal{O}_{X\times X}/\mathcal{J}^m_\Delta \to 0$$

gives an exact sequence

$$0 \to (\Omega^1_X)^{\otimes m} \otimes \mathcal{L} \to \mathcal{P}^m_X(\mathcal{L}) \to \mathcal{P}^{m-1}_X(\mathcal{L}) \to 0. \tag{10.57}$$

We will be interested in the case when $X_0 = C_0$ is an irreducible curve of genus g and $X = C$ is its normalization. By induction, the sheaf $\mathcal{P}^m_C(\mathcal{L})$ is a locally free sheaf of rank $m+1$, and

$$\deg \mathcal{P}^m_C(\mathcal{L}) = (m+1)\deg \mathcal{L} + m(m+1)(g-1). \tag{10.58}$$

For any subspace $V \subset H^0(C, \mathcal{L})$, there is a canonical homomorphism

$$V \to H^0(\Delta^m, q^*\mathcal{L}) = H^0(C, p_*q^*\mathcal{L}) = H^0(C, \mathcal{P}^m_C(\mathcal{L}))$$

which defines a morphism of locally free sheaves

$$\alpha_m : V_C := \mathcal{O}_C \otimes V \to \mathcal{P}^m_C(\mathcal{L}). \tag{10.59}$$

Note that the fiber of $\mathcal{P}^m_C(\mathcal{L})$ at a point x can be canonically identified with $\mathcal{L}/\mathfrak{m}^{m+1}_{C,x}\mathcal{L}$ and the map α_m at a point x is given by assigning to a section $s \in V$ the element $s \mod \mathfrak{m}^{m+1}_{C,x}\mathcal{L}$. If $m = 0$, we get $\mathcal{P}^m_C(\mathcal{L}) = \mathcal{L}$ and the map is the usual map given by evaluating a section at a point x.

Suppose that $(V, \mathcal{L})$ defines a morphism $f : C \to \mathbb{P}(V)$ such that the induced morphism $f : C \to f(C) = C_0$ is the normalization map. We have $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}}(1))$. Let $\mathcal{P}^m \subset \mathcal{P}^m_C(\mathcal{L})$ be the image of α_m. Since the composition of α_1 with the projection $\mathcal{P}^1_C \to \mathcal{L}$ is generically surjective (because C_0 spans $\mathbb{P}(V)$), the map α_1 is generically surjective. Similarly, by induction, we show that α_m is generically surjective for all m. Since C is a smooth curve,

this implies that the sheaves $\mathcal{P}^m$ are locally free of rank $m+1$. They are called the *osculating sheaves*. Let

$$\sigma_m : C \to G(m+1, V^\vee)$$

be the morphisms defined by the surjection $\alpha_m : V_C \to \mathcal{P}^m$. The morphism σ_m can be interpreted as assigning to each point $x \in C$ the m-th *osculating plane* of $f(C)$ at the point $f(x)$. Recall that it is a m-dimensional subspace of $\mathbb{P}(V)$ such that it has the highest-order contact with the branch of C_0 defined by the point $x \in C$. One can always choose a system of projective coordinates in $\mathbb{P}(V) \cong \mathbb{P}^n$ such that the branch of C_0 corresponding to x can be parameterized in the ring of formal power series by

$$t_0 = 1,\ t_i = t^{i+s_1+\cdots+s_i} + \text{ highest-order terms},\ i = 1, \ldots, n, \qquad (10.60)$$

where $s_i \geq 0$. Then the osculating hyperplane is given by the equation $t_n = 0$. The codimension 2 osculating subspace is given by $t_{n-1} = t_n = 0$ and so on. A point $x \in C$ (or the corresponding branch of $f(C)$) with $s_1 = \ldots = s_n = 0$ is called an *ordinary point*, other points are called *hyperosculating* or *stationary* points . It is clear that a point x is ordinary if and only if the highest order of tangency of a hyperplane at x is equal to n. For example, for a plane curve, a point is ordinary if the corresponding branch is nonsingular and not an inflection point.

The image $\sigma_m(C)$ in $G_m(\mathbb{P}^n)$ is called the m-th *associated curve*. Locally, the map σ_m is given by assigning to a point $x \in C$ the linear subspace of $\mathbb{C}^{n+1}$ generated by $\tilde{f}(x), \tilde{f}'(x), \ldots, \tilde{f}^{(m)}(x)$, where $\tilde{f} : C \to \mathbb{C}^{n+1}$ is a local lift of the map f to a map to the affine space, and $\tilde{f}^{(k)}$ are its derivatives (see [267], Chapter II, §4).

Let $\mathbb{P}(\mathcal{P}^m) \to C \times \mathbb{P}(V)$ be the morphism corresponding to the surjection α_m. The projection of the image to $\mathbb{P}(V)$ is called the *m-th osculating developable* of $(C, \mathcal{L}, V)$ (or of C_0). For $m = 1$ it is a ruled surface, called the *developable surface* or *tangential surface* of C_0.

Let r_m be the degree of $\mathcal{P}^m$. The $(n-1)$-rank r_{n-1} is called the *class* of C_0. If we consider the $(n-1)$-th associated curve in $G(n, n+1)$ as a curve in the dual projective space $|V|$, then the class of C_0 is its degree. The $(n-1)$-th associated curve $C^\vee$ is called the *dual curve* of C_0. Note that the dual curve should not be confused with the dual variety of C_0. The latter coincides with the $(n-2)$-th osculating developable of the dual curve.

Proposition 10.4.13 *Let $r_0 = \deg \mathcal{L} = \deg f(C)$. For any point $x \in C$ let*

$s_i(x) = s_i$, *where the* s_i*'s are defined in* (10.60)*, and* $k_i = \sum_{x\in C} s_i(x)$. *Then*

$$r_m = (m+1)(r_0 + m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i$$

and

$$\sum_{i=1}^{n}(n-i+1)k_i = (n+1)(r_0 + n(g-1)).$$

In particular,

$$r_1 = 2(r_0 + g - 1) - k_1.$$

Proof Formula (10.58) gives the degree of the sheaf of principal parts $\mathcal{P}^m_C(\mathcal{L})$. We have an exact sequence

$$0 \to \mathcal{P}^m \to \mathcal{P}^m_C(\mathcal{L}) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a skyscraper sheaf whose fiber at $x \in C$ is equal to the cokernel of the map $\alpha^m(x) : V \to \mathcal{L}/\mathfrak{m}_{C,x}\mathcal{L}$. It follows from formula (10.60) that $\dim \mathcal{F}(x)$ is equal to $s_1+(s_1+s_2)+\cdots+(s_1+\cdots+s_m) = \sum_{i=1}^{m}(m-i+1)s_i$. The standard properties of Chern classes give

$$\deg \mathcal{P}^m = \deg \mathcal{P}^m_C(\mathcal{L}) - h^0(\mathcal{F}) = (m+1)(r_0+m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i.$$

The second formula follows from the first one by taking $m = n$ in which case $r_n = 0$ (the surjection of bundles of the same rank $V_C \to \mathcal{P}^n$ must be an isomorphism). □

Adding up r_{m-1} and r_{m+1} and subtracting $2r_m$, we get the following.

Corollary 10.4.14

$$r_{m-1} - 2r_m + r_{m+1} = 2g - 2 - k_{m+1}, \quad m = 0, \ldots, n-1, \qquad (10.61)$$

where $r_{-1} = r_n = 0$.

The previous formulas can be viewed as the *Plücker formulas* for space curves. Indeed, let $n = 2$ and C is a plane curve of degree d and class $d^\vee$. Assume that the dual curve $C^\vee$ has δ ordinary nodes and κ ordinary cusps. Applying Plücker's formula, we have

$$d = d^\vee(d^\vee-1)-2\delta-3\kappa = 2d^\vee+(d^\vee(d^\vee-3)-2\delta-2\kappa)-\kappa = 2d^\vee+2g-2-\kappa.$$

In this case $d^\vee = r_1, d = r_0$ and $\kappa = k_1$, so the formulas agree.

Example 10.4.15 If R_n is a rational normal curve in $\mathbb{P}^n$, then it has no hyperosculating hyperplanes (since no hyperplane intersects it with multiplicity $> n$). So $r_m = (m+1)(n-m) = r_{n-m-1}$. Its dual curve is a rational normal curve in the dual space. Its tangential surface is of degree $r_1 = 2(n-1)$ and the $(n-1)$-th osculating developable is the discriminant hypersurface for binary forms of degree n. For example, for $n = 3$, the tangential surface of R_3 is a quartic surface with equation $Q_0Q_1 + Q_2^2 = 0$, where Q_0, Q_1, Q_2 are some quadrics containing R_3. To see this, one considers a rational map $\mathbb{P}^3 \dashrightarrow \mathcal{N}^\vee \cong \mathbb{P}^2$ defined by the net $\mathcal{N}$ of quadrics containing R_3. After we blow-up $\mathbb{P}^3$ along R_3, we obtain a regular map $Y \to \mathbb{P}^2$ which blows down the proper transform of the tangential surface to a conic in $\mathbb{P}^2$. Its equation can be chosen in the form $t_0t_1 + t_2^2 = 0$. The preimage of this conic is the quartic surface $Q_0Q_1 + Q_2^2 = 0$. It contains R_3 as its double curve. Also, it is isomorphic to the discriminant hypersurface for binary forms of degree 3.

Conversely, assume that C has no hyperosculating hyperplanes. Then all $k_i = 0$, and we get

$$\begin{aligned} &\sum_{m=0}^{n-1}(n-m)(r_{m-1} - 2r_m + r_{m+1}) = -(n+1)r_0 \qquad (10.62)\\ &= \sum_{m=0}^{n-1}(n-m)(2g-2) = n(n+1)(g-1). \end{aligned}$$

This implies $g = 0$ and $r_0 = n$.

The computation from the previous example (10.62) can be used to obtain the formula for the number W of hyperosculating points of a curve C embedded in $\mathbb{P}^n$ by a linear series of degree d (see also [280], Lemma 5.21).

Proposition 10.4.16 *The number of hyperosculating points, counting with multiplicities, is equal to*

$$W = (n+1)(d + n(g-1)). \qquad (10.63)$$

Proof Applying (10.61), we obtain

$$\begin{aligned} W &= \sum_{x\in C}\sum_{i=1}^{n}(s_1(x) + \cdots + s_i(x)) = \sum_{i=0}^{n}(n-i+1)k_i = \sum_{i=0}^{n-1}(n-i)k_{i+1}\\ &= \sum_{i=0}^{n-1}(n-i)(2g-2) - \sum_{i=0}^{n-1}(n-i)(r_{i-1} - 2r_i + r_{i+1})\\ &= (n+1)n(g-1) + (n+1)d = (n+1)(d + n(g-1)). \end{aligned}$$

The number $\sum_{i=1}^{n}(s_1(x)+\cdots+s_i(x))$ should be taken as the definition of the multiplicity of a hyperosculating point x. A simple hyperosculating point satisfies $s_i(x)=\ldots=s_{n-1}(x)=0, s_n(x)=1$. □

Example 10.4.17 Let C be an elliptic curve embedded in $\mathbb{P}^n$ by a complete linear system $|(n+1)x_0|$, where x_0 is a point on C. Then the degree of E is equal to $n+1$ and formula (10.63) gives $W=(n+1)^2$. This is equal to the number of $(n+1)$-torsion points of C in the group law defined by the choice x_0 as the zero point. Of course, each such point x satisfies $(n+1)x\in|(n+1)x_0|$, and hence is a hyperosculating point. The formula shows that there are no other hyperosculating points.

In particular, we see that $k_i=0$ for $i<n$, hence the degree r_1 of the tangential surface is equal to $2(n+1)$. Also, if $n>2$, the dual of C is a curve of degree $r_2=3(n+1)$. It has $(n+1)^2$ singular points corresponding to $(n+1)^2$ hyperosculating planes.

Example 10.4.18 Assume C is a canonical curve in $\mathbb{P}^{g-1}$. Recall that a *Weierstrass point* of a smooth curve of genus $g>1$ is a point x such that

$$\sum_{i=1}^{g}(h^0(x)+\cdots+h^0(ix)-i)>0.$$

Let $a_i=h^0(x)+\cdots+h^0(ix)$. We have $a_1=1$ and $a_i=i$ if and only if $h^0(x)=\ldots=h^0(ix)=1$. By Riemann-Roch, this is equivalent to that $h^0(K_C-ix)=g-i$, i.e. the point x imposes the expected number of conditions for a hyperplane to have a contact with C of order i at x. A point x is a Weierstrass point if and only if there exists $i\le g$ such that the number of such conditions is less than expected by the amount equal to a_i-i. With notation (10.60), this shows that

$$s_1+\cdots+s_{i-1}=a_i-i,\ i=2,\ldots,g.$$

In particular, the point x is hyperosculating if and only if it is a Weierstrass point. Applying formula (10.63), we obtain the number of Weierstrass points

$$W=g(g^2-1). \tag{10.64}$$

Since $h^0(2x)=1$ for all points on C (because C is not hyperelliptic), we get $k_1=0$. Applying Proposition 10.4.13, we obtain that the rank r_1 of C is equal to $6(g-1)$.

Assume that C is general in the sense that all Weierstrass points x are simple, i.e. $W(x)=1$. It follows from the proof of Proposition 10.4.16 that $s_i(x)=0, i<g-1$, and $s_{g-1}(x)=1$. Thus $k_m=0, m<g-1$, and $k_{g-1}=W=g(g^2-1)$. It follows from Proposition 10.4.13 that $r_m=(m+2)(m+1)(g-1)$

for $1 \leq m < g-2$ and $r_{g-2} = g(g-1)^2$. The latter number coincides with the class of C. For example, if $g = 3$, we get $r_1 = 12$ and the 24 Weierstrass points are flex points of C. If $g = 4$, we get $r_1 = 18$ and $r_2 = 36$. We have 60 hyperosculating planes at Weierstrass points. The linear system of cubics through C defines a birational map from $\mathbb{P}^3$ to a cubic hypersurface in $\mathbb{P}^4$ with an ordinary double point. The image of the tangential surface is the enveloping cone at the node, the intersection of the cubic with its first polar with respect to the node. Its degree is equal to 6, so the tangential surface is the proper inverse image of the cone under the rational map.

We refer for the proof of the following Proposition to [437].

Proposition 10.4.19 *Let $f^\vee : C \to (\mathbb{P}^n)^\vee$ be the normalization of the $n-1$-th associated curve of $f : C \to \mathbb{P}^n$, the dual curve of $f(C)$. Then*

(i) $r_m(f^\vee(C)) = r_{n-m-1}(f(C))$;
(ii) $(f^\vee)^\vee = f$;
(iii) $k_i(f^\vee) = k_i(f)$.

Recall from Chapter 1 that the dual variety of C_0 is the closure in $(\mathbb{P}^n)^\vee$ of the set of tangent hyperplanes to smooth points of C_0. If $t_0 = f(x)$ is a smooth point, the set of tangent hyperplanes at x is a codimension 2 subspace of the dual space equal to $(n-2)$-th developable scroll of the dual curve. By the duality, we obtain that the dual of the $(n-2)$-th developable scroll of a curve C_0 is the dual curve of C_0. In particular, if $n = 3$, we obtain that the dual of the tangential surface to a nondegenerate curve C_0 in $\mathbb{P}^3$ is the dual curve of C_0, and the dual of a nondegenerate curve C_0 in $\mathbb{P}^3$ is the tangential surface of its dual curve.

Proposition 10.4.20 *Let S be a ruled surface in $\mathbb{P}^3$. The following properties are equivalent:*

(i) S is a developable surface;
(ii) S is a tangential surface corresponding to some curve C_0 lying on S;
(iii) the tangent lines of the curve $C_0 \subset G_1(\mathbb{P}^3)$ parameterizing the rulings are contained in $G_1(\mathbb{P}^3)$.

Proof (i) $\Rightarrow$ (ii). Consider the Gauss map $\gamma : S \to (\mathbb{P}^3)^\vee$ which assigns to a smooth point $x \in S$ the embedded tangent plane $\mathbb{T}_x(C)$. Obviously, γ blows down generators of S, hence the image of S is a curve $\check{C}_0$ in the dual space. This curve is the dual variety of S. Its dual variety is our surface S, and hence coincides with the tangential surface of the dual curve C_0 of $\check{C}_0$.

(ii) $\Rightarrow$ (iii) Let $q_C : Z_C \to C$ be the projection from the incidence variety

and $D \in |\mathcal{O}_{Z_C}(1)|$. The tangent plane at points of a ruling ℓ_x cuts out the ruling with multiplicity 2. Thus the linear system $|D - 2\ell_x)|$ is non-empty (as always, we identify a ruling with a fiber of q_C). The exact sequence

$$0 \to \mathcal{O}_{Z_C}(D - 2\ell_x) \to \mathcal{O}_{Z_C}(D - \ell_x) \to \mathcal{O}_{\ell_x}(D - \ell_x) \to 0$$

shows that $h^0(\mathcal{O}_{l_x}(D-\ell_x)) = 1$, i.e. $|D-\ell_x|$ has a base point $y(x)$ on ℓ_x. This means that all plane sections of S containing ℓ_x have residual curves passing through the same point $y(x)$ on ℓ_x. Obviously, this implies that the point $y(x)$ is a singular point of S and the differential of the projection $p_C : Z_C \to S$ at $y(x)$ is not surjective. Applying Proposition 10.1.14, we obtain that the tangent line $\mathbb{T}_x(C)$ is contained in the α-plane $\Omega(y(x)) \subset G_1(\mathbb{P}^3)$.

(iii) $\Rightarrow$ (i) Applying Proposition 10.1.14, we obtain that each ℓ_x has a point $y(x)$ such that its image in S is a singular point and the differential of p_C at $y(x)$ is not surjective. This implies that $y(x)$ is a base point of the linear system $|D - \ell_x|$ on ℓ_x. As above, this shows that $|D - 2\ell_x|$ is not empty and hence there exists a plane tangent to S at all points of the ruling ℓ_x. □

The set of points $y(x) \in \ell_x, x \in C$ is a curve C_0 on S such that each ruling ℓ_x is tangent to a smooth point on C_0. So S is the tangential surface of C_0. The curve C_0 is called the *cuspidal edge* of the tangent surface. It is a curve on S such that at its general point s the formal completion of $\mathcal{O}_{S,s}$ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 + z_2^3)$.

10.4.4 Quartic ruled surfaces in $\mathbb{P}^3$

Here we will discuss the classification of quartic ruled surfaces in $\mathbb{P}^3$ due to A. Cayley and L. Cremona. Note that we have already classified ruled surfaces of degree 3 in Chapter 9. They are non-normal cubic surfaces and there are two kinds of them. The double curve Γ is a line and the map $D(f) \to \Gamma$ is an irreducible (reducible) degree 2 cover. The surface Z_C is isomorphic to $\mathbf{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. The linear system $|h|$ that gives the map $f : \mathbf{F}_1 \to \mathbb{P}^3$ is equal to $|\mathfrak{e}_0 + 2\mathfrak{f}|$, where $\mathfrak{e}_0$ is the divisor class of the exceptional section E_0 and $\mathfrak{f}$ is the class of a fibre. The curve $D(f) \in |h - \mathfrak{f}| = |\mathfrak{e}_0 + \mathfrak{f}|$. In the first case the surface S has ordinary singularities and $D(f)$ is an irreducible curve. In the second case $D(f) \in |h|$ and consists of the exceptional section and a fibre. Now let us deal with quartic surfaces. We do not assume that the surface has only ordinary singularities. We start with the following.

Proposition 10.4.21 *The genus of a ruled quartic surface is equal to* 0 *or* 1.

Proof A general plane section H of S is a plane quartic. Its geometric genus

g is the genus of S. If $g = 3$, the curve H is nonsingular, hence S is normal and therefore nonsingular. Since $K_S = 0$, it is not ruled. If $g = 2$, the singular curve of S is a line. The plane sections through the line form a linear pencil of cubic curves on S. Its preimage on the normalization X of S is a pencil of elliptic curves. Since X is a $\mathbb{P}^1$-bundle over a curve of genus 2, a general member of the pencil cannot map surjectively to the base. This contradiction proves the assertion. □

So, we have two classes of quartic ruled surfaces: rational ruled surfaces ($g = 0$) and *elliptic ruled surfaces* ($g = 1$). Each surface S is defined by some curve C_0 of degree 4 in $G_1(\mathbb{P}^3)$. We denote by X the minimal ruled surface $\mathbb{P}(\mathcal{E})$ obtained from the universal family Z_{C_0} by the base change $\nu : C \to C_0$, where ν is the normalization map. We will denote by E_0 an exceptional section of X defined by choosing a normalized vector bundle $\mathcal{E}_0$ with $\mathbb{P}(\mathcal{E}_0)$ isomorphic to X.

We begin with classification of rational quartic ruled surfaces.

Proposition 10.4.22 *A rational quartic ruled surface is a projection of a rational normal scroll $S_{2,5}$ or $S_{1,5}$ of degree 4 in $\mathbb{P}^5$.*

Proof Let $|h|$ be the linear system of hyperplane sections on the quartic rational normal scroll $S_{a,n} \cong \mathbb{F}_e$. We have $|h| = |\mathfrak{e}_0 + k\mathfrak{f}|$, where $k > e$. Since $h^2 = 4$, we get $2k + e = 4$. This gives two solutions $(e, k) = (0, 2), (2, 1)$. In the first case we get the scroll $S_{2,5} \cong \mathbb{F}_0$, in the second case we get the scroll $S_{1,5} \cong \mathbf{F}_2$. □

Let $\mathbb{D}(f)$ be the double-point divisor class. We know that the singular curve Γ on S is the image of a curve $D(f)$ from $\mathbb{D}(f)$ on X, where $X = S_{2,5}$ or $S_{1,5}$. Applying (10.52), this gives

$$D(f) \sim 2h - 2\mathfrak{f} = \begin{cases} 2\mathfrak{e}_0 + 2\mathfrak{f} & \text{if } X = S_{2,5}, \\ 2\mathfrak{e}_0 + 4\mathfrak{f} & \text{if } X \cong S_{1,5}. \end{cases}$$

Since a general plane section of S is a rational curve, $D(f)$ and Γ consists of at most three irreducible components. The linear system

$$|h| = \begin{cases} |\mathfrak{e}_0 + 2\mathfrak{f}| & \text{if } X = S_{2,5}, \\ |\mathfrak{e}_0 + 3\mathfrak{f}| & \text{if } X = S_{1,5}, \end{cases}$$

maps a component D_i of $D(f)$ to an irreducible component Γ_i of Γ of degree $d_i = \frac{1}{m_i} H \cdot D_i$, where m_i is the degree of the map $D_i \to \Gamma_i$. The number m_i is equal to the multiplicity of a general point on Γ_i as a singular point of the surface unless Γ_i is a curve of cusps. In the latter case $m_i = 1$, but D_i enters

D with multiplicity 2. A fiber $F_x = \pi^{-1}(x)$ could be also a part of D. In this case Γ has a singular point at $\nu(x)$. If it is an ordinary double point, the fiber component enters with multiplicity 1, if it is a cusp, it enters with multiplicity 2. Other cases will not occur. Finally, we use that $\dim |h - D_i| > 0$ if Γ_i is contained in a plane, i.e. a line or a conic.

This gives us the following cases making a "rough classification" according to the possible singular locus of the surface.

1. $X = S_{2,5}$:
   - (i) $D(f) = D_1, d_1 = 3$;
   - (ii) $D(f) = D_1 + D_2, D_1 \in |\mathfrak{e}_0|, D_2 \in |\mathfrak{e}_0 + 2\mathfrak{f}|, d_1 = 1, d_2 = 2$;
   - (iii) $D(f) = D_1 + D_2 + F_1 + F_2, D_1, D_2 \in |\mathfrak{e}_0|, d_1 = d_2 = 1$;
   - (iv) $D(f) = 2D_1, D_1 \in |\mathfrak{e}_0 + \mathfrak{f}|, d_1 = 1$;
   - (iv)' $D(f) = 2D_1, D_1 \in |\mathfrak{e}_0 + \mathfrak{f}|, d_1 = 3$;
   - (v) $D(f) = 2D_1 + 2F_1, D_1 \in |\mathfrak{e}_0|, d_1 = 1$;
   - (vi) $D(f) = 2D_1 + F_1 + F_2, D_1 \in |\mathfrak{e}_0|, d_1 = 2$;
   - (vi)' $D(f) = 2D_1 + 2F_1, D_1 \in |\mathfrak{e}_0|, d_1 = 2$.
2. $X = S_{1,5}$:
   - (i) $D(f) = D_1, d_1 = 3$;
   - (ii) $D(f) = E_0 + D_1 + F, D_1 \in |\mathfrak{e}_0 + 3\mathfrak{f}|, d_1 = 1, d_2 = 2$;
   - (iii) $D(f) = 2E_0 + 2F_1 + 2F_2, d_1 = 1$;
   - (iv) $D(f) = 2D_1, D_1 \in |\mathfrak{e}_0 + 2\mathfrak{f}|, d_1 = 1$.

Theorem 10.4.23 *There are 12 different types of rational quartic ruled surfaces corresponding to 12 possible cases from above.*

Proof It suffices to realize all possible cases from above. By Proposition 10.4.22, the different types must correspond to different choices of the center of the projection in $\mathbb{P}^5$.

Let us introduce some special loci in $\mathbb{P}^5$ which will play a role for choosing the center of the projection.

We will identify curves on $\mathbf{F}_0$ with their images in $S_{2,5}$. A conic directrix is a curve $E \in |\mathfrak{e}_0|$. Consider the union of planes spanning the E's. It is a scroll Σ_1 of dimension 3 parameterized by $|\mathfrak{e}_0| \cong \mathbb{P}^1$. Let us compute its degree. Fix two generators F_1 and F_2 of $\mathbf{F}_0$. Then $|h - F_1 - F_2| = |\mathfrak{e}_0|$. If we fix three pairs of generators $F_1^{(i)}, F_2^{(i)}, i = 1, 2, 3$, each spanning a $\mathbb{P}^3$, then we can establish a correspondence Γ of tridegree $(1, 1, 1)$ on $|\mathfrak{e}_0| \times |\mathfrak{e}_0| \times |\mathfrak{e}_0|$ such that the point $(x, y, z) \in \Gamma$ corresponds to three hyperplanes from each linear system $|h - F_1^{(i)} - F_2^{(i)}|$ which cut out the same curve $E \in |\mathfrak{e}_0|$. The three hyperplanes intersect along the plane spanning E. This shows that our scroll is

the join of three disjoint lines in the dual $\mathbb{P}^5$ which can be identified with the same $\mathbb{P}^1$. Applying formula (10.49), we obtain that the degree of Σ_1 is equal to 3.

The next scroll we consider is the union Σ_2 of 3-dimensional spaces spanned by tangent planes of $S_{2,5}$ along points on a fixed generator. This 3-dimensional space is spanned by the tangent lines of two fixed conic directrices at the points where they intersect the generator. Thus our scroll is the join of the tangential scroll of the two directrices with respect to the correspondence between the directrices defined by the generators. The degree of this scroll is given by the formula in Example 10.4.3. Since the tangent lines of a conic are parameterized by the conic, and the two conics are disjoint, the degree of Σ_2 is equal to 4. Obviously, Σ_1 is a 2-directrix of Σ_2. Since the tangent plane to $S_{2,5}$ at a point x is spanned by the generator passing through this point and the tangent line of the conic directrix passing through this point, we obtain that Σ_2 coincides with the tangential scroll of $S_{2,5}$.

One more scroll is constructed as follows. Consider directrices of $S_{2,5}$ defined by the images of curves $\Gamma_3 \in |\mathfrak{e}_0 + \mathfrak{f}|$. We identify them with the images. These are directrices of degree 3. Let Σ_3 be the union of tangent planes to $S_{2,5}$ at the points of Γ_3. These tangent planes can be obtained as joins of tangent lines of Γ_3 at points $x \in \Gamma_3$ and the points x' on a conic directrix E such that the points x, x' lie on the same generator. Thus Σ_3 is obtained by construction from Example 10.4.3 as the join of the tangential scroll of Γ_3 and the conic. The degree of the tangential scroll has been computed there; it is equal to 4. Thus the degree of Σ_3 is equal to $4+2-1 = 5$, where we subtracted 1 because the conic and Γ_3 meet at one point dropping the dimension of the join by 1.

Let $p_\ell : S_{2,5} \to S$ be the projection map from a line ℓ. We will use the fact that any two points x_1, x_2 in the double locus $D(f)$ which are projected to the same point must lie on a secant of $D(f)$ that passes through these points and intersects ℓ. The secant becomes a tangent line if $x_1 = x_2$ is a critical point of p_ℓ.

- Type 1 (i).

Take a line ℓ in $\mathbb{P}^5$ which intersects the quartic scroll Σ_2 at four distinct points and is not contained in any 3-dimensional space spanned by a cubic directrix $\Gamma_3 \in |\mathfrak{e}_0 + \mathfrak{f}|$. Let D be an irreducible component of $D(f)$ and let x be a general point of D. We know from the classification of all possible components of $D(f)$ that the degree of the projection map must be 2 or 3. If the degree is equal to 3, then $D \in |\mathfrak{e}_0 + \mathfrak{f}|$ is a cubic directrix and its projection is a line. This implies that ℓ belongs to the linear span of D. By assumption on ℓ this does not happen. So the degree is equal to 2. The map which assigns to a point

$x \in D$ the intersection point of ℓ and the secant passing through x is a degree 2 map $D \to \ell$. The intersection points of ℓ with Σ_2 are the branch points of this map. By Hurwitz's formula, the normalization of D is a genus 1 curve, hence the arithmetic genus is ≥ 1. The classification of possible D's shows that this could happen only if D is a nonsingular curve from $|2\mathfrak{e}_0 + 2\mathfrak{f}|$. So this realizes type 1(i).

The quartic scroll S can be described as follows. Consider a normal rational cubic curve R_3 in $\mathbb{P}^3$ and let S be the set of its secants contained in a non-special linear line complex. The set of secants of R_3 is a surface in $G = G_1(\mathbb{P}^3)$ of degree 4 in its Plücker embedding. This can be seen by computing its cohomology class in G. A general α-plane $\Omega(p)$ contains only one secant. A general β-plane $\Omega(\Pi)$ contains three secants. This shows that the degree of the surface of secants is equal to 4. The surface must be a Veronese surface in $\mathbb{P}^5$ because it does not contain lines. The intersection of the surface with a general linear line complex is a curve C of degree 4. It defines a quartic ruled surface S_C. Take a point $p \in R_3$ and consider the set of secants $\ell_x, x \in C$ such that $p \in \ell_x$. The intersection of the Schubert plane $\Omega(p, \mathbb{P}^3)$ with the Veronese surface is a conic. Its intersection with the linear line complex must consist of two lines. Thus each point of R_3 lies on two generators of the surface S_C. The curve R_3 is the double curve of S.

- Type 1 (ii).

In this case we take ℓ intersecting Σ_1 at some point x_0 in the plane spanned by some conic directrix $D \in |\mathfrak{e}_0|$. The projection of D is a line and the map is 2:1. Note that in this case the point x_0 is contained in two tangents to D so two of the four intersection points of ℓ and Σ_2 coincide. It also shows that Σ_1 is contained in the singular locus of Σ_2. The remaining two points in $\ell \cap \Sigma_2$ are the branch points of the double cover $E' \to \ell$, where $D' \in |\mathfrak{e}_0 + 2\mathfrak{f}|$ is the residual component of $D(f)$. Arguing as in the above we see that D' is a smooth rational curve of degree 4. Its projection is a conic.

- Type 1 (iii).

This time we take ℓ intersecting Σ_1 at two points p_1, p_2. These points lie in planes Π_1 and Π_2 spanned by directrix conics E_1 and E_2. The projection from ℓ maps these conics to disjoint double lines of S. Let us now find two generators F_1 and F_2 which are projected to the third double line. Consider the pencil $\mathcal{P}_i$ of lines in the plane Π_i with base point p_i. By intersecting the lines of the pencil with the conic E_i, we define an involution on E_i and hence an involution γ_i on the pencil $|\mathfrak{f}| \cong \mathbb{P}^1$ (interchanging the generators intersecting E_i at two points in the involution). Now we have two involutions on $|\mathfrak{f}|$ whose

graphs are curves of type $(1,1)$. They have two common pairs in the involution which give us two generators on $S_{2,5}$ intersecting E_i at two points on a line ℓ_i through p_i. The 3-dimensional subspace spanned by ℓ, ℓ_1 and ℓ_2 contains the two generators. They are projected to a double line of S.

- Type 1 (iv).

The image of D_1 on $S_{2,5}$ is a rational normal cubic R_3 spanning a 3-plane M of $\mathbb{P}^5$. We project from a general line contained in M. The restriction of the projection to D_1 is a degree 3 map. So the projection of D_1 is a triple line of S.

Another possibility here is to project from a line directrix ℓ of the tangential scroll Σ_3. Each point on ℓ lies in a tangent plane to a cubic directrix $\Gamma_3 \in |E_0 + F|$. So the projection from ℓ maps Γ_3 to a rational curve R_3 of degree 3 and maps the tangent lines to Γ_3 to tangent lines to R_3. Thus the scroll S is a developable quartic surface considered in Example 10.4.15. Let us find a line directrix on Σ_3. We know that Σ_3 is equal to the image of a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$ of rank 3 and degree 5. Thus $\deg \mathcal{E}^\vee(1) = -5 + 3 = -2$, and applying Riemann-Roch, we obtain $h^0(\mathcal{E}^\vee(1)) \geq \deg \mathcal{E}^\vee(1) + 3 > 0$. This implies that there is a nontrivial map of sheaves $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\mathcal{L}$ be the image of this map. It defines a section $\sigma : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ such that $\sigma^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \mathcal{L}$. Thus the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ to $\mathsf{D} = \sigma(\mathbb{P}^1)$ is of degree ≤ 1. Since Σ_3 is a scroll in our definition, the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample, hence the degree must be equal to 1. So, the image of D in Σ_3 is a line directrix.

- Type 1 (v).

This is a degeneration of the previous case. The rational normal cubic degenerates into the union of a directrix conic and a generator. The projection is a degree 2 map on the conic and degree 1 on the line. The double curve Γ is a triple line. It is a generator and a directrix at the same time. Through each point on Γ passes two generators other than itself. As in the previous case a plane containing Γ contains only one of other generators.

- Type 1 (vi).

Consider a hyperplane section $L \cap \Sigma_1$, where L contains two generators F_1 and F_2 of $S_{2,5}$. The quartic curve $L \cap S_{2,5}$ consists of the two generators and a directrix conic D from $|\mathfrak{e}_0|$. Thus the cubic surface $L \cap \Sigma_1$ contains a plane, and the residual surface is a quadric Q containing D. Take a line ℓ in the 3-dimensional subspace M spanned by F_1 and F_2 that is tangent to the quadric $M \cap Q$. The projection from ℓ maps $S_{2,5}$ to a quartic ruled surface with double

line equal to the image of the two generators F_1 and F_2 and the cuspidal conic equal to the image of the directrix conic D.

- Type 1 (vi)'.

The same as the previous case, but we choose L to be tangent along a generator F_1. The double locus is a reducible cuspidal cubic.

- Type 2 (i).

Type 2 corresponds to a projection of the rational normal quartic scroll $S_{1,5} \cong \mathbf{F}_2$ in $\mathbb{P}^5$. The exceptional section E_0 is a line directrix on $S_{1,5}$. The curves from the linear system $|\mathfrak{e}_0 + 2\mathfrak{f}|$ are cubic directrices. The analog of the tangential scroll Σ_2 here is the join Σ_2' of the tangential surface of a cubic directrix D with the line E_0. It is the union of 3-dimensional spaces spanned by a tangent line to D and E_0. We know that the the tangential scroll of rational normal cubic is of degree 4. Thus the degree of Σ_2' is equal to 4. The rest of the argument is the same as in case 1 (i). We take ℓ intersecting Σ_2' at four distinct points and not contained in a 3-space spanned by a cubic directrix. The double curve is a smooth elliptic curve of degree 6 from $|2\mathfrak{e}_0 + 4\mathfrak{f}|$.

- Type 2 (ii).

This time we take ℓ intersecting the plane Π spanned by E_0 and a generator F. We also do not take it in any 3-plane spanned by a cubic directrix. Then E_0 and F will project to the same line on S, the double line. The residual part of the double locus must be a curve E from $|\mathfrak{e}_0 + 3\mathfrak{f}|$. Since no cubic directrix is a part of the double locus, we see that E is an irreducible quartic curve. Its image is a double conic on S.

- Type 2 (iii).

We choose a line ℓ intersecting two planes as in the previous case. Since the two planes have a common line E_0, they span a 3-dimensional subspace. It contains three lines which are projected to the same line on S, a triple line of S.

- Type 2 (iv).

Take a cubic curve from $|\mathfrak{e}_0 + 2\mathfrak{f}|$ and a line in the 3-dimensional space spanned by the cubic. The cubic is projected to a triple line.

□

Remark 10.4.24 We have seen that a developable quartic surface occurs in case 1 (iv). Let us see that this is the only case when it may occur.

The vector bundle of principal parts $\mathcal{P}^1_C(\mathcal{L})$ must be given by an extension

$$0 \to \Omega^1_C \otimes \mathcal{L} \to \mathcal{P}^1_C(\mathcal{L}) \to \mathcal{L} \to 0, \tag{10.65}$$

where C is a rational cubic in $\mathbb{P}^3$ and $\mathcal{L} = \mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$. It is known that the extension

$$0 \to \Omega^1_C \to \mathcal{P}^1_C \to \mathcal{O}_C \to 0,$$

from which the previous extension is obtained by twisting with $\mathcal{L}$, does not split. Its extension class is defined by a nonzero element in $\mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C) \cong H^1(C, \Omega^1_C) \cong \mathbb{C}$ (this is the first Chern class of the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$). After tensoring (10.65) with $\mathcal{O}_{\mathbb{P}^1}(-2)$ we get an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{P}^1_{\mathbb{P}^1}(\mathcal{L})(-2) \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

The locally free sheaf $\mathcal{E} = \mathcal{P}^1_C(\mathcal{L})(-2)$ has 2-dimensional space of global sections. Tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1)$ and using that the coboundary homomorphism

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$$

is nontrivial, we obtain that $\mathcal{E}(-1)$ has no nonzero sections, hence $\mathcal{E}$ is a normalized vector bundle of degree 0 defining the ruled surface $\mathbb{P}(\mathcal{E})$. There is only one such bundle over $\mathbb{P}^1$, the trivial bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Untwisting $\mathcal{E}$, we obtain that the sheaf $\mathcal{P}^1_R(\mathcal{L})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, so $\mathbb{P}(\mathcal{P}^1_R(\mathcal{L})) \cong \mathbf{F}_0$ and the complete linear system defined by the tautological invertible sheaf corresponding to $\mathcal{P}^1_R(\mathcal{L})$ embeds $\mathbf{F}_0$ in $\mathbb{P}^5$ as the rational normal scroll $S_{2,5}$. The double locus class $\mathbb{D}(f)$ must be divisible by 2, and the only case when it happens is type 1 (iv)'.

We can also distinguish the previous cases by a possible embedding of the quartic curve C_0 parameterizing generators of S in $\mathbb{G} = G_1(\mathbb{P}^3)$. Since $\deg C_0 = 4$ in the Plücker embedding, the curve is always contained in a hyperplane L on $\mathbb{P}^5$. If, furthermore, C_0 lies in a codimension 2 subspace, then this subspace is either contained in one tangent hyperplane of $\mathbb{G}$ or is equal to the intersection of two tangent hyperplanes (because the dual variety of $\mathbb{G}$ is a quadric). So we have the following possibilities:

I C_0 is a rational normal quartic contained in a hyperplane L that is not tangent to $\mathbb{G}$;

II C_0 is a rational normal quartic contained in a hyperplane L which is tangent to $\mathbb{G}$ at a point O not contained in C_0;

III C_0 is a rational normal quartic contained in a hyperplane L which is tangent to $\mathbb{G}$ at a point O contained in C_0;

IV C_0 is a rational quartic curve contained in the intersection of two different tangent hyperplanes of $\mathbb{G}$;

V C_0 is a rational quartic curve contained in a 3-dimensional subspace through which passes only one tangent hyperplane of $\mathbb{G}$. The tangency point is an ordinary node of C_0.

A quartic surface of type 1 (i) or 1 (iv)' from Theorem 10.4.23 belongs to type I. Following W. Edge [200] we redenote types 1 (i) and 1 (iv)' with I.

In type 1 (ii) the line component of the double curve is a directrix, so all generators belong to a linear complex tangent to $G_1(\mathbb{P}^3)$ at the point O representing this directrix. This is Edge's type II. Through any point p on the directrix passes two generators, the point O belongs to a secant of C_0 formed by the line $\Omega(p, \Pi)$, where Π is the plane spanned by the two generators. It is a nonsingular point of C_0. We have Edge's type II (C).

In type 1 (iii) we have two directrices which are not generators. This means that C is contained in the intersection of two special linear line complexes tangent to $G_1(\mathbb{P}^3)$ at two points. This is type IV (B). The tangency points correspond to the line directrices on S. The curve C_0 is contained in the intersection of two special linear line complexes that is a nonsingular quadric. The curve C_0 has an ordinary node at the point corresponding to two generators mapped to a double line on F.

In type 1 (iv), the triple line is a directrix of S, so we are again in case II but in this case the point O intersects the α-plane $\Omega(p)$ at three non-collinear points and intersects the β-plane $\Omega(\Pi)$ at one point. This is Edge's type II (A).

In case 1 (v) the double curve is a triple line. One of the generators F is contained in $D(f)$ with multiplicity 2 and is mapped to the triple line. Thus S is contained in a unique special line complex which is tangent to G at a cusp of C_0. Since C_0 is singular, it is contained in a 3-dimensional space. So C is contained in a quadric cone equal to the intersection of $G_1(\mathbb{P}^3)$ with two linear line complexes. The singular point of this cone is the singular point of C_0. This is Edge's Type III (A).

In type 1 (vi) two generators on $S_{2,5}$ are projected to a double generator of S. The curve C_0 has an ordinary double point, hence it lies in two linear line complexes. The double generator is the only line directrix on S. Thus there is only one special linear line complex containing S and its tangency point is an ordinary double point of C_0. This is Edge's type V (A). In case 1 (vi)', we also have type V (A), only this time the singular point of C_0 is a cusp.

In type 2 (i) the line directrix ℓ corresponding to E_0 defines a line complex

containing C. Thus we are in type II. The Schubert plane $\Omega(p, \mathbb{P}^3), p \in \ell$, contains only one point, the α-plane $\Omega(\pi), \ell \subset \pi$, contains three points. This is Edge's type II (B).

In type 2 (ii) we have a line directrix which is at the same time a generator g. This shows that we are in type III. The curve C has a cuspidal singularity at the point O corresponding to the generator g. The curve C intersects any plane $\Omega(p, \mathbb{P}^3), p \in g$, in one point and every plane $\Omega(\pi, \mathbb{P}^3), g \subset \pi$, at two points. This is Edge's type III (B).

In type 2 (iii) we have a triple line on S formed by the projection of the line directrix E_0 of $S_{2,5}$ and its two generators. We are in case V, where the singular point of C is the singular point of the quadric cone. This is Edge's type V (B).

In type 2 (iv) we have a triple line projected from a rational cubic curve. We have two line directrices of S, one is a triple line. The curve C is nonsingular. This is Edge's type IV (A).

Next, we have to classify elliptic ruled quartic surfaces in $\mathbb{P}^3$. Let $\pi : X \to C$ be a minimal ruled surface with a base C. We write X in the form $X = \mathbb{P}(\mathcal{E}_0)$, where $\mathcal{E}_0$ is a normalized rank 2 locally free sheaf. Since $K_C = 0$ in our case, the canonical class formula (1.33) gives

$$K_X = -2\mathfrak{e}_0 + \pi^*(\mathfrak{a}). \tag{10.66}$$

By the adjunction formula, $0 = E_0^2 + K_X \cdot E_0 = -E_0^2 + \deg \mathfrak{a}$. Thus, $a = \deg \mathfrak{a} = \mathfrak{e}_0^2 \leq 0$.

Let $|h|$ be the linear system on X which defines the normalization map $f : X \to S$. We can write $h \equiv \mathfrak{e}_0 + m\mathfrak{f}$, where $\mathfrak{f}$ is the class of a fibre. Since h is ample, intersecting both sides with $\mathfrak{e}_0$, we get $m + a > 0$. We also have $h^2 = 2m + a = 4$. This gives two possibilities $a = 0, m = 2$ and $a = -2$ and $m = 3$. In the second case $h \cdot \mathfrak{e}_0 = 1$, hence $|h|$ has a fixed point on E_0. This case is not realized (it leads to the case when S is a cubic cone). The formula for the double-point locus gives $\mathbb{D}(f) \equiv 2h - \pi^*(\mathfrak{d})$, where $d = \deg \mathfrak{d} = 4$. Thus we obtain

$$H \equiv \mathfrak{e}_0 + 2\mathfrak{f},\ \mathfrak{e}_0^2 = 0,\ D(f) \equiv 2\mathfrak{e}_0.$$

By Riemann-Roch, $\dim |h| = 3$. Since $\dim |h - \mathfrak{e}_0| = \dim |2\mathfrak{f}| = 1$, we obtain that the image of E_0 is a line. Since the restriction of $|h|$ to E_0 is a linear series of degree 2, the image of E_0 is a double line. We have two possibilities: $D(f)$ consists of two curves $E_0 + E_0'$, or $D(f)$ is an irreducible curve D with $h \cdot D = 4$. Since $|h - D| = \emptyset$, we obtain that the image of D is a space quartic,

so it cannot be the double locus. This leaves us with two possible cases: $D(f)$ is the union of two disjoint curves $E_0 + E_0'$, or $D(f) = 2E_0$.

In the first case $H \cdot E_0 = H \cdot E_0' = 2$ and $\dim |h - E_0| = \dim |h - E_0'| = \dim |2\mathfrak{f}| = 1$. This shows that the images of E_0 and E_0' are two skew double lines on S. The curve C is a nonsingular elliptic curve in $G_1(\mathbb{P}^3)$. It spans a 3-dimensional subspace equal to the intersection of two special linear line complexes.

Since $X = \mathbb{P}(\mathcal{E})$ has two disjoint sections with self-intersection 0, the sheaf $\mathcal{E}$ splits into the direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2$ of invertible sheaves of degree 0. This easily follows from [282], Chapter V, Proposition 2.9. One of them must have a nonzero section, i.e. must be isomorphic to $\mathcal{O}_C$. So we obtain

$$X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(\mathfrak{a})),$$

where $\deg \mathfrak{a} = 0$. Note that X cannot be the direct product $C \times \mathbb{P}^1$ because in this case the image of any $C \times \{x\}$ must be a double line, in other words, in this case $|H|$ defines a degree 2 map. So, we have $\mathfrak{a} \sim 0$.

In the second case, two double lines come together forming the curve of tacnodes. In this case the curve C lies only in one special linear line complex. The pencil of hyperplanes containing C intersects the dual Klein quadric at one point.

Let $\sigma : \mathcal{E} \to \mathcal{O}_C(\mathfrak{e})$ be the surjective map of sheaves corresponding to the section E_0. Since $\deg \mathcal{E} = \deg \mathfrak{a} = 0$, we have $\deg \operatorname{Ker}(\sigma) = 0$. Thus $\mathcal{E}$ can be given as an extension of invertible sheaves

$$0 \to \mathcal{O}_C(\mathfrak{b}) \to \mathcal{E} \to \mathcal{O}_C(\mathfrak{a}) \to 0,$$

where $\deg \mathfrak{b} = 0$. Suppose this extension splits, then X has two disjoint sections with self-intersection zero. By the above, we see that the map defined by the linear system $|h|$ maps each section to a double line of S. This leads to the first case. So in our case, there are no disjoint sections, and hence the extension does not split. This implies that $\operatorname{Ext}^1(\mathcal{O}_C(\mathfrak{a}), \mathcal{O}_C(\mathfrak{b})) = H^1(C, \mathcal{O}_C(\mathfrak{e} - \mathfrak{b})) \neq \{0\}$. This is possible only if $\mathfrak{b} \sim \mathfrak{a}$. Since $\mathcal{E}$ has a nonzero section, we also have $H^0(C, \mathcal{O}_C(\mathfrak{a})) \neq \{0\}$, i.e. $\mathfrak{e} \sim 0$. Thus we obtain that $\mathcal{E}$ is given by a non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0.$$

In fact, it is known that any elliptic ruled surface with $\mathfrak{e}_0^2 = 0$ which corresponds to a non-split vector bundle, must be isomorphic to the ruled surface $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is defined by the above extension (see [282], Chapter V, Theorem 2.15).

Let us summarize our classification in the following Table 10.1.

Type	Double cuve	Edge	Cremona	Cayley	Sturm
I (i),(iv)'	R_3	I	1	10	III
I (ii)	L+K	II (C)	2	7	V
I (iii)	L+L'+G	IV (B)	5	2	VII
I (v)	3L	II (A)	8	9	IX
I (iv)	3L	III (A)	3	-	XI
I (vi),(vi)'	2L+G	V (A)	6	5	VIII
II (i)	R_3	II (B)	7	8	IV
II (ii)	L+K	III (B)	4	-	VI
II (iii)	3L	V (B)	10	6	XII
II (iv)	3L	IV (A)	9	3	X
$g = 1$	L+L'	VI(A)	11	1	I
$g = 1$	2L	VI(B)	12	4	II

Table 10.1 *Quartic ruled surfaces*

Here R_3 denotes a curve of degree 3, L denotes a line, K is a conic and G is a generator.

A finer classification of quartic ruled surfaces requires to describe the projective equivalence classes. We refer to [451] for a modern work on this. Here we explain, following [50], only the fine classification assuming that the double curve is a Veronese cubic R_3. First, by projective transformation we can fix R_3 which will leave us only with the 3-dimensional subgroup G of $\mathrm{PGL}(4)$ leaving R_3 invariant. It is isomorphic to $\mathrm{PSL}(2)$.

Let $\mathcal{N}$ be the net of quadrics in $\mathbb{P}^3$ that contains R_3. It defines a rational map $\alpha : \mathbb{P}^3 \dashrightarrow \mathcal{N}^\vee$. The preimage of a point s in $\mathcal{N}^\vee$, i.e. a pencil in $\mathcal{N}$, is the base locus of the pencil. It consists of the curve R_3 plus a line intersecting R_3 at two points. This makes an identification between points in $\mathcal{N}^\vee \cong \mathbb{P}^2$ and secants of R_3. The preimage of a conic K in $\mathcal{N}^\vee$ is a quartic surface which is the union of secants of R_3. It is a quartic ruled surface. Conversely, every quartic ruled surface S containing R_3 as its double curve is obtained in this way. In fact, we know that S is the union of secants of R_3 and hence the linear system of quadrics containing R_3 should blow down each secant to a point in $\mathcal{N}^\vee$. The preimage of a general line in the plane is a quadric that cuts out on S a curve of degree 8 that consists of the curve R_3 taken with multiplicity 2 and two lines. This shows that the image of S is a conic. Thus we find a bijection between quartic surfaces with double curve R_3 and conics in the plane. The group G is naturally isomorphic to the group of projective transformations of $\mathcal{N}^\vee$. It is well known that the projective representation of $\mathrm{PSL}(2)$ in $\mathbb{P}^2$ leaves a nonsingular conic $\mathfrak{C}$ invariant. The quartic surface corresponding to $\mathfrak{C}$ is the only quartic surface invariant under G. This is, of course, the developable

quartic ruled surface (see Example 10.4.15). In this way our classification is reduced to the classification of orbits in the space of nonsingular conics $\mathbb{P}^5$ under the action of the group $\mathrm{PSL}(2)$ of projective automorphisms leaving $\mathfrak{C}$ invariant. The orbit space is of dimension 2. Let K be a conic different from $\mathfrak{C}$. There are five possible cases for the intersection $K \cap \mathfrak{C}$: four distinct points; one double coincidence, two double coincidences, one triple coincidence and one quartuple coincidence. Together with $\mathfrak{C}$ it gives six different types. The first type has two parameters, the cross ratio of four points and a point in the pencil of conics with the same cross ratio. The second type is a one-parameter family. All other types have finitely many orbits. We refer for explicit equations to [451] and [50].

There are many direct geometric constructions of quartic ruled surfaces. The first historical one uses Cayley's construction of a ruled surface as the union of lines intersecting three space curves (see Example 10.4.11). For example, taking $(d_1, d_2, d_3) = (2, 2, 1)$ and $(a_{12}, a_{13}, a_{23}) = (2, 0, 1)$ gives a quartic ruled surface with a double conic and a double line that intersect at one point. Another construction is due to L. Cremona. It is a special case of the construction from Example 10.4.3, where we take the curves C_1 and C_2 of degree 2. If the two conics are disjoint, a correspondence of bidegree $(1, 1)$ gives a quartic ruled surface. In the next Subsection we will discuss a more general construction due to B. Wong [604].

Finally, we reproduce equations of quartic ruled surfaces (see [200], p. 69).

$$
\begin{aligned}
I &: Q(xz-y^2, xw-yz, yw-z^2)=0,\\
&\quad \text{where } Q=\sum_{1\le i\le j\le 3} a_{ij}t_it_j \text{ is a nondegenerate quadratic form;}\\
II(A) &: zy^2(ay+bx)+wx^2(cy+dx)-ex^2y^2=0;\\
II(B) &: \text{same as in (I) with } a_{22}^2+a_{22}a_{13}-4a_{12}a_{23}+a_{11}a_{33}=0;\\
II(C) &: (cyz+bxz+axy+zw-wx)^2-xz(ax-by+cz)^2=0;\\
III(A) &: ax^2y^2-(x+y)(x^2w+y^2z)=0;\\
III(B) &: (xw+yz+azw)^2-zw(x+y)^2=0;\\
IV(A) &: x(az+bw)w^2-y(cz+dw)z^2=0;\\
IV(B) &: y^2z^2+axyzw+w^2(bz+cx)x=0,\\
&\quad (yz-xy+awx)^2-xz(x-z+bw)^2=0;\\
V(A) &: : (yz-xy+axw)^2-xz(x-z)^2=0;\\
V(B) &: (az^2+bzw+cw^2)(yz-xw)-z^2w^2=0;\\
VI(A) &: ax^2w^2+xy(bz^2+czw+dw^2)+ey^2z^2=0;\\
VI(B) &: (xw-yz)^2+(ax^2+bxy+cy^2)(xw-yz)+\\
&\quad (dx^3+ex^2y+fxy^2+gy^3)x=0.
\end{aligned}
$$

10.4.5 Ruled surfaces in $\mathbb{P}^3$ and the tetrahedral line complex

Fix a pencil $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$ with a nonsingular base curve. The pencil contains exactly four singular quadrics of corank 1. We can fix coordinate systems to transform the equations of the quadrics to the diagonal forms

$$\sum_{i=0}^{3} a_it_i=0, \quad \sum_{i=0}^{3} b_it_i=0.$$

The singular points of four singular quadrics in the pencil are the reference points $p_1=[1,0,0,0]$, $p_2=[0,1,0,0]$, $p_3=[0,0,1,0]$, $p_4=[0,0,0,1]$. For any point not equal to one of these points, the intersection of polar planes $P_x(Q), Q\in\mathcal{Q}$, is a line in $\mathbb{P}^3$. This defines a rational map $f:\mathbb{P}^3 \dashrightarrow G_1(\mathbb{P}^3)\subset\mathbb{P}^5$ whose image is a tetrahedral line complex K (see the end of Section 10.3.6). The Plücker coordinates p_{ij} of the line $f([t_0,t_1,t_2,t_3])$ are

$$p_{ij}=(a_ib_j-a_jb_i)t_it_j.$$

For any space curve C of degree m not passing through the reference points, its image under the map f is a curve of degree $d = 2m$ in the tetrahedral complex. It defines a ruled surface S_C in $\mathbb{P}^3$ of degree $2d$, the union of lines $f(x), x \in C$. If we consider the graph $G_f \subset \mathbb{P}^3 \times G_1(\mathbb{P}^3)$ of f, its projection to $G_1(\mathbb{P}^3)$ is the universal family Z_C. Its projection to $\mathbb{P}^3$ is our ruled surface.

Let Π be a plane in $\mathbb{P}^3$ not containing any of the points p_i. The restriction of f to Π is given by the complete linear system of conics. Thus, its image $f(\Pi)$ is a Veronese surface embedded in $G_1(\mathbb{P}^3)$ as a congruence of secant lines of a rational normal curve R_Π in $\mathbb{P}^3$. The curve R_Π is the image of the map

$$\phi_\Pi : \mathcal{Q} \cong \mathbb{P}^1 \to \mathbb{P}^3 \tag{10.67}$$

which assigns to a quadric $Q \in \mathcal{Q}$ the intersection of polars $P_x(Q), x \in \Pi$. For any line ℓ in Π, the ruled surface S_ℓ is a quadric containing R_Π. So, one can identify the net of quadrics containing R_Π with the dual plane $\Pi^\vee$. More generally, for any curve C in Π of degree m, the ruled surface S_C is a surface of degree $2m$ containing R_Π. Consider a point $x \in \Pi$ as the intersection point of two lines ℓ_1 and ℓ_2 in Π. Then the line $f(x)$ is contained in the intersection of the two quadrics S_{ℓ_1} and S_{ℓ_2}. Hence it coincides with a secant of the curve R_Π. Thus, we obtain that generators of S_C are secants of R_Π. If $m = 2$, this gives that $f(C)$ is the intersection of a Veronese surface with a linear line complex, a general choice of Π gives us quartic surfaces of type I (i).

Take a line ℓ in Π. The quadric S_ℓ comes with a ruling on the quadrics whose generators are secants of R_Π. The set of lines in Π that parameterizes singular quadrics containing R_Π is a conic in $\Pi^\vee$. The dual conic $\mathfrak{C}$ in Π parameterizes pencils of quadrics containing R_Π and its line tangent. The corresponding ruled quartic surface is the developable quartic surface, a special case of type I (iii). The points on the line are pencils of quadrics containing Q_ℓ. If ℓ is tangent to $\mathfrak{C}$, then the tangency point is a pencil of quadrics which all tangent to $R_3(\Pi)$ at one point. The point is the singular point of a unique singular quadric in the pencil.

The lines $f(x), x \in \ell$, are generators of the quadric S_ℓ which intersect R_Π at two points. If ℓ is tangent to $\mathfrak{C}$ then S_ℓ is a singular quadric and all the lines $f(x), x \in \ell$, pass through its singular point. The curve R_Π also passes through this point. In this case, the line ℓ intersects a curve C of degree m in Π at m points different from $\mathfrak{C}$, all the generators of S_C corresponding to these points must pass through one point on R_Π. The converse is also true, if the generators $f(x), x \in C$, all pass through the same point on R_Π, then these points lie on a line tangent to $\mathfrak{C}$. Thus we obtain that R_Π is m-multiple curve on S_C. This agrees with type $1(i)$ of quartic ruled surfaces. Also note that C intersects $\mathfrak{C}$ at

$2m$ points corresponding to generators tangent to R_Π. If $m = 2$, we get four torsal generators.

Now, let us see what happens if we choose special plane Π. For example, let us take Π passing through one of the points $p_1, \dots, p_4$, say p_1. Then the map ϕ_Π defined in (10.67) is not anymore of degree 3. In fact, it is not defined at the quadric Q which has p_1 as its singular point. The map extends to a map of degree 2. Thus the cubic R_Π degenerates to a conic. The lines in Π correspond to quadrics containing the conic R_Π and some line intersecting the conic. This is a degeneration of the singular curve to the union of a conic and a line.

Finally, let us see how elliptic quartic surfaces arise. Take Π passing through the points p_1 and p_2. Take a nonsingular cubic C in the plane which passes through p_1 and p_2. The linear system of quadrics defining the rational map f has two of its base points on C. Thus its image in $G_1(\mathbb{P}^3)$ is a quartic elliptic curve. We see that a ruled surface of degree 6 which corresponds to a general cubic degenerates in this case to the union of a quartic surface and two planes (the images of the blow-ups of p_1 and p_2). The cubic R_Π degenerates to a line, one of the two double lines of S. A quadric corresponding to a line through p_1 or p_2 degenerates to a plane with a choice of a pencil of lines in it. This plane does not depend on the line, but the pencil of lines in the plane does. The line passing through p_1 and p_2 is blown down under f to a point in $G_1(\mathbb{P}^3)$ defining the second double line of S_C. This is the intersection line of the planes corresponding to p_1 and p_2.

Exercises

10.1 Let $P_n \subset \mathbb{C}[t]$ be the space of polynomials of degree $\leq n$. Let $f_0, \dots, f_m$ be a basis of a subspace L of P_n of dimension $m + 1$. Consider the *Wronskian* of the set $(f_0, \dots, f_m)$

$$W(f_0, \dots, f_m) = \det \begin{pmatrix} f_0 & f_1 & \dots & f_m \\ f_0'(t) & f_1'(t) & \dots & f_m'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(m)}(t) & f_1^{(m)}(t) & \dots & f_m^{(m)}(t) \end{pmatrix}.$$

Show that the map

$$G_m(\mathbb{P}^n) \to \mathbb{P}^{(m+1)(n-m)}, \quad L \mapsto [W(f_0, \dots, f_m)]$$

is well defined and is a finite map of degree equal to the degree of the Grassmannian in its Plücker embedding.

10.2 Show that any $\binom{n+1}{2} - 1$ lines in $G_1(\mathbb{P}^n), n \geq 3$, lie in a linear line complex. Using this, prove that one can choose coordinates in $\mathbb{P}^n$ so that any linear line complex

can be given by Plücker equations $p_{12} + \lambda p_{34} = 0$, where $\lambda = 0$ if and only if the line complex is special.

10.3 Show that the tangent lines of any smooth curve of genus g and degree d in $\mathbb{P}^n$ is contained in a linear line complex if $2(d+g-1) < \binom{n+1}{2}$.

10.4 Show that any k-plane Λ of $G_m(\mathbb{P}^n)$ coincides with the locus of m-planes in $\mathbb{P}^n$ containing a fixed $(m-1)$-plane and contained in a fixed $(m+k)$-plane or with the locus of m-planes contained in a fixed $(k+1)$-plane and containing a fixed $(k-m)$-plane. Identify these loci with appropriate Schubert varieties.

10.5 Using the previous exercise, show that any automorphism of $G_r(\mathbb{P}^n)$ arises from a unique projective automorphism of $\mathbb{P}^n$ unless $n = 2r+1$, in which case $\mathrm{PGL}(n+1)$ is isomorphic to a subgroup of index 2 of $\mathrm{Aut}(G_r(\mathbb{P}^n))$.

10.6 How many lines intersect a set of m general k-planes in $\mathbb{P}^n$?

10.7 Show that $\mathrm{Sec}_k(G_1(\mathbb{P}^n))$ is equal to the set of singular points of $\mathrm{Sec}_{k+1}(G_1(\mathbb{P}^n))$ for all $k = 0, \ldots, [\frac{n-3}{2}]$.

10.8 Using Schwarzenberger vector bundles, prove that the projective plane embedded in $G_1(\mathbb{P}^n)$ as the surface of secants of a normal rational curve of degree d in $\mathbb{P}^d$ is isomorphic to the Veronese surface V^2_{n-1}.

10.9 Let $Q_1$and Q_2 be two nonsingular quadrics in $\mathbb{P}^3$ with a choice of a ruling of lines on each of them. Any general line ℓ intersects $Q_1 \cup Q_2$ at four lines, two from each ruling. Together with ℓ, these lines span four planes in the pencil of planes through ℓ. Show that the closure of the locus of lines ℓ such that the four planes is projectively equivalent to the four intersection points of ℓ with Q_1 and Q_2 form a Battaglini line complex. Also show that any general Battaglini line complex can be obtained in this way [525].

10.10 Show that the linear system of quadrics in $\mathbb{P}^4$ passing through a normal rational quartic curve R_4 defines a rational map $\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ whose image is a nonsingular quadric in $\mathbb{P}^5$ identified with the Klein quadric $G(2,4)$. Show that:

(i) the secant variety $S_1(R_4)$ is mapped to a Veronese surface;
(ii) the map Φ extends to a regular map of the blow-up of $\mathbb{P}^4$ along R_4 that maps the exceptional divisor to a ruled hypersurface of degree 6 which is singular along the Veronese surface;
(iii) the image of a hyperplane in $\mathbb{P}^4$ is a tetrahedral line complex;
(iv) the image of a plane in $\mathbb{P}^4$ not intersecting R_4 is a Veronese surface;
(v) the image of a trisecant plane of R_4 is a plane in $G(2,4)$. Show that planes from another family of planes are the images of cubic ruled surfaces singular along R_4.

10.11 Show that four general lines in $\mathbb{P}^4$ determine the unique fifth one such that the corresponding points in $G_1(\mathbb{P}^4) \subset \mathbb{P}^9$ lie in the same three-dimensional subspace. Any plane which meets four lines meets the fifth line (called the *associated line*).

10.12 Show that two linear line complexes $\mathfrak{C}_\omega$ and $X_{\omega'}$ in $G_1(\mathbb{P}^3)$ are apolar to each other if and only if $i_\omega(X_{\omega'}) = \mathfrak{C}_\omega$.

10.13 Show that a general web of linear line complexes in $G_1(\mathbb{P}^4)$ contains five special line complexes.

10.14 Show that the projection of the Segre cubic primal from its nonsingular point is a double cover with branch locus isomorphic to a Kummer surface.

10.15 Using the Schubert calculus, show that the variety of lines contained in a cubic hypersurface in $\mathbb{P}^4$ with isolated singularities is a surface of degree 45 in the Plücker

embedding of $G_1(\mathbb{P}^4)$. Show that the variety of lines contained in the Segre cubic primal S_3 is a surface of degree 45 that consists of 15 planes and six del Pezzo surfaces of degree 5.

10.16 Let $\mathcal{N}$ be a general 2-dimensional linear system of quadrics in $\mathbb{P}^3$. Show that the union of lines contained in quadrics $Q \in \mathcal{N}$ is parameterized by a cubic line complex (called a *Montesano line complex*) [392].

10.17 Let $p_1, \ldots, p_{nd+1}$ be points in $\mathbb{P}^n$ in general linear position. A *monoidal line complex* consists of all codimension 2 linear subspaces Π of $\mathbb{P}^n$ for which there exists a monoidal hypersurface with singular locus containing to Π. Using the isomorphism $G_{n-2}(\mathbb{P}^n) \cong G_1(\mathbb{P}^n)$, we consider it as a line complex. Show that the degree of a monoidal line complex is equal to $\frac{1}{2}3d(d-1)$ and it coincides with a Montesano line complex when $n = d = 3$ [178].

10.18 Consider a smooth curve C of degree d and genus g in $\mathbb{P}^3$ and choose two general lines ℓ and ℓ'. Find the degree of the scroll of lines that intersect C, ℓ and ℓ'.

10.19 Let F be a surface of degree 6 in $\mathbb{P}^3$ which has the edges of the coordinate tetrahedron as its double lines. Find an equation of F and show that its normalization is an Enriques surface.

10.20 Show that the Hessian of a developable quartic ruled surface is equal to the surface itself taken with multiplicity 2. The Steinerian in this case is the whole space [590].

10.21 Consider the embedding of the Klein quartic curve of genus 3 in $\mathbb{P}^3$ given by the linear system $|3\theta|$, where θ is the unique even theta characteristic invariant with respect to the group of automorphisms of the curve. Show that each hyperosculating point is of multiplicity 2 and is equal to the image of an inflection point.

10.22 Show that a generator intersecting the double curve of a ruled surface at a pinch point is a torsal generator.

10.23 Classify all ruled surfaces in $\mathbb{P}^3$ which have two line directrices.

10.24 For each type of a quartic ruled surface, find the type of its dual quartic ruled surface.

10.25 Find projective equivalence classes of quartic ruled surfaces with a triple line.

Historical Notes

The main sources for these notes are [362], [369], [524], [432], and [611]. Line Geometry originates from J. Plücker who was the first to consider lines in 3-space as elements of a new four-dimensional space. These ideas had appeared first in [445] and the details were published much later in [446]. The study of lines in $\mathbb{P}^3$ was very much motivated by mechanics and optics. An early differential geometrical treatment of line geometry can be found in the works of E. Kummer [349] and [350]. The six Plücker coordinates p_{ij} of a line were first introduced by H. Grassmann in 1844 [260] in a rather obscure notation. Unaware of the work of Grassmann, in 1859 A. Cayley introduced the coordinates in its modern form as six determinants of a 2×4-matrix and exhibited the quadric equation satisfied by the coordinates [80]. In a subsequent paper,

under the same title, he introduced, what is now called the Chow form of a space curve. The notions of a linear line complex of lines and a congruence of lines (the intersection of two linear line complexes) are due to Plücker, and the first proofs of some of his results were given by G. Battaglini [30]. Among other earlier contributers to theory of general line complexes we cite M. Pash [433].

Plücker began the study of quadratic line complexes by introducing a singular quartic surface with 16 nodes. Although, in a special case, many Plïcker's results about quadratic line complexes were independently obtained by Battaglini. In his dissertation, and later published paper [336], Klein introduced the coordinate system determined by six mutually apolar linear line complexes and showed that the singular surface can be identified with a Kummer surface. The notion of the singular surface of a quadratic complex is due to Klein. We refer to [306] and [319] for the history of Kummer surfaces and their relationship with Line Geometry. We followed [319] in deriving the equation of a Kummer surface in Klein coordinates.

Plücker defined a linear complex as we understand it now, i.e. as a set of lines whose coordinates satisfy a linear equation. The set of lines in a linear complex passing through a point x lies in a plane $\Pi(x)$; this defines a linear correlation from the space to the dual space. The correlations arising in this way satisfy the property $x \in \Pi(x)$. They were first considered by G. Giorgini [246] and A. Möbius [391] and were called Nullsystems by von Staudt ([540], p. 191). The notions of a null-line and a null-plane belong to Möbius. Chasles' Theorem 10.2.10 gives a purely geometric definition of a Nullsystem [93]. Linear systems of linear line complexes were extensively studied in Sturm's book [552].

In 1868, in his Inauguraldissertation at Bonn published later in [336], [339], F. Klein pointed out that Weierstrass's theory of canonical forms for a pair of quadratic forms can be successfully used for the classification of quadratic line complexes. This was accomplished later by A. Weiler (see also [593], [519]). The classification consists of 49 different types of line complexes corresponding to different Segre symbols of the pencil of quadrics. As we have already noticed earlier, the Segre symbol was first introduced by A. Weiler [598] and Segre acknowledges this himself in [519]. In each case the singular surface is described. For example, some of the ruled quartic surfaces can be obtained as singular surfaces of a degenerate quadratic complex. A full account of the classification and the table can be found in Jessop's book [319]. Many special quadratic line complexes were introduced earlier by purely geometric means. Among them are the tetrahedral line complexes and Battaglini's harmonic line complexes [31] considered in the present Chapter. A complete historical ac-

count of tetrahedral line complexes can be found in Lie's book [362]. Its general theory is attributed to T. Reye [461] and even they are often called Reye line complexes. However, in different disguises, tetrahedral line complexes appear in much earlier works, for example, as the locus of normals to two confocal surfaces of degree 2 [42] (see a modern exposition in [527], p. 376), or as the locus of lines spanned by an argument and the value of a projective transformation [94], or as the locus of secants of twisted cubics passing through the vertices of a tetrahedron [402]. We refer to [482] and [285] for the role of tetrahedral line complexes in Lie's theory of differential equations and groups of transformations.

Modern multi-linear algebra originates in Grassmann's work [259], [260]. We refer to [51] for the history of multilinear algebra. The editorial notes for the English translation of [260] are very helpful for understanding Grassmann's work. As a part of Grassmann's theory, a linear k-dimensional subspace of a linear space of dimension n corresponds to a decomposable k-vector. Its coordinates can be taken as the coordinates of the linear subspace and of the associated projective subspace of $\mathbb{P}^{n-1}$. In this way Grassmann was the first to give a higher-dimensional generalization of the Cayley-Plücker coordinates of lines in $\mathbb{P}^3$. Equations (10.3) of Grassmann varieties could not be found in his book. The fact that any relation between the Plücker coordinates follows from these relations was first proven by G. Antonelli [9] and much later by W. Young [605]. In [504] and [505] H. Schubert defines what we now call Schubert varieties, and computes their dimensions and degrees in the Plücker embedding. In particular, he finds the formula for the degree of a Grassmann variety. A modern account of Schubert's theory can be found in Hodge-Pedoe's book [302], v. II and Fulton's book [231].

The study of linear line complexes in arbitrary $[n]$ (the classical notation $[n]$ for $\mathbb{P}^n$ was introduced by Schubert in [504]) was initiated in the work of S. Kantor [328], F. Palatini [424] and G. Castelnuovo [68] (in case $n = 4$). The Palatini scroll was first studied in [425] and appears often in modern literature on vector bundles (see, for example, [419]). Quadratic line complexes in $\mathbb{P}^4$ were extensively studied by B. Segre [514]. Although ruled surfaces were studied earlier (more from differential point of view), A. Cayley was the fist who laid the foundations of the algebraic theory of ruled surfaces [75], [82], [83]. The term scroll belongs to Cayley. The study of non-normal surfaces in $\mathbb{P}^3$ and, in particular, ruled surfaces, began with G. Salmon [488], [489]. Salmon's work was extended by A. Cayley [87]. The formulas of Cayley and Salmon were revised in a long memoir of H. Zeuthen [609] and later in his book [610]. A modern treatment was given by R. Piene [438]. The fact that the class of a ruled surface is equal to its degree is due to Cayley. The degree of a

ruled surface defined by three directrices from Example 10.4.11 was first determined by G. Salmon [487]. Cubic ruled surfaces were classified by A. Cayley in [83], Part II, and, independently, by L. Cremona [139]. The classification of quartic ruled surfaces were started by A. Cayley [83], Parts II and III. However, he had missed two types. A complete classification was given later by L. Cremona [145]. An earlier attempt for this classification was made by M. Chasles [94]. The classification based on the theory of tetrahedral line complexes was given by B. Wong [604]. Ruled surfaces of degree 5 were classified by H. Schwarz [507]. Much later this classification was extended to surfaces of degree 6 by W. Edge [200]. Edge's book and Sturm's book [551], vol. 1, give a detailed exposition of the theory of ruled surfaces. The third volume of Sturm's book contains an extensive account of the theory of quadratic line complexes.

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